

Directed polymers and Randomness

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Abstract

The effects of two types of randomness on the behaviour of directed polymers are discussed in this chapter. The first part deals with the effect of randomness in medium so that a directed polymer feels a random external potential. The second part deals with the RANI model of two directed polymers with heterogeneity along the chain such that the interaction is random. The random medium problem is better understood compared to the RANI model.

1 Directed polymers

A long flexible elastic string, to be called a polymer, has several features of a critical system, showing power law behaviours without much fine tuning [1, 2]. An important quantity for a polymer is its size or the spatial extent as the length N becomes large. For a translationally invariant system with one end ($z = 0$) fixed at origin, the average position at $z = N$ is zero but the size is given by the rms value

$$\langle \mathbf{r}_N \rangle = 0, \quad \langle r_N^2 \rangle^{1/2} \sim N^\nu, \quad (1)$$

with $\nu = 1/2$, for the free case. Similar power laws can be defined in other properties also. In general, such exponents like the size exponent ν define the polymer universality class and these depend only on a few basic elements of the polymer. In addition to the geometric properties, the usual thermodynamic quantities, e.g. free energy (or energy at temperature $T = 0$), entropy etc., are also important, especially if one wants to study phase transitions.

Power laws generally imply absence of any length scale in the problem. Given a microscopic Hamiltonian with its own small length scales like the bond length, interaction range etc, power laws occur only in the long distance limit (large N) for thermally averaged quantities which require summing over all possible configurations. As a result, in the long distance limit, these power laws become insensitive to minute details at the microscopic level, a feature that may be exploited to choose appropriate simplified models to describe a polymer.

In thermal equilibrium, the Boltzmann distribution ultimately determines the macroscopic behaviour. In most cases one may avoid the issue of probability distribution but instead may concentrate only on the first few, may be the first two, moments or cumulants.

For example, one needs to know the average energy, entropy etc and the various response functions which come from the width of the distribution. Thermodynamic descriptions do not generally go beyond that.

In random physical systems, one encounters an extra problem. If the randomness is not thermalizable (“quenched”), any quantity of interest becomes realization dependent. As a result, an additional disorder averaging has to be done over and above the usual thermal averaging for each realization. It is therefore necessary to know if and how this extra averaging alters the behaviour of the system.

Critical-like systems [3] become a natural choice for studying the effect of quenched randomness because it is hoped that at least some of the properties would be insensitive to the minute details of the randomness. Since for a critical system, the influence of the randomness on a long scale is what matters, it transpires that the critical behaviour will be affected if the disorder is a relevant variable. In the renormalization group language, a coupling is classified as *relevant, irrelevant or marginal*, if, with increasing length scale, it grows, decays or remains invariant, because the contribution of a relevant quantity cannot be ignored at long distances even if it is numerically small to start with.

With relevant disorder, there is the obvious possibility of a change in the critical properties (e.g. new set of critical exponents). More complex situations may include emergence of new or extra length scales. One may recall that approach to criticality, most often, is described by a diverging length scale. Developing a description of the system in terms of this large length scale only goes by the name of *scaling* theory. Emergence of any new or extra length scale would then alter the corresponding scaling description. In case it is possible to change the nature of disorder from relevant to irrelevant (say by changing temperature), then a phase transition would occur which will have no counterpart in the pure problem. For the disorder-dominated phase, on a large scale, there are possibilities of rare events (see Appendix A) which necessitates a distinction between the average value and the typical (e.g. most probable) value. In such situations higher moments of the quantity concerned become important. These are some of the aspects that make disorder problems important, interesting and difficult.

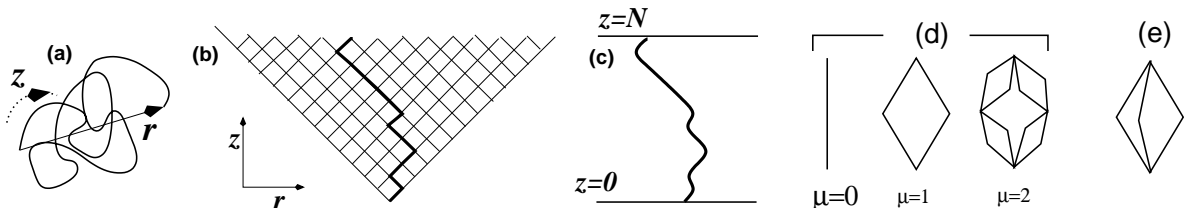


Figure 1: (a) A random walk in d dimensions with z as the variable along the contour of the polymer i.e. giving the location of the monomers. (b) Directed polymer on a square lattice. A polymer as of (a) can be drawn in $d + 1$ dimensions. This is like a path of a quantum particle in nonrelativistic quantum mechanics. (c) A situation where both the transverse space (\mathbf{r}) and z are continuous. (d) The directed polymers on a hierarchical lattice. Three generations are shown for 4 bonds. (e) A general motif of $2b$ bonds.

The problem of a polymer in a random medium was initiated by Chakrabarti and Kertesz [4, 5] with the application of the Harris criterion. This problem has enriched our overall understanding of polymers and random systems in general, but still a complete understanding

remains elusive. Not surprisingly, the look out for simpler problems that capture the basic essence of the original complex system gained momentum. In this context, directed polymers played a very crucial role.

Let us define the problem here. Consider a polymer where each monomer sees a different, independent, identically distributed random potentials. Geometrically this can be achieved if the monomers live in separate spaces. One way to get that is to consider the polymer to be a $d + 1$ dimensional string with the monomers in d dimensional planes but connected together in the extra dimension. As shown in Fig. 1, this is a polymer which is directed in one particular direction. Hence the name directed polymer [6, 7, 8, 9].

For a directed polymer, the size would now refer to the size in the transverse d -directions and so Eq. (1) refers to the transverse size as the length in the special z -direction increases. For long enough chains, it is this size that matters and enters the scaling description.

The significance of directed polymer lies in the fact that the pure system is very well understood and exactly solvable in all dimensions while the random problem can be attacked in several different ways, a luxury not affordable in most situations.

Two types of randomness can arise in the context of directed polymers. One type would involve imposition of a random external potential (random medium problem). In the second type, the interaction (between say two chains) is random (RANI model). In the random medium problem, the random potential would like to have a realization dependent ground state which may not match with the zero-field state. In the RANI model, the randomness in the interaction may lead to a change in the phase transition behaviour shown by the polymers. These two classes are discussed separately.

2 Outline

We first consider the random medium problem and then the random interaction (RANI) model. In both cases, disorder turns out to be marginally relevant though at two different dimensions. The quantities of interest in a disordered system are introduced in Sec. 3. For the random medium problem, the question of relevance of disorder, the size exponent ν and the free energy fluctuation exponent θ are discussed in Secs. 4 and 5. Sec 4 deals with the renormalization group (RG) for the moments of the partition function and also the special Bethe ansatz results for the $1 + 1$ dimensional problem. A Flory approach and scaling ideas are also summarized there. Sec. 5 deals with the RG approach for the free energy via the Kardar-Parisi-Zhang equation. The behaviour of the overlap especially near the transition to strong disorder phase may be found in Sec. 6. We briefly mention the unzipping behaviour in presence of a force and summarize some recent results for the pure case in Sec. 7. More on unzipping of a two chain system may be found in Sec. 9.4. These results and scaling arguments are then used to visualize the nature of the ground state in Sec. 8. Various aspects of the RANI model can be found in Secs. 9 and 10. The question of relevance, and annealed versus quenched disorder in multi chain system are analyzed in Sec. 9. The two different types of randomness or heterogeneity on hierarchical lattices are studied in Sec. 10. In the last part of this section, one may find the general validity and extension of the Harris criterion as applied to this polymer problem. Various technical issues are delegated to the Appendixes. An example of rare events is given in App. A. A short review of the pure polymer behaviour can be found in App. B. The issue of self-averaging and some

recent results about it for disordered systems are surveyed in App. C. The renormalization group approach to polymers as used in Sec. 4.2.1 is spelt out in App. D in the minimal subtraction scheme with dimensional regularization. The momentum shell RG approach for the free energy is dealt with in App. E. All the possible flow diagrams are reviewed in App. F. A short introduction to the transfer matrix approach for the directed polymer problem is given in App. G. The exact RG for the RANI model can be found in App. H.

On Notation and convention:

- To avoid proliferation of symbols, we reserve the symbol f to denote an arbitrary or unspecified function, not necessarily same everywhere.
- By a *sample* or a *realization* we would mean one particular arrangement of the random parameters over the whole system. For a thermodynamic (infinitely large) system the sample space is also infinite.
- *Sample averaging* is denoted by $[\dots]_{\text{av}}$ while *thermal averaging* is denoted by $\langle \dots \rangle$.
- The Boltzmann constant is set, most often, to one, $k_B = 1$.
- “Disorder” and “randomness” will be used interchangeably.

3 Hamiltonian and Randomness

By definition, a directed polymer is defined in D dimensions out of which one direction is special that represents the sense of direction of the polymer. It is then useful to show that explicitly by writing $D = d + 1$ where d is the dimension of the transverse space. One may consider various possible models of the underlying space as shown in Fig. 1.

- One may consider a lattice (square lattice in the Fig. 1b) with the polymer as a random walk on the lattice with a bias in the diagonal z -direction, never taking a step in the $-z$ direction. The length of the polymer is then the number of steps on the lattice.
- Instead of a lattice model, one may consider continuum where both the space and the z -direction are continuous as shown in Fig. 1c. The polymer itself may consist of small rods or bonds whose length provides us with a small scale cut-off. In many situations, it helps to consider polymers without any intrinsic small scale cutoff.
- Quite often it is useful to consider very special lattices, e.g. hierarchical lattices as shown in Figs. 1d, and 1e, because of the possibility of exact analysis. Here one starts with a bond and then replaces the bond iteratively by a predetermined motif (“diamond” of $2b$ bonds) and the process can be iterated *ad infinitum*. This is a lattice with a well-defined dimension (see Sec. 10) but unfortunately without any proper Euclidean distance. Consequently geometric properties are not useful here. The effective dimension of the lattice is $d_{\text{eff}} = (\ln 2b) / \ln 2$, if there are $2b$ bonds per motif. A directed polymer can be taken as a random walk on this lattice starting from say the bottom point, going up, and ending at the top end.

3.1 Pure case

Taking the polymer as an elastic string, one may define a Hamiltonian

$$H_0 = \frac{d}{2}K \int dz \left(\frac{\partial \mathbf{r}}{\partial z} \right)^2 \quad (2)$$

which gives a normalized probability distribution of the position vector \mathbf{r} at length z from the end at $(\mathbf{0}, 0)$

$$P(\mathbf{r}, z) = \frac{1}{(2\pi z)^{d/2}} e^{-r^2/2z} \quad (Kd/k_B T = 1). \quad (3)$$

Here k_B is the Boltzmann constant and T is the temperature. One can even write down the distribution for any two intermediate points (\mathbf{r}_i, z_i) and (\mathbf{r}_f, z_f) as

$$G(\mathbf{r}_f, z_f | \mathbf{r}_i, z_i) = \frac{1}{[2\pi(z_f - z_i)]^{d/2}} \exp \left(-\frac{(\mathbf{r}_f - \mathbf{r}_i)^2}{2(z_f - z_i)} \right) \quad (4)$$

For the lattice random walk, there is no “energy” and the elastic Hamiltonian of Eq. (2) just simulates the entropic effect at non-zero temperatures. One needs to look at the lattice problem in case one is interested in low or zero temperature behavior. A recapitulation of a few properties of polymers is done in Appendix B.

For a polymer of length N the probability distribution gives

$$\langle \mathbf{r} \rangle = 0, \quad \langle r^2 \rangle = N, \quad (Kd/k_B T = 1) \quad (5)$$

so that the transverse size of the polymer is given by

$$R_0 \sim \langle r^2 \rangle^{1/2} \sim N^\nu \text{ with } \nu = 1/2. \quad (6)$$

The power law growth of the size of a polymer as the length increases is a reflection of the absence of any “length scale” in the Hamiltonian.

3.2 Random medium

Let us now put this polymer in a random medium. In the lattice model of Fig. 1, each site has an independent random energy and the total energy of the lattice polymer is the sum of the energies of the sites visited. In continuum, the Hamiltonian can be written as

$$H = H_0 + \int_0^N dz \int d\mathbf{r} \eta(\mathbf{r}, z) \delta(\mathbf{r}(z) - \mathbf{r}) = H_0 + \int_0^N dz \eta(\mathbf{r}(z), z) \quad (7)$$

where $\eta(\mathbf{r}, z)$ is an identical, independent Gaussian distributed random variable with zero mean and variance $\Delta > 0$,

$$[\eta(\mathbf{r}, z)]_{\text{av}} = 0, \quad [\eta(\mathbf{r}, z) \eta(\mathbf{r}', z')]_{\text{av}} = \Delta \delta(\mathbf{r} - \mathbf{r}') \delta(z - z'). \quad (8)$$

The averaging over η is to be called *sample averaging*, denoted by $[\dots]_{\text{av}}$ (as opposed to thermal averaging, denoted by $\langle \dots \rangle$). With this distribution of random energies, we see

$[H]_{\text{av}} = H_0$ and so the average Hamiltonian is not of much use. Disorder averaging of sample dependent thermal averages is to be called *quenched averaging* while disorder averaging done at the partition function level is to be called *annealed averaging*.

We shall consider the situation with one end point ($z = 0$) fixed. Otherwise, the polymer may drift in the medium to locate the best possible region that would minimize its free energy. Such a case, eventhough formally quenched in nature, is tantamount to an annealed case.

3.2.1 Partition function

The partition function for a polymer in a random medium or potential is given by

$$Z = \int \mathcal{DR} e^{-\beta H}. \quad (9)$$

This is a symbolic notation (“path integral”) to denote sum over all configurations and is better treated as a continuum limit of a well-defined lattice partition function

$$Z = \sum_{\text{paths}} e^{-\beta \eta(\mathbf{r}, z)} \quad (10)$$

where the sum is over all possible paths of N steps starting from $\mathbf{r} = \mathbf{0}$ at $z = 0$. It often helps to define the partition function such that $Z(\{\eta = 0\}) = 1$ to avoid problems of going to the continuum limit (see Eq. (3)). This is done by dividing (or normalizing) Z by $Z_0 = \mu^N$, Z_0 being the partition function of the free walker with μ as the connectivity constant (= 2 for Fig. 1b).

Let us define the free energy

$$F = -T \ln Z, \quad (11)$$

for a polymer of length N where the end point at $z = N$ is free. A more restricted free energy may be defined as

$$F(\mathbf{r}, N) = -T \ln Z(\mathbf{r}, N), \quad (12)$$

when the end at $z = N$ is at \mathbf{r} .

3.3 Unzipping and response

Quite often it is useful to force a system to change its overall configuration by applying an external field. The response function then tells us about the rigidity of the system against such external perturbations. E.g., a magnetic field may be applied on a magnet and the magnetic susceptibility is the corresponding response function. A similar applied force for directed polymer is an unzipping force or a pulling force applied at one end (see Fig. 2). We call such a force an unzipping force because of its role in unzipping of DNA-type double stranded polymers [10]. There could be two different ensembles. One is a fixed force ensemble where one applies a force at the free end $z = N$ and studies the change in the size and shape of the polymer or its response. The position of the end point is given by $r = -T \partial F(g, N) / \partial g$. The second is the fixed distance ensemble where the free end is at point \mathbf{r} and then what is

the force required to maintain it at that point. Using the constrained free energy, we may write $\mathbf{g} = -T\nabla F(\mathbf{r}, N)$. (Here we used the same notation F to denote the free energies of the two ensembles. The arguments and the context would distinguish the two.) The two ensembles behave differently in a disordered system.¹

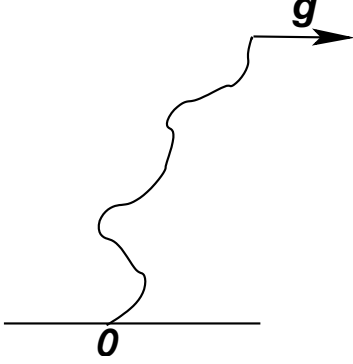


Figure 2: A directed polymer with an unzipping force.

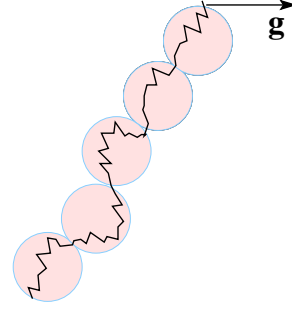


Figure 3: A blob picture of the polymer under a force. Though drawn as sphere, the z -direction is elongated with isotropy in the transverse direction.

If we consider the response of a directed polymer to the unzipping force, the response function comes from the Hamiltonian

$$\begin{aligned} H &= \frac{d}{2}K \int dz \left(\frac{\partial \mathbf{r}}{\partial z} \right)^2 + \int dz \eta(\mathbf{r}(z), z) - \mathbf{g} \cdot \int \frac{\partial \mathbf{r}}{\partial z} dz \\ &= \frac{d}{2}K \int dz \left(\frac{\partial}{\partial z} \left(\mathbf{r} - \frac{\mathbf{g}z}{Kd} \right) \right)^2 + \int dz \eta(\mathbf{r}(z), z) - \frac{1}{2} \frac{g^2 N}{Kd}. \end{aligned} \quad (13)$$

The disorder is Gaussian-distributed as in Eq. (8). The general response function for the force is

$$C_T|_{ij} = \left. \frac{\partial^2 [\ln Z]_{\text{av}}}{\partial g_i \partial g_j} \right|_{g=0} = [\langle r_i r_j \rangle - \langle r_i \rangle \langle r_j \rangle]_{\text{av}}, \quad (14)$$

with i, j representing the components. It is known in statistical mechanics that the response of a system in equilibrium is determined by the fluctuations.

3.3.1 Exact result on response: pure like

By a redefinition of the variables and using the δ -correlation of the disorder in the z direction, we have

$$[\ln Z(\mathbf{g})]_{\text{av}} = [\ln Z(\mathbf{g} = 0)]_{\text{av}} + \frac{g^2 N}{Kd}, \quad (15)$$

from which it follows that

$$C_T = \frac{TN}{Kd}, \quad (16)$$

¹The inequivalence of the two ensembles is known also for pure case if the force is applied at some intermediate point [11]

as one would expect in a pure system, Eq. (3). And there are no higher order correlations.

Two things played important roles in getting this surprising pure-like result: (i) The disorder correlation has a statistical translational invariance coming from the delta function in the z -coordinate, and (ii) the quadratic nature of the Hamiltonian. If disorder had any correlation along the length of the polymer, Eq. (16) will not be valid.

The significance of the result is that the conventional thermal fluctuation, averaged over randomness, superficially does not say much about the effect of disorder. We shall see later that this innocuous result however contains important information.

3.3.2 Quantities of interest

Let us list some of the quantities which are of interest for a disordered system.

- A random system needs to be described by the probability distribution of various physical quantities or by the averages and moments (over realizations). The moments are useful, especially in absence of full information on the probability distribution and also for characterization of the properties of the polymer. Since there is no unique partition function, one of the important probability distributions would be of the partition function, $P(Z)$. Any quantity of interest needs to be averaged over such a distribution. Similarly the probability distribution $P(F)$ of the free energy is also of interest. The thermodynamic behaviour is obtained from $[F]_{\text{av}}$. In case the probability distribution (over the realizations) of a thermal averaged quantity X becomes very sharp, especially in the large size limit, one may avoid doing the disorder averaging. This may happen for extensive quantities because of additivity over subsamples. Such quantities are called self-averaging. Certain aspects of self-averaging is discussed in Appendix C.
- The first thing to determine is the relevance of disorder. To do so, we may write

$$[\ln Z]_{\text{av}} = [\ln\{[Z]_{\text{av}} + (Z - [Z]_{\text{av}})\}]_{\text{av}} = \ln [Z]_{\text{av}} + \frac{[Z^2]_{\text{av}} - [Z]_{\text{av}}^2}{2[Z]_{\text{av}}^2} + \dots \quad (17)$$

This shows the importance of the variance of the partition function. If the variance remains small, in the limit $N \rightarrow \infty$, then the polymer can be described by the average partition function which is more or less like a pure problem. Otherwise not. We see that the relevance of the disorder may be inferred from the nature of the variance of the partition function.

- For the partition function we may use the simple identity

$$[Z^n]_{\text{av}} = [e^{n \ln Z}]_{\text{av}} = \exp\left(\sum \frac{1}{m!} n^m [(\ln Z)^m]_{\text{av}}^{(c)}\right), \quad (18)$$

where $[(\ln Z)^m]_{\text{av}}^{(c)}$ are the cumulants. In contrast to Eq. (17), it is now the fluctuations of the free energy that become important. One may introduce a scaling behaviour, namely

$$[(\ln Z)^2]_{\text{av}}^{(c)} \equiv [(\ln Z)^2]_{\text{av}} - [\ln Z]_{\text{av}}^2 \sim N^{2\theta}, \quad (19)$$

defining a new exponent θ . Obviously, for a pure problem $\theta = 0$. If higher order fluctuations (or cumulants) do not require any new exponent, then it is fair to expect $[(\ln Z)^m]_{\text{av}}^{(c)} \sim N^{m\theta}$. This free energy fluctuation exponent is one of the new quantities required to describe the random system.

- A simpler form of Eq. (18) is the basis of the replica approach for disordered systems, namely

$$[\ln Z]_{\text{av}} = \lim_{n \rightarrow 0} \frac{[Z^n]_{\text{av}} - 1}{n}. \quad (20)$$

so that to compute the average free energy we may consider a case of n -replicas of the original system or after averaging, an n -polymer problem with extra interactions induced by the disorder though an $n \rightarrow 0$ limit is to be taken at the end. A few possible paths to take the limit for long chains are shown in Fig. 4.

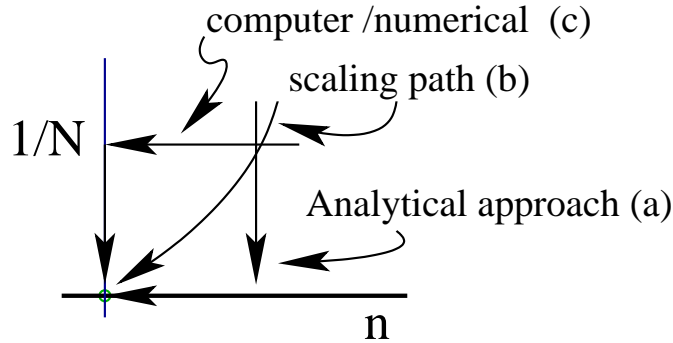


Figure 4: Paths for replica approach

Nontrivial results are expected if and only if the origin in Fig. 4 is a singular point so that the limits $n \rightarrow 0$ and $N \rightarrow \infty$ become non-interchangeable. In other words, the n and N dependences should be coupled so that the appropriate path is a scaling path like (b) in the figure 4.

- If we demand that $\ln [Z^n]_{\text{av}}$ is proportional to N for large N , then, apart from the extensive term ($\propto nN$), there will be corrections which may be assumed to involve a scaling variable $x = nN^\theta$. For $x \rightarrow \infty$, $\mathcal{F}(x) \sim x^{1/\theta}$ so that

$$\ln [Z^n]_{\text{av}} = n\epsilon N + \mathbf{a}n^{1/\theta}N. \quad (N \rightarrow \infty) \quad (21)$$

This is for path (a) of Fig. 4. In contrast, if we take $n \rightarrow 0$ for finite N , path (c), a Taylor series expansion gives

$$\ln [Z^n]_{\text{av}} = nN\epsilon + \mathbf{a}n^2N^{2\theta} + \dots \quad (n \rightarrow 0) \quad (22)$$

Eqs. 21,22 can be used to calculate θ , the free energy fluctuation exponent.

- So far as the geometrical properties are concerned, we first note the lack of translational invariance for a particular realization of disorder and therefore $\langle \mathbf{r} \rangle \neq 0$, but on averaging over randomness, translational invariance will be restored statistically and

so $[\langle \mathbf{r} \rangle]_{\text{av}}$ has to be zero. One may therefore consider the size of the polymer by the “disorder” correlations

$$C_{\text{dis}} \equiv [\langle \mathbf{r}^2 \rangle]_{\text{av}} \text{ or } C_{\text{dis}} \equiv [|\langle \mathbf{r} \rangle|]_{\text{av}}^2 \text{ with } C_{\text{dis}} \sim N^{2\nu} \quad (23)$$

We have already seen in Eq. (16) that the usual correlation function is the thermal correlation of the pure problem and has no signature of the disorder. Of course, the disordered averaged probability distribution $P(\mathbf{r}, N) = [Z(\mathbf{r}, N) / \int d\mathbf{r} Z(\mathbf{r}, N)]_{\text{av}}$ is also of importance.

- For the pure case ($\Delta = 0$), there is no “energy”, only configurational entropy of the polymer. But with $\Delta \neq 0$, there may be one or more lowest energy states. The nature of the ground state is an important issue for any disordered problem. For the lattice problem, the energy is the sum of $N (\rightarrow \infty)$ random energies of the sites visited. But it is the minimization over a connected path that makes the problem difficult. If the low temperature behaviour of the polymer is same as that at $T = 0$, the fluctuation of the ground-state energy will also be described by the exponent θ of Eq. (19). Same is true for the size exponent also. Such a situation requires that the disorder dominated phase is controlled by the “zero temperature” fixed point.
- The problem we face in a disordered system is that there is no well defined ground state - the ground state is sample dependent. There is therefore no predefined external field that will force the system into its ground state (e.g., a magnetic field in a ferromagnetic problem). This is a generic problem for any random system.

But, suppose, we put in an extra fictitious (ghost) polymer and let it choose the best path. Now we put in the actual polymer in the same random medium but with a weak attraction v with the ghost. At $T = 0$, this polymer will then sit on top of the ghost. In absence of any interaction ($v \rightarrow 0$), the polymer would go over the ghost in any case if there is a unique ground state, not otherwise. At non-zero temperatures there will be high energy excursions and how close to the ground state we are will be determined by the number of common points of the two polymers. This is called overlap which may be quantitatively defined as

$$q_i = \frac{1}{N} \int \delta(\mathbf{r}_1(z) - \mathbf{r}_2(z)), \quad (24)$$

for a given sample i and then one has to average over the disorder samples, $q \equiv [q_i]_{\text{av}}$.

In case of a repulsive interaction, the situation will be different. If there is more than one ground state, the two chains will occupy two different paths and there is no energetic incentive to collapse on top of each other when the repulsion $v \rightarrow 0+$. In such a scenario, the overlap $q(v \rightarrow 0+) \neq q(v \rightarrow 0-)$. Conversely, a situation like this for the overlap would indicate presence of degenerate ground states. For a unique ground state, the second chain would follow a nearby excited path with certain amount of overlaps with the ground state. A little reflection shows that overlap is associated with the second moment of the partition function.

4 Relevance of disorder

Let us first see if disorder is at all relevant.

4.1 Annealed average: low temperature problem

If the disorder is irrelevant, then the average partition function $[Z]_{\text{av}}$ (annealed average) is expected to give the thermodynamic behaviour. However this extra averaging of the partition function may lead to a violation of the laws of thermodynamics, questioning the validity of the annealed averaging. This happens for directed polymers. With a Gaussian distribution for the random energy, from Eq. (10),

$$[Z]_{\text{av}} = \exp(\beta^2 \Delta N/2) \exp(N \ln \mu), \quad (25)$$

so that $F/N = -T(\ln \mu + \beta^2 \Delta/2)$. The entropy obtained from this partition function ($S = -\partial F/\partial T$) by definition has to be positive which requires $T \geq T_A \equiv \sqrt{\frac{\Delta}{2 \ln \mu}}$. Annealed averaging will not work at very low temperatures.

This does not necessarily mean that something like a phase transition has to happen, because this problem will occur for any disordered system, even finite in size. However, for the directed polymer problem, this does signal a phase transition, though the proof comes from other analysis.

4.2 Moments of Z

The moments can be written as the partition function of an n -polymer problem with an extra interaction induced by the disorder. Starting from H as given by Eq. (7), and averaging over the Gaussian distribution of Eq. (8), we have

$$[Z^n]_{\text{av}} = \int \mathcal{D}\mathcal{R}_1 \mathcal{D}\mathcal{R}_2 \dots \mathcal{D}\mathcal{R}_n e^{-\beta H_n},$$

where $H_n = \frac{Kd}{2} \int_0^N dz \sum_i^n \left(\frac{\partial \mathbf{r}_i}{\partial z} \right)^2 - \beta \Delta \int_0^N dz \sum_{i < j} \delta(\mathbf{r}_i(z) - \mathbf{r}_j(z)).$ (26)

This particular form can be understood in terms of two polymers, to which one may restrict if the interest lies only in the second cumulant of Z . These two polymers start from the same point and do their random walk as they take further steps. If there is a site which is energetically favourable, both the polymers would like to be there. The effect is like an attractive interaction between the two polymers - an interaction induced by the randomness. For the many polymer problem (for the n -th moment), there is nothing beyond two polymer interaction. This has to do with the nature of correlation of disorder.

4.2.1 Bound state: two polymer problem and RG

For the second moment, we have a two polymer problem. The analogy with quantum mechanics tells us that for $d < 2$, any binding potential can form a bound state but a critical strength is required for a bound state for $d > 2$. In the polymer language, this means that any small disorder will change the behaviour of the free (pure) chain for $d < 2$ (*disorder is always relevant*) but for $d > d_c = 2$, if $\beta \Delta < (\beta \Delta)_c$, the chain remains pure-like (*disorder is irrelevant*). Actually in higher dimensions ($d > 2$) the delta-function potential needs to be regularized appropriately (e.g., by a ‘‘spherical’’ well). Such cases are better treated by renormalization group (RG) which also helps in making the definitions of relevance/irrelevance more precise. We discuss this below [12].

In short, the second term (fluctuation in partition function) in the expansion of Eq. (17) cannot be ignored if $d < 2$ or if $\beta\Delta$ is sufficiently large for $d > 2$. This signals a disorder dominated phase for all disorders in low dimensions or at low temperatures (strong disorder) in higher dimensions.

4.2.2 Expansion in potential

We do an expansion in the interaction potential and just look at the first contributing term. The full series can of course be treated exactly. On averaging, the first order terms drop out, yielding

$$[Z^2]_{\text{av}} - [Z]_{\text{av}}^2 = \int \mathcal{D}\mathcal{R}_1 \int \mathcal{D}\mathcal{R}_2 e^{-\beta H_{0,1}} e^{-\beta H_{0,2}} \int dz \beta^2 \Delta \delta(r_1(z) - r_2(z)) + \dots \quad (27)$$

This is the first order term if Eq. (26) is used. A diagrammatic representation is often helpful in book-keeping as shown in Fig. 5. Some details of the renormalization group approach is given in Appendix D.

4.2.3 Reunion

For this two polymer problem, the interaction contributes whenever there is a meeting or reunion of the two polymers at a site [13]. At the order we are considering, there is only one reunion but this reunion can take place anywhere along the chain and anywhere in the transverse direction. The second order term as shown on the right side of Fig. 5 can be thought of as two walkers starting from origin have a reunion anywhere, thereby forming a loop.

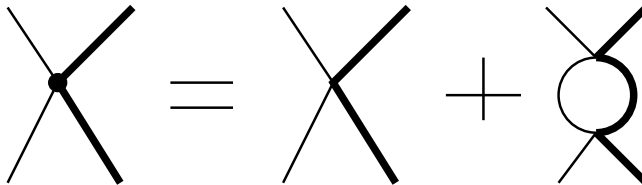


Figure 5: Renormalization of the interaction. The two polymers are represented by the two lines of different thickness and an intersection represents an interaction. A heavy circle on the left hand side represents the effective interaction that is to be used for renormalization.

The probability that two walkers starting from origin would meet at \mathbf{r} at z is given by $P^2(\mathbf{r}, z)$ (Eq. (3)) so that a reunion anywhere is given by a space integral of this probability which gives

$$\mathcal{R}_z \equiv \int d\mathbf{r} P^2(\mathbf{r}, z) = (4\pi z)^{-\Psi}, \quad \text{with } \Psi = d/2. \quad (28)$$

This exponent Ψ will be called the reunion exponent. The occurrence of a power law is again to be noted. The eventual renormalization group approach hinges on this power law behaviour.

It should be noted that the value of the reunion exponent above is that of simple Gaussian chains. This need not be the case with interaction. For example for repulsive interaction between two directed polymers, $\Psi = 3/2$ in $d = 1$ but $\mathcal{R}_N \sim N^{-1}(\ln N)^{-2}$ in $d = 2$. The Gaussian behaviour is recovered in $d > 2$ [13].

4.2.4 Divergences

The contribution in Eq. (27) at the next one loop order ($O(\Delta^2)$) as shown in Fig. 5 involves, apart from some constants, an integral over the reunion behaviour given in Eq. (28). This integral in the limit $N \rightarrow \infty$ is

$$\int_a^N dz \frac{1}{z^{d/2}} \sim a^{1-d/2} \quad \text{for } d > 2 \text{ but } \sim N^{1-d/2} \quad \text{for } d < 2. \quad (29)$$

For a finite cut-off, as is usually the case, the integral is finite for $d > 2$, and therefore $[(Z - [Z]_{\text{av}})^2]_{\text{av}} \approx O(\beta^2 \Delta)$. This however is not the case for $d < 2$ with $d = 2$ as a *borderline case*.

The divergence we find for $d < 2$ comes from the large N behaviour of the probability distribution and is therefore ignorant of the details at the microscopic level. In other words, a lattice model will also show this divergence in low dimensions. This forms the basis of a rigorous analysis done in Ref. [14], but we pursue a renormalization group approach here.

4.2.5 RG flows

The problem of divergence we face here is ideal for a renormalization group approach. Let us introduce an arbitrary length scale L in the transverse direction and define a dimensionless “running” coupling constant

$$u(L) = (\beta^3 K \Delta) L^\epsilon, \quad \epsilon = 2 - d. \quad (30)$$

The purpose behind this length scale is to choose a tunable scale at which we may probe the system. One may then study the RG flow of the coupling constant as the scale L is changed.

In the dimensional regularization scheme we adopt here (see Appendix D for details) the divergence seen in Eq. 29 are handled by analytic continuation in d . The problem of convergence of the integral then appears as singularities at specific d . One then tries to remove these divergences in ϵ by absorbing them in the coupling constants, thereby renormalizing the coupling. This in a sense takes care of reunions at small scales to define the effective coupling on a longer scale. One then has to rescale the system to preserve the original length scales. With this rescaling, one ends up with a description on a coarser scale with small scale fluctuation effects getting absorbed in the redefined parameters. The fact that the process can be implemented without any need of additional parameters is linked to renormalizability of the Hamiltonian.

The effect of renormalization is best expressed by the variation of the parameters or coupling constants with length scale. This is called a flow equation. For the problem in hand, the eventual flow equation

$$L \frac{du}{dL} = (2 - d)u + u^2, \quad (31)$$

where the first term follows from the definition of u while the second one is from the loop. The magnitude of the coefficient of the u^2 term is not very crucial because at this order, this coefficient can be absorbed in the definition of the u itself. *What matters is the sign of the u^2 term.* General cases are discussed in Appendix F.

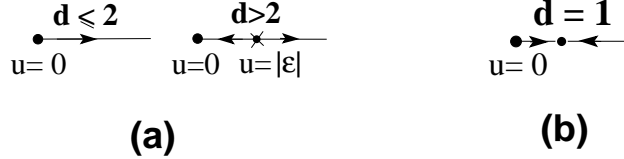


Figure 6: RG Fixed points for u . (a) Based on the second moment of the partition function. (b) Based on the KPZ equation. Arrows show the flow of u . $\epsilon = 2 - d$. General flow diagrams are discussed in App. F.

Since u emanates from the variance of the disorder distribution, it cannot be negative. We therefore need to concentrate only on the $u \geq 0$ part with the initial condition of $u(L = a) = u_0$. What we see is that for $d < 2$, the flow on the positive axis goes to infinity indicating a strong disorder phase for any amount of disorder provided we look at long enough length scale. An estimate of this length scale may be obtained from the nature of divergence for a given u_0 . An integration of the flow equation gives $L \sim u_0^{1/(2-d)}$ ($d < 2$), a crossover length beyond which the effect of the disorder is appreciable.

For $d > 2$, there is a fixed point at $u^* = |\epsilon|$ where $\epsilon = 2 - d$. For $u < u^*$, the disorder strength goes to zero and one recovers a “pure”-like behaviour. This is a *weak disorder* limit. But, if $u > u^*$ the disorder is relevant.

Based on the fixed point analysis, we conclude, as already mentioned, that disorder can be relevant depending on the dimensions we are in (i.e. the value of d) and temperature or strength of disorder. In particular, one finds

1. A disorder-dominated or strong disorder phase for all temperatures for $d \leq 2$.
2. A disorder dominated or strong disorder phase at low temperatures for $d > 2$.

For $d > 2$, one sees a phase transition by changing the strength of the disorder or equivalently temperature for a given Δ . This is an example of a *phase transition induced by disorder* which cannot exist in a pure case.

It is to be noted that the phase transition (the unstable $O(|\epsilon|)$ fixed point) occurs because of the positive u^2 term in the flow equation of Eq. (30). At $d = 2$, u is marginal (no L dependence) but renormalization effects lead to an eventual growth. Such parameters are called marginally relevant. Any marginally relevant variable will produce an unstable fixed point, and hence a phase transition, in dimensions higher than the dimension in which it is marginal. A general statement can then be made: Disorder is expected to produce a new phase transition if it is marginally relevant at its critical dimension.

The new phase transition (a critical point) is to be characterized by its own set of exponents. An important quantity is the length scale behaviour. The flow equation around the fixed point for $d > 2$ shows that one may define a diverging “length-scale” associated with the critical point as

$$\xi \sim |u - u^*|^{-\zeta}, \text{ with } \zeta = 1/|2 - d|. \quad (32)$$

In the critical dimension $d = 2$, there are log corrections. In the weak disorder phase where the disorder is irrelevant, $[\ln Z]_{\text{av}} \approx \ln[Z]_{\text{av}}$, and therefore one may put a bound on the transition temperature T_c for a lattice model as $T_c \geq T_A$ as defined below Eq. (25).

Attempts were made to determine ζ by numerical methods and verify the RG prediction. However, the results from specific heat [15] and size calculations [16] agree neither with each other nor with the RG result of Eq. (32). This remains an open problem.

We come back to the strong disorder phase in Sec. 5.6.

4.3 Bethe ansatz and θ

For the directed polymer problem, a mapping to a quantum problem helps in the evaluation of $[Z^n]_{\text{av}}$ at least in $d = 1$. For a Gaussian distributed, delta-correlated disorder, $[Z^n]_{\text{av}}$ corresponds to the partition function of an n -polymer system with the Hamiltonian given by Eq. (26). Noting the similarity with the quantum Hamiltonian with z playing the role of imaginary time, finding $N^{-1} \ln [Z^n]_{\text{av}}$ for $N \rightarrow \infty$ is equivalent to finding the ground state energy E of a quantum system of n particles. This problem can be solved exactly only in one dimension ($d = 1$) using the Bethe ansatz [23, 24]. This gives the ground state energy as

$$E = -K(n - n^3) \text{ in } d = 1, \quad (33)$$

which gives

$$\theta = \frac{1}{3}, \quad (34)$$

from Eq. (21). As we shall see below, this implies $\nu = 2/3$ so that the polymer has swollen far beyond the random walk or Gaussian behaviour. What looks surprising in this approach is that there is no “variance” (2nd cumulant) contribution. It is just not possible to have a probability distribution whose variance vanishes identically. This is a conspiracy of the $N \rightarrow \infty$ limit inherent in the quantum mapping and the value of the exponent θ that suppressed the second cumulant contribution (see Eq. (21)).

4.3.1 Flory approach

Using the quantum analogy, we may try to estimate the ground state energy in a simple minded dimensionally correct calculation based on the assumption of only one length scale. Such an approach generally goes by the generic name of “Flory approach”. The elastic energy is like the kinetic energy of quantum particles which try to delocalize the polymers (random walk) while the attractive potential tries to keep the polymer together. For n polymers there are $n(n - 1)/2$ interactions. We take the large n limit so that if the particles are bound in a region of size R , the energy is (using dimensionally correct form with R as the only length scale)

$$E = \frac{n}{R^2} - \frac{n^2 \Delta}{R} \quad (35)$$

which on minimization gives $E \sim n^3$ consistent with the Bethe ansatz solution. At this point we see the problem of the replica approach if the limit is taken too soon. Since our interest is eventually in $n \rightarrow 0$, we could have used in this argument the linear term of the combinatorics. That would have made energy “extensive” with respect to the number of particle and replaced the disorder-induced attraction by a repulsion (note the negative sign).

The end result would however have no n^3 dependence. This is a real danger and any replica calculation has to watch out of these pitfalls. Quite strangely we see that the correct answer comes by taking $n \rightarrow \infty$ first and then $n \rightarrow 0$ or, probably better to say, by staying along the “attractive part” of the interaction only.

4.3.2 Confinement energy

Suppose we confine the polymer in a tube of diameter D . This is like the localization length argument used to justify the energy in the quantum formulation. The polymer in the random medium won't feel the wall until its size is comparable to that, $D \sim N_0^\nu$ which gives the length at which the polymer feels the wall. Elastic energy of a blob is D^2/N_0 . But because of the tube, the polymer will be stretched in the tube direction. One may then consider the polymer as consisting of free N/N_0 blobs aligning with the force, so that the energy is

$$\frac{N}{N_0} \frac{D^2}{N_0} = N \left(\frac{D}{N_0} \right)^2 = N \frac{1}{D^{2(1-\nu)/\nu}}. \quad (36)$$

This gives the known form $1/D^2$ used in the quantum analogy, Eq. (35), (and consistent with dimensional analysis) but for $\nu = 2/3$, this gives $1/D$.

A cross-check of this comes from the energy of a blob. Each blob has the fluctuation energy N_0^θ and so free energy per unit length $F/N \sim N_0^\theta/N_0 \sim D^{-2(1-\nu)/\nu}$.

5 Analysis of free energy: specialty of directed polymer

Another unique feature of this directed polymer problem is that there is a way to study the average free energy and implement RG directly for the free energy bypassing the $n \rightarrow 0$ problem of the replica analysis completely, giving an independent way of checking the results of replica approach.

5.1 Free energy and the KPZ equation

For a polymer, the partition function satisfies a diffusion or Schrodinger-like equation. This equation can be transformed to an equation for the free energy $F(\mathbf{r}, z) = -T \ln Z(\mathbf{r}, z)$. This is the free energy of a polymer whose end point at z is fixed at \mathbf{r} . To maintain the distance fixed at \mathbf{r} a force is required which is given by $\mathbf{g} = -\nabla F$. If we want to increase the length of a polymer by one unit, we need to release the constraint at the previous layer (think of a lattice). The change in free energy would then depend on the force at that point, and of course the random energy at the new occupied site. The change $\partial F(\mathbf{r})/\partial z$ being a scalar can then depend only on the two scalars $\nabla \cdot \mathbf{g}$ and g^2 . A direct derivation of the differential equation for the free energy shows that these are the three terms required. The differential equation, now known as the Kardar-Parisi-Zhang equation [7], is

$$\frac{\partial F}{\partial z} = \frac{T}{2K} \nabla^2 F - \frac{1}{2K} (\nabla F)^2 + \eta(\mathbf{r}, z). \quad (37)$$

If we can solve this *exact* equation and average over the random energy η , we get all the results we want.

One may also write down the equation for the force in this “fixed distance” ensemble as

$$\frac{\partial \mathbf{g}}{\partial z} = \frac{T}{2K} \nabla^2 \mathbf{g} - \frac{1}{2K} \mathbf{g} \cdot \nabla \mathbf{g} + \nabla \eta(\mathbf{r}, z). \quad (38)$$

This equation is known as the Burgers equation.

5.2 Free energy of extension: pure like

We want to know the free energy cost in pulling a polymer of length N from origin (where the other end is fixed) to a position \mathbf{r} . For the pure case, the free energy follows from Eq. (3) (with K inserted) as

$$F(\mathbf{r}, N) - F(\mathbf{0}, N) = \frac{d}{2} K \frac{r^2}{N}. \text{ (pure)} \quad (39)$$

The probability distribution for the free energy can be obtained by choosing $\mathbf{g} = K d\mathbf{r}_N/N$ in Eq. (13) as.

$$P(F(\mathbf{r}_N)) = P(F(0) + \frac{K d r_N^2}{2N}). \quad (40)$$

One then gets the surprising result,

$$[F(\mathbf{r}, N) - F(\mathbf{0}, N)]_{\text{av}} = \frac{d}{2} K \frac{r^2}{N}. \text{ (disorder)} \quad (41)$$

Therefore, on the average the stretching of a chain is pure-like (elastic) with the same elastic constant though the fluctuation is anomalous ($\theta \neq 0$). This is analogous to the pure-like result for the correlation function, Eq. (16). These results have a far reaching consequence that in a renormalization group procedure, the elastic constant must remain an invariant (independent of length scale). As we shall see, this invariance condition puts a constraint on ν and θ , making only one independent.

5.3 RG of the KPZ equation

To analyze the nonlinear KPZ equation, an RG procedure may be adopted. This RG is based on treating the nonlinear term in an iterative manner by starting from the linear equation. This is a bit unusual because here we are not starting with a “Gaussian” polymer problem, rather, a formal linear equation[25] that *does not* necessarily represent a polymer. Leaving aside such peculiarity, one may implement the coarse-graining of RG to see how the couplings change with length scale.

5.3.1 Scale transformation and an important relation

Under a scale transformation $x \rightarrow bx$, $z \rightarrow b^{1/\nu} z$, and $F \rightarrow b^{\theta/\nu} F$, the randomness transforms like $\Delta \rightarrow b^{-d-(1/\nu)} \Delta$. This transformation done on Eq. (41) shows that for K to be an invariant (no b -dependence) we must have,

$$\theta + 1 = 2\nu. \quad (42)$$

This is trivially valid for the Gaussian pure polymer problem but gives a relation between the free energy fluctuation and the size of the polymer. This is borne out by the intuitive picture we develop below. This relation gives the size exponent $\nu = 2/3$ in $d = 1$.

The equation in terms of the transformed variables is then

$$\frac{\partial F}{\partial z} = \frac{T}{2K} b^{(1-2\nu)/\nu} \nabla^2 F - \frac{1}{2K} b^{(\theta-2\nu+1)/\nu} (\nabla F)^2 + b^{(1-d\nu-2\theta)/\nu} \eta(\mathbf{r}, z) \quad (43)$$

The b -dependent factors can be absorbed to define new parameters, except for K .

The temperature however gets renormalized as $T \rightarrow T b^{(1-2\nu)/\nu}$. Its flow is described by the flow equation

$$L \frac{\partial T}{\partial L} = \frac{1-2\nu}{\nu} T \text{ (to leading order)} \quad (44)$$

For $\nu > 1/2$, $T(L) \rightarrow 0$. The disorder dominated phase is therefore equivalent to a zero temperature problem. In other words, the fluctuation in the ground state energy and ground state configurations dominate the behaviour at low temperatures in situations with $\nu > 1/2$. It is this renormalization that was missing in Sec 4.2.1.

5.4 RG flow equation

The nonlinearity (or g^2) contributes further to the renormalization of the temperature through the appropriate dimensionless variable $u = (K\Delta/T^3)L^{2-d}$ (same as in Eq. (30)). As in the previous section, the important flow equation is for this parameter u (upto constant factors). The renormalization of temperature acquires subtle d -dependence that introduces a new element in the flow equation. Some details may be found in Appendix E. We quote the flow equation here as

$$L \frac{du}{dL} = (2-d)u + \frac{2d-3}{2d}u^2. \quad (45)$$

General cases of such a flow equation are discussed in Appendix F.

One immediately sees a major difference with the flow equation Eq. (31) for $d = 1$. Because of the change of sign of the quadratic term in Eq. (45), there is now a fixed point for $d = 1$ as shown in Fig. 6(b). This flow equation does not behave properly in a range $d \in [1.5, 2)$ but that is more of a problem of implementation of RG than directed polymer *per se*, and so, may be ignored here. Note also that no extra information can be obtained from Eq. (45) for $d \geq 2$ other than what we have obtained so far in Sec. 4, namely the existence of a critical point.

This approach however has the advantage of getting the renormalization of temperature by u . Some details of this RG is given in Appendix E. Eq. (44) then gets modified to

$$L \frac{\partial T}{\partial L} = \left(\frac{1-2\nu}{\nu} + \frac{2-d}{4d}u \right) T. \quad (46)$$

If we now demand scale invariance at a fixed point of u , we can determine the exponent ν . At $d=1$, the stable fixed point $u^* = 2$ from Eq. (45) then gives the exact exponents at $d = 1$

$$\nu = \frac{2}{3}, \theta = \frac{1}{3}, \quad (47)$$

in agreement with the Bethe ansatz results mentioned above. To get θ the exponent relation Eq. (42) (from invariance of K) has been used. The RG results for $d = 1$ are expected to be exact.

5.5 Critical point for $d > 2$

For $d > 2$, the unstable fixed point is $u^* = O(|\epsilon|)$, ($d = 2 - \epsilon$). This gives at the critical point $\nu = 1/2 + O(\epsilon^2)$ indicating the possibility of

$$\nu = 1/2, \text{ and } \theta = 0 \tag{48}$$

to be exact.

One may argue [26] for $\theta = 0$ in the following way. At $T = T_c$, thermal fluctuation enables the polymer to get out of the trap set by the random potential (“ground state”). Just above the critical point, on a scale determined by the correlation length of the critical point, the random potential scaling is set by b^θ with the value of θ at T_c . As $T \rightarrow T_c+$, the length scale diverges and therefore the relevant energy scale would also grow with the same exponent. However a critical point implies the energy scale to be $O(T_c)$ which is finite. These can be reconciled if $\theta = 0$ at the critical point. One then gets $\nu = 1/2$. Though this is the same as that of a Gaussian polymer, we shall see later that the polymer has extra non-Gaussian features.

5.6 Strong disorder phase for $d \geq 2$

Unlike the strong disorder phase at $d=1$, the absence of any fixed point for the strong disorder phase for $d > 2$ in this approach forbids quantitative results about the phase itself. In addition, the behaviour of the strong disorder phase in $d = 1$ can be obtained by various methods. Unfortunately, there are few concrete results in higher dimensions $d \geq 2$. Most reliable values of the exponents come from various numerical approaches based on the KPZ equation. A recent estimate for $d = 2$ is [17]

$$1/\nu = 1.67 \pm 0.025, \text{ and } \theta = 0.229 \pm 0.05. \tag{49}$$

Numerical studies indicate that ν decreases as d increases. A question arises about the existence of an upper critical dimension $d = d_{\text{UCD}}$ such that $\nu = 1/2$ for $d > d_{\text{UCD}}$. For example, higher loop contributions in the RG of Sec. 4.2.5 show singularities at $d = 4$ which could indicate $d = 4$ as another critical dimension. Various analytical approaches [6, 18, 19] suggest $d_{\text{UCD}} = 4$, or even nonintegral dimension [20]. But numerical simulations and other arguments [21, 22, 9] suggest $d_{\text{UCD}} = \infty$. This issue is yet to be resolved.

The fact that the size exponent ν (often called wandering exponent) is different from $1/2$ has important implications in various applications, especially for flux lines in superconductors. For example, confinement of a flux line in presence of many other flux lines would lead to a steric repulsion[27] (similar to the confinement energy in Sec. 4.3.2) and the interaction of the vortices may lead to an attractive fluctuation induced (van der Waals type) interaction[28].

6 Overlap

In a replica approach, occupancy of different ground states may be achieved by “replica symmetry breaking” (i.e. various replicas occupying various states) but the difficulty arises from the $n \rightarrow 0$ limit. In the case of directed polymer, we have argued that the degenerate states occur only rarely and therefore the effect of “replica symmetry breaking” if any has to be very small. This is why the Bethe ansatz gave correct results without invoking replica symmetry breaking.

The method to compute the overlap was developed by Mukherji [29]. By introducing a repulsive potential $v \int dz \delta(\mathbf{r}_1(z) - \mathbf{r}_2(z))$ for the two polymers in the same random medium,

$$\begin{aligned} \mathbf{H} = & \frac{1}{2} \int_0^N dz \left[\left(\frac{\partial \mathbf{r}_1(z)}{\partial z} \right)^2 + \left(\frac{\partial \mathbf{r}_2(z)}{\partial z} \right)^2 \right] + \int_0^N dz \eta(\mathbf{r}_1(z), z) \\ & + \int_0^N dz \eta(\mathbf{r}_2(z), z) + \int_0^N dz v_0 \delta(\mathbf{r}_{12}(z)). \end{aligned} \quad (50)$$

The free energy $F(\mathbf{r}_1(z), \mathbf{r}_2(z), z, v)$ satisfies a modified KPZ type equation

$$\frac{\partial F}{\partial z} = \sum_i \left(\frac{T}{2K} \nabla_i^2 F - \frac{1}{2K} (\nabla_i F)^2 + \eta(\mathbf{r}_i, z) \right) + v \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (51)$$

which can be studied by RG. The mutual interaction has no effect on the single chain behaviour but the interaction gets renormalized. The flow equation for the dimensionless parameter u of Eq. (45) remains the same. The exponent relation of Eq. (42) also remains valid. The interaction gets renormalized as

$$L \frac{\partial v}{\partial L} = \left(\frac{1 - \theta}{\nu} - d + \frac{u}{2} \right) v, \quad (52)$$

where v is in a dimensionless form. For the pure problem ($\theta = 0, \nu = 1/2$) this reduces to the expected flow equation of Eq. (31) for repulsive interaction ($u \rightarrow -u$). For overlap one needs only the first order term because we need $v \rightarrow 0$.

The overlap can be written in a polymer-type scaling form $q = N^\Sigma f(vN^{-\phi\nu})$, where $\Sigma = \theta - \phi\nu - 1$. The above RG equation for v shows that the exponent $\Sigma = 0$ at the stable fixed point for u of Fig 6(b) at $d = 1$. However, $\Sigma < 0$ at the transition point for $d > 2$. This means that the overlap vanishes at the transition point from the strong disorder side as $q \sim |T - T_c|^{|\Sigma|\zeta}$.

This approach to overlap can be extended to m -chain overlaps also, which show a non-linear dependence on m at the transition point [30]. This suggests that eventhough the size exponentis $\nu = 1/2$ Gaussian like, there is more intricate structure than the pure Gaussian chain. Overlaps of directed polymers on trees have been considered in Ref. [31]. A case of cross-correlation of randomness (each polymer seeing a different noise) has been considered by Basu in Ref. [32].

7 Unzipping: pure case

Unzipping was first considered in the context of DNA [10]. However the same ideas play a role here. Let us consider a pure case of a directed polymer with one end fixed at origin

and with an attractive interaction with a line at $\mathbf{r}(z) = 0$ (instead of being in a random medium). The Hamiltonian for a delta-function interaction can be written as

$$H = \frac{d}{2}K \int_0^N dz \left(\frac{\partial \mathbf{r}}{\partial z} \right)^2 - v \int_0^N dz \delta(\mathbf{r}(z)) - \mathbf{g} \cdot \int_0^N dz \frac{\partial \mathbf{r}}{\partial z}, \quad (53)$$

very similar in form with Eq. (13) except here we have an attractive interaction instead of a random medium.

For the zero force case, there is a critical unbinding transition at $v = v_c$. For $d \leq 2$, $v_c = 0$. The pulling force would like to align the polymer in the direction of the force while the interaction would like to keep the polymer attached to the rod. At zero temperature the unzipping transition takes place at a force where the binding energy is compensated by the force term. Upto a geometric factor a , this is given by $Nv = agN$. At nonzero temperature, the entropic effects are to be taken into account, which may be done by using the quantum analogy.

The problem can be mapped on to a quantum Hamiltonian, albeit non-hermitian, for a particle of co-ordinate \mathbf{r}

$$H_q(\mathbf{g}) = \frac{1}{2}(\mathbf{p} + i\mathbf{g})^2 + V(\mathbf{r}), \quad (54)$$

in units of $\hbar(\equiv k_B T) = 1$ and mass = 1, with \mathbf{p} as momentum. For long chains $N \rightarrow \infty$ the free energy is the ground-state energy of this non-hermitian Hamiltonian. A phase transition takes place whenever the ground state is degenerate. The analysis done in Ref. [10] shows that if the ground state energy (i.e. the binding energy of the polymer per unit length) is E_0 , then the critical force is given by

$$g_c = 2\sqrt{E_0} \sim |v - v_c|^{1/|2-d|}. \quad (55)$$

where the v -dependences of E_0 close to v_c , for general d , is used. In fact if the bound state has extensive entropy, then there is a possibility of a re-entrance at low temperatures (see Sec. 9.4. This however is not possible in this case in hand.

8 Nature of ground states and excitations

Powered by the quantitative estimates of the free energy fluctuation and size exponents, we now try to generate a physical picture.

8.1 Rare events

We have seen that there is a low temperature region (in lower dimensions for all T) where randomness results in a new phase but the response to an unzipping force is the same as for the pure system. For the pure case as $N \rightarrow \infty$ the width of $P(\mathbf{r}_N, N)$ increases. Hence the increase of C_T with N . With randomness, for $T \rightarrow 0$ we need to look for the minimum energy path. Let us suppose that there is a unique ground state, i.e. $E(\mathbf{r}_N)$ or $F(\mathbf{r}_N)$ is a minimum for a particular path. This tells us that as the temperature is changed, T still low, the polymer explores the nearby region so that the probability distribution gains some

width which is determined by the thermal length. Susceptibility would be the width of the distribution and this is independent of N . This cannot satisfy the relation given by Eq. (42). If we invoke the the unzipping argument, then we need to exert a force exceeding the critical force to take the polymer out of the bound state and so the response to a small force ($g \rightarrow 0$) would be insignificant. The situation will not be any better even on averaging over the random samples if every sample has a unique ground state.

However, it may happen that most of the samples have unique ground-states but once in a while (rare samples) there is more than one ground state which happens to be far away from each other. Suppose there are such rare samples, whose probabilities decay as $N^{-\kappa}$, where the paths are separated by N^ν , then the contribution to the fluctuation from these samples would be $N^{2\nu-\kappa}$. In case $\kappa = 2\nu - 1$, we get back the exact result. The relation of Eq. (42) tells us $\kappa = \theta$. The rare events control the free energy fluctuation. From the unzipping point of view, the threshold in such rare cases is at zero force because a small force can take a polymer from one ground state to another one, gaining energy in the process. Following Ref. [33], one may argue that the gain in energy from the force should be similar to the energy fluctuation. Assuming a scaling of the force, $g \sim \hat{g}N^\sigma$, then $\hat{g}N^\sigma N^\nu \sim N^\theta$ which gives $\sigma = \theta - \nu = 1/3$ in $d = 1$. This argument implies that if the average stretching is proportional to g and to N , i.e., $[\langle r \rangle]_{\text{av}} \sim gN$, then one should get a linear plot if $[\langle r \rangle]_{\text{av}}/N^\nu$ is plotted against $gN^{\nu-\theta}$. The surprising feature is the sample dependence of such a plot. For a directed polymer, these quantities can be obtained by a transfer matrix calculation which is exact for a given sample and finite N . Some details on the transfer matrix approach are given in Appendix G. As shown in Fig. 7, one sees steps with an overall linear dependence. The susceptibility within a flat step is zero as seen in the plot of the fluctuations.

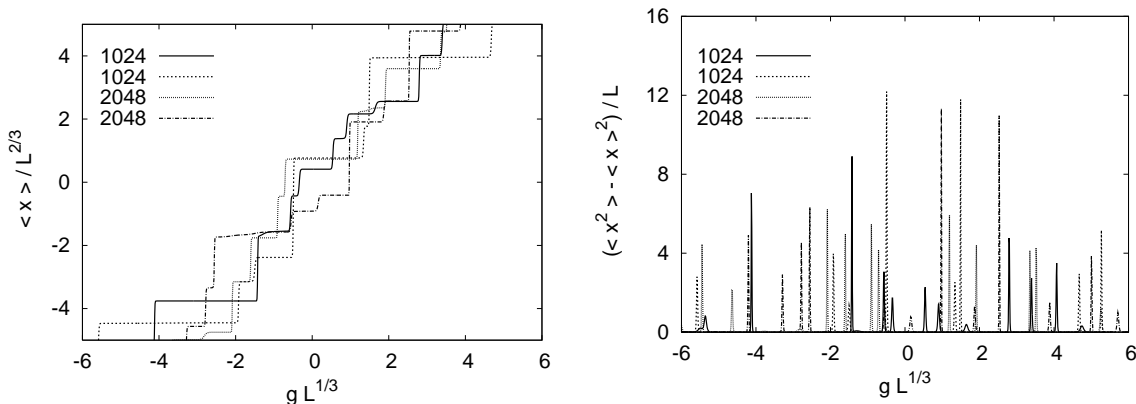


Figure 7: (a) Plot of $\langle r \rangle / N^\nu$ vs $gN^{\nu-\theta}$ for $d = 1$. Two different values of N and in different realizations of disorder. (b) Corresponding fluctuation in position. From Ref. [34].

What we see here is that though the average behaviour is the same as that of the pure system, the underlying phenomenon is completely different; the average thermal response is determined by the rare samples that have widely separated degenerate ground state and the probability of such states also decays as a power law of the chain length. This picture also shows the ensemble dependence. What we have discussed is the fixed force ensemble. In the fixed distance ensemble, if we keep the end point fixed at \mathbf{r} and try to determine the force to maintain it at that point, then by definition, the force comes from a small

displacement around \mathbf{r} . Such small displacements will never lead to the big jumps that ultimately contribute to the average susceptibility. This difference in behaviour in the two ensembles is one of the important features of quenched randomness.

In a given sample elastic energy $\sim r^2/N \sim N^{2\nu-1}$. The pinning energy would also grow with length say as $N^{\tilde{\theta}}$. We see, $\tilde{\theta} = 2\nu - 1 = \theta$. One way to say this is that the sample to sample fluctuation and the energy scale for a given sample are the same.

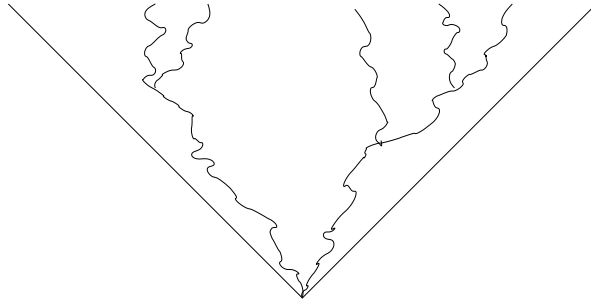


Figure 8: Various paths for various locations of the end point.

These results can now be combined for an image of the minimum energy paths. If the end points at $z = N$ are separated by $r \ll N^\nu$, the paths remain separated (each path exploring an independent disordered region) until they join at $\Delta z \sim r^{1/\nu}$, after which they follow the same path. If the end points are separated by a distance $r \gg N^\nu$, the two paths explore independent regions and they need not meet. This picture (Fig.8) is often alluded to as the river-basin network.

8.2 Probability distribution

For a pure polymer, the probability distribution of the end point is Gaussian but it need not be so for the disordered case. One way to explore the probability distribution is to study the response of the polymer as we take it out of its optimal or average position, e.g. by applying a force. In a previous section we used the fixed distance ensemble where the end point was kept fixed and we looked at the force g required to maintain that distance (see Eq. (38)). Here we consider the conjugate fixed force ensemble.

8.2.1 Response to a force

Let us apply a force that tries to pull the end of the polymer beyond the equilibrium value $r \sim R_0$. In equilibrium, the average size R or extension by the force can be expressed in a scaling form

$$R = R_0 f\left(\frac{gR_0}{T}\right). \quad (56)$$

This is because for zero force one should get back the unperturbed size while the force term may enter only in a dimensionless form in the above equation where the quantities available are the size R_0 and the thermal energy. For small g , linearity in g is expected. This requirement gives

$$R = R_0 \frac{R_0}{T} g \quad (k_B = 1), \quad (57)$$

R is not proportional to N if $\nu \neq 1/2$ ($R_0 \sim N^\nu$). The polymer acts as a spring with T/R_0^2 as the effective spring constant.

8.2.2 Scaling approach

Let us try to develop a physical picture and the corresponding algebraic description (called a scaling theory). The polymer in absence of any force has some shape of characteristic size R_0 . The force stretches it in a way that it breaks up into blobs of size $\xi_g = T/g$. For size $< \xi_g$ the polymer looks like a chain without any force and these blobs, connected linearly by geometry, act as a “new” polymer to respond to the force by aligning along it. We now have two scales R_0 and $\xi_g = T/g$. A dimensionless form is then

$$R = R_0 f\left(\frac{R_0}{\xi_g}\right) \sim \frac{R_0^2}{T} g \quad (58)$$

Now each blob is of length N_g so that $\xi_g = N_g^\nu$ and there are N/N_g blobs. We therefore expect

$$R = \frac{N}{N_g} \xi_g = N \xi_g^{1-1/\nu} = N \left(\frac{g}{T}\right)^{(1-\nu)/\nu} \quad (59)$$

This gives a susceptibility $\chi = \partial R / \partial g \sim g^{(1-2\nu)/\nu}$.

8.2.3 Probability Distribution

Let us try to get the susceptibility of Eq. (59) in another way. Let us assume that the probability distribution for large R is

$$P(r) \sim \exp\left(-\left(r/R_0\right)^\delta\right). \quad (60)$$

The entropy is given by $S(r) = -\ln P$. The free energy in presence of a force which stretches the polymer to the tail region is given by

$$F = T\left(r/R_0\right)^\delta - gr. \quad (61)$$

This on minimization gives

$$g = \frac{T}{R_0} \left(\frac{r}{R_0}\right)^{\delta-1}. \quad (62)$$

By equating this form with Eq. (59), we get

$$\delta = \frac{1}{1-\nu} \quad (63)$$

and

$$P(R) \sim \exp\left[-\left(\frac{r}{R_0}\right)^{1/(1-\nu)}\right]. \quad (64)$$

For $\nu = 1/2$ we do get back the Gaussian distribution.

The above analysis, done routinely for polymers, relies on the fact that there is only one length scale in the problem, namely, the size of the polymer. If we are entitled to do the same for the disorder problem, namely only one scale, $R_0 \sim N^\nu$, matters, then the blob picture goes through in toto. The chain breaks up into “blobs” and the blobs align as dictated by the force. Each blob is independent and the polymer inside a blob is exploring its environment like a directed polymer pinned at one end. The probability distribution is therefore given by Eq. (64) which for $d = 1$ is

$$P(r) \sim \exp(-|r|^3/N^2) \quad (d = 1). \quad (65)$$

If we use the relation $\Delta F \equiv F(x, N) - F(0, N) \sim x^2/N$, then the above probability distribution can be mapped to the distribution of the free energy as

$$P(F) \sim \exp \left[- \left(\frac{|\Delta F|}{N^\theta} \right)^{1/2(1-\nu)} \right] \sim \exp \left(- \frac{|\Delta F|^{3/2}}{N^{1/3}} \right) \quad (d = 1). \quad (66)$$

This has been tested numerically [6]. See, e.g. Ref. [35] for more recent work.

9 Random interaction - RANI model

So far we have been considering the problem of random medium. A different situation arises if there is randomness in the interaction of polymers. This is the RANI model [36, 37]. Consider the problem of two directed polymers interacting with a short range interaction as in Eq. (51) but take v to be random. Such problems are of interest, especially in the context of DNA where the base sequence provides heterogeneity along the chain. In this DNA context, the randomness can be taken to be dependent only on the z coordinate and not on others like the transverse position \mathbf{r} . It can be written as

$$H_{\text{int}} = \int_0^N dz v_0 [1 + b(z)] \delta(\mathbf{r}_1(z) - \mathbf{r}_2(z)), \quad (67)$$

where the randomness is introduced through $b(z)$. We take uncorrelated disorder with a Gaussian distribution

$$[b(z)]_{\text{av}} = 0, [b(z_1)b(z_2)]_{\text{av}} = \Delta \delta(z_1 - z_2). \quad (68)$$

This would correspond to uncorrelated base sequence of a DNA problem. The full Hamiltonian can be written as

$$\mathbf{H} = \frac{1}{2} \int_0^N dz \left[\left(\frac{\partial \mathbf{r}_1(z)}{\partial z} \right)^2 + \left(\frac{\partial \mathbf{r}_2(z)}{\partial z} \right)^2 \right] + \int_0^N dz v_0 [1 + b(z)] V(\mathbf{r}_{12}(z)). \quad (69)$$

where $\mathbf{r}_i(z)$ is the d -dimensional position vector of a point of chain i at a contour length z , and $\mathbf{r}_{12}(z) = \mathbf{r}_1(z) - \mathbf{r}_2(z)$. Though written for a general potential $V(\mathbf{r})$, we shall consider only short-range interaction that, when convenient, can be replaced by a δ -function.

9.1 Annealed case: two chain

As expected, the average partition function (annealed averaging) would correspond to a pure-type problem. This however is not the case always as we see in Sec. 9.3 for more than two chains.

A straightforward averaging of $Z = \int Dr_1 Dr_2 \exp(-\mathbf{H})$ using the probability distribution of Eq. (68) defines an effective Hamiltonian \mathcal{H}_{eff} such that

$$[Z]_{\text{av}} = \int Dr_1 Dr_2 \exp(-\mathcal{H}_{\text{eff}}), \quad (70)$$

and it is given by

$$\mathcal{H}_{\text{eff}} = \frac{1}{2} \int_0^N dz \sum_{i=1}^2 \left(\frac{\partial \mathbf{r}_i(z)}{\partial z} \right)^2 + v_0 \int_0^N dz V(\mathbf{r}_{12}(z)) - \frac{v_0^2 \Delta}{2} \int_0^N dz V^2(\mathbf{r}_{12}(z)). \quad (71)$$

It appears from the above expression of the effective Hamiltonian that an attraction is generated between the two chains. Now, since any short range potential under renormalization maps onto a δ function potential, we can take the “minimal” effective Hamiltonian for $\langle Z \rangle$ as

$$\mathcal{H}_2 = \frac{1}{2} \int_0^N dz \left[\left(\frac{\partial \mathbf{r}_1(z)}{\partial z} \right)^2 + \left(\frac{\partial \mathbf{r}_2(z)}{\partial z} \right)^2 \right] + \bar{v}_0 \int_0^N dz \delta(\mathbf{r}_{12}(z)). \quad (72)$$

where \bar{v}_0 is the reduced coupling constant which takes care of the attraction described earlier. We believe that the large length scale properties as described by Eq. (72) is same as that of Eq. (71). If necessary, we can restrict the strength of the disorder so that \bar{v}_0 , which represents the effective coupling between the two chains, is positive (i.e. repulsive interaction). Now the problem reduces to a relatively simple situation where the two chains interact with a pure δ -function interaction with a reduced coupling constant \bar{v}_0 . The solution of this pure problem is known [12] as discussed in the Appendix D.

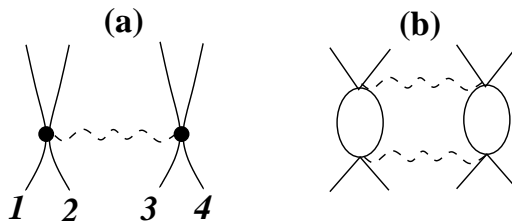


Figure 9: (a) Inter-replica interaction in the RANI model. The dotted wavy line indicates the “ r ”-type interaction between the pairs (1,2) and (3,4). (b) indicates a loop formed by the disorder induced interaction.

9.2 Marginal Relevance of disorder

When we consider the second cumulant of the partition function, we require four chains. On averaging over the disorder, a new “inter-replica” interaction is generated that couples the

original chains with the replica,

$$H_{\text{rep}} = -\bar{r}_0 \int_0^N dz \delta(\mathbf{r}_{12}(z))\delta(\mathbf{r}_{34}(z)), \quad (73)$$

with $\bar{r}_0 = v_0^2 \Delta$ and 3, 4 representing the two new chains, and $\mathbf{r}_{ij}(z) = \mathbf{r}_i - \mathbf{r}_j$. This is a special four chain interaction in the sense that this interaction favours a contact for chains 3 and 4 at z if chains 1 and 2 also enjoy a contact there, though not necessarily at the same point in the transverse space (see Fig. 9). In addition to this four chain interaction, the disorder also creates an effective two body attraction that changes the starting or bare interaction. So far as the effective two body interaction is concerned its effect on the long length scale is given by the RG flows of Eq. (31) except that u can now be negative also. Assuming that we are at the critical point of this two body interaction, e.g. for low dimensions, at $u^* = 0$, the effect of the disorder can be obtained from the RG flow of \bar{r}_0 . Defining the dimensionless coupling constant $r_0 = \bar{r}_0 L^{2\epsilon'}$ where $\epsilon' = 1 - d$ with $r(L)$ as the dimensionless running coupling constant, the RG equation is given by [36, 37]

$$L \frac{dr}{dL} = 2(\epsilon' r + r^2). \quad (74)$$

There are two fixed points (i) $r = 0$, and (ii) $r^* = -\epsilon'$. A nontrivial fixed point becomes important for $\epsilon' < 0$, i.e., for $d > 1$. See Appendix H for details.

We see certain similarities of disorder or randomness becoming marginally relevant at some dimension: $d_c = 2$ for the random medium problem, but $d_c = 1$ for the RANI problem. A new fixed point emerges above this critical dimension. For the random medium problem, this implies the existence of a new phase and a disorder induced phase transition, but for the random interaction problem it defines a new type of critical behaviour. These results based on the exact RG analysis [36, 37] were later on also recovered from a dynamic renormalization group study [38].

9.3 Annealed case: three and four chains

With pairwise random interaction, the RANI problem can be formulated for more than two chains also. In such cases, even the annealed averaging problem is not just the equivalent pure type problem [37]. The Hamiltonian for a four chain system is given by

$$\mathbf{H} = \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \left(\frac{\partial \mathbf{r}_i(z)}{\partial z} \right)^2 + \int_0^N dz v_0 (1 + b(z)) \sum_{\substack{i,j=1 \\ i < j}}^4 \delta(\mathbf{r}_{ij}(z)), \quad (75)$$

where $\mathbf{r}_{ij} = \mathbf{r}_i(z) - \mathbf{r}_j(z)$. After averaging of the partition function, using the Gaussian distribution of $b(z)$, one gets the following effective Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{eff}} = & \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \left(\frac{\partial \mathbf{r}_i(z)}{\partial z} \right)^2 + \bar{v}_0 \int_0^N dz \sum_{\substack{i,j=1 \\ i < j}}^4 \delta(\mathbf{r}_{ij}(z)) \\ & - 2v_0^2 \Delta \int_0^N \sum_{\substack{i,j,k \\ i < j < k}} \delta(\mathbf{r}_{ij})\delta(\mathbf{r}_{jk}) - 2v_0^2 \Delta \int_0^N dz \sum_{\substack{i,j,k,l=1 \\ i < j < k < l}}^4 \delta(\mathbf{r}_{ij})\delta(\mathbf{r}_{kl}). \end{aligned} \quad (76)$$

The remarkable feature of the effective Hamiltonian is that there are now two new types of attractive interaction one of which involves three chains, while the other one couples four chains together, similar to the 2-chain quenched problem.

If we take a three chain system, the corresponding effective Hamiltonian will involve only the three chain term but no four chain interaction of Eq. (76). There is now the possibility of a disorder induced multicritical behavior. The four chain attractive interaction is marginal at $d = 1$ and so is the three chain interaction. The presence of these two marginal operators (at $d = 1$) however remain decoupled mainly because for directed polymers, higher order interactions (order \equiv number of chains involved) do not renormalize lower order interactions. This has already been seen in the overlap problem for the random medium case in Sec. 6. Therefore the resulting renormalization of the two new couplings are independent of each other, and, in fact they are the same by virtue of the nature of the interaction. Because of the four body interaction, we expect a disorder induced criticality as for the two chain quenched case, but here this happens for a real four chain system - no replica is involved.

9.4 Unzipping

If we consider a pure problem, there will be an unzipping transition as described in Sec. 7 above. The transition is first order in nature. In this particular case there is no thermal unbinding in one dimension but if we forbid crossing of the polymers, then a thermal unbinding is possible. On a lattice this model can be solved exactly with a phase boundary given by [39]

$$g_c(T) = T \cosh^{-1} \left(\frac{1}{\sqrt{1 - \exp(v/T)} - 1 + \exp(v/T)} - 1 \right). \quad (77)$$

as shown in Fig. 10. One of the noteworthy features of the phase diagram is the re-entrance in the low temperature region [39, 40]. At low temperatures, the unzipped chains are stretched by the force so that on a lattice their entropy is exponentially suppressed. Though one gains energy by unzipping, one loses both the binding energy and the entropy of the bound chain. If $\ln \mu_B$ is the entropy per site of the bound state in the ground-state then, the transition occurs at $g_c = v + T \ln \mu_B$ with a positive slope. In the case of a single chain interacting with a rod, the bound state has no entropy ($\mu_B = 1$) and therefore no re-entrance. This argument also shows that the transition is first order.

It has been argued that the unzipping transition for the quenched averaged RANI case is second order [41]. However for real DNA, it is not the quenched averages that matter. There is strong ensemble dependence and sample to sample variation. This has been exploited to identify point mutants by a comparison of the unzipping force in a fixed distance ensemble [42]. An experimental determination of the unzipping phase boundary for a real DNA has been reported in Ref. [43]

10 Hierarchical lattice

Useful information can be obtained from models amenable to exact analysis even if they look artificial. Real space renormalization group approach can be handled in an exact fashion for a class of tailor-made lattices called hierarchical lattice. Such lattices are constructed

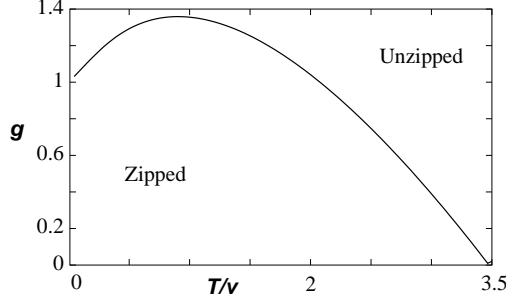


Figure 10: Unzipping phase diagram (exact) for two mutually avoiding directed polymers with an attractive interaction v in 1+1 dimensions. Note the reentrance at low temperatures. From Ref. [39].

in a recursive fashion as shown in Fig. 1. The problem of a directed polymer in a random medium on hierarchical lattices has been considered in Ref. [44, 45]. Here we consider the RANI problem on hierarchical lattices. As already noted, the effective dimensionality is $d_{\text{eff}} = (\ln 2b)/\ln 2$ for a motif of $2b$ bonds. Two different situations can arise, the polymers interacting on shared bonds [46] or on contacts at vertices [47]. Two cases are different. Aperiodic variation of the interaction has also been considered [48].

10.1 Randomness on sites

Let us now consider interaction on the vertices. The problem is different from the bond case because by construction there are sites with large degrees. In other words, all sites are not equivalent, unlike the bond case where all bonds are on the same footing.

Let Z_μ be the partition function of a given realization of randomness and let $S_\mu = b^{L_\mu - 1}$ the number of single-chain configurations, at the μ th generation. Here $L_\mu = 2^\mu$ is the length of a directed polymer. We define $\mathcal{Z}_\mu(n) = [Z_\mu^\nu]_{\text{av}} / S_\mu^{2n}$, to factor out the free chain entropy. We call $\mathcal{Z}_\mu(n)$ the moments. For a given temperature, there is a critical value $n = n_c(\bar{y})$ below which all moments are in their high temperature phase, in the thermodynamic limit of course (overbar indicating disorder average). In this limit, $\mathcal{Z}_\mu(n)$ approaches a fixed point value for $n < n_c$, whereas, for $n > n_c$, the moments diverge but with a finite “free energy” density $f_\mu(n) = (nL_\mu)^{-1} \ln \mathcal{Z}_\mu$. The approach to the thermodynamic limit can be written generically as

$$\mathcal{Z}_\mu(n)^{1/n} = \mathcal{Z}(n)^{1/n} + B_Z(n)L_\mu^{-\psi} + \dots \quad (78)$$

where $\mathcal{Z}(n)$ is the thermodynamic limit ($\mu \rightarrow \infty$) and $B_Z(n)$ is the amplitude of the finite size correction.

For a given realization of disorder, the partition function can be written as (see Fig.1)

$$Z_{\mu+1} = bZ_\mu^{(A)}yZ_\mu^{(B)} + b(b-1)S_\mu^4 \quad (79)$$

The first term originates from the configurations that require the two directed polymers to meet at a vertex. In contrast, the second term counts the “no encounter” cases. There are no energy costs at the two end points. The Boltzmann weight is random and for a Gaussian

distribution of energy, $\overline{y^m} = \overline{y}^{m^2}$. The moments of the partition function can be written as

$$\mathcal{Z}_{\mu+1}(n) = b^{-n} \sum_{m=0}^n P_{nm} Z_{\mu}^2(m) \quad \text{with } P_{nm} = \binom{n}{m} (b-1)^{n-m} \overline{y^m}, \quad (80)$$

and the initial condition $Z_0(n) = 1$ for all the moments because there is no interaction in the zeroth generation (one single bond).

By iterating the recursion relations, the moments can be calculated exactly to arbitrary precision from which $B_Z(n)$ can be estimated. One finds that $B_Z(n)$ blows up as

$$B_Z(n) \sim (n_c - n)^{-r} \text{ as } n \rightarrow n_c-, \quad (81)$$

with $r = 0.71 \pm 0.02$. This exponent is independent of temperature but depends on b .

As mentioned, in the site version there are special sites with large degree. Whether such vertices lead to a Griffiths' type singularity needs to be explored. Griffiths singularity in the context of random interactions has recently been discussed in Ref. [49].

10.2 Bond randomness: Problem with Harris criterion

According to the Harris criterion, disorder is irrelevant at a critical point if the pure specific heat exponent $\alpha < 0$. We show that a more general criterion is required for the directed polymer problem [46].

We place two interacting polymers on a hierarchical lattice. They start from one end (bottom) and meet at the other end (top). See Fig. 1. There is an attractive interaction $-v$ ($v > 0$) if a bond of the lattice is shared by two polymers. As in real space, here also the polymers undergo a binding-unbinding transition for $b > 2$.

Randomness is introduced by allowing the interaction energy to be random on each and every bond. The first model, model A, has independent random energy on all the $2b$ bonds. The randomness in the second model, model B, is taken only along the longitudinal direction so that equivalent bonds on all directed paths have identical random energy. Model B is a hierarchical lattice version of the continuum RANI model.

The pure problem can be solved easily by a Real space renormalization approach where one needs only the renormalization of the Boltzmann factor $y = \exp(v/T)$. Let $y_n, Z_n,$ and E_n be the renormalized weight, partition function and energy at the n th generation. By decimating the diamonds the recursion relations are given by

$$y_{n-1} = (y_n^2 + b - 1)/b, \quad (82)$$

$$Z_{n+1} = (Z_n^2 + b - 1)/b, \quad (83)$$

$$E_{n+1} = \frac{2}{b} \frac{Z_n^2 E_n}{Z_{n+1}}. \quad (84)$$

The two fixed points of the quadratic recursion relation, Eq. (82), are 1 and $b - 1$ of which the larger one is the unstable fixed point representing the transition point. Since $y > 1$ there is a transition at $y_c = b - 1$ only for $b > 2$. The other fixed point $y^* = 1$ corresponds to the high temperature limit.

The length scale exponent ζ , and the specific heat exponent, α , can be obtained from Eqs. (82)-(84) as

$$\zeta = \frac{\ln 2}{\ln[2(b-1)/b]}, \text{ and } \alpha = 2 - \zeta. \quad (85)$$

Note that $\alpha < 0$ for $b < 2 + \sqrt{2}$. It is clear that hyperscaling holds good with $d = 1$ and not the effective dimension d_{eff} of the lattice. It is also gratifying to observe that the same ν and α describe the finite size scaling form of E_n .

For the disordered case, the recursion relation for the Boltzmann weight can be written as

$$y_{n-1} = b^{-2}(y^{(11)}y_n^{(12)} + y^{(21)}y_n^{(22)} + \dots y^{(b1)}y_n^{(b2)}) + (b-1)b^{-1}, \quad (86)$$

where $y_n^{(ij)}$ is the Boltzmann weight in the n th generation for the disorder on the upper ($j = 1$) or lower ($j = 2$) part of the i th branch as in Fig. 1. To understand the effect of the disorder at the pure critical point we introduce a small disorder $y_n^{(ij)} = y_c + \varepsilon_n^{(ij)}$. The average of the disorder, $[\varepsilon]_{\text{av}}$, acts like the temperature as it measures the deviation from the pure critical point. The second moment is the measure of disorder. In principle, one should look at the variance of ε , but at the pure critical point the variance would be the same as the second moment. Since our motivation is to find the flow of the disorder at the pure fixed point, we need only study the first two moments, starting from a finite and small variance.

The crossover exponent for the disorder is defined through the homogeneity of the singular part of the free energy in terms of the scaling fields μ_1 (temperature) and μ_2 (disorder). Under decimation, the free energy behaves

$$f_{\text{sing}}(\mu_1, \mu_2) = \mu_1^{2-\alpha} f(\mu_2/\mu_1^\phi) \quad (87)$$

defining ϕ which can be obtained from the RG recursion relations for the first two moments. This crossover exponent determines the relevance of disorder at the critical point and can be computed exactly for both models A and B. One finds a striking difference between the two models as

$$\phi = \frac{\ln(2y_c^2/b^3)}{\ln[2(b-1)/b]} \quad (\text{model A}), \quad \text{and} \quad \phi = \frac{\ln[2(b-1)^2/b^2]}{\ln[2(b-1)/b]} \quad (\text{model B}). \quad (88)$$

For model A, ϕ is negative for all $b > 2$, implying irrelevance of disorder and $\phi \neq \alpha$ but it is equal to $2 - d_{\text{eff}}\nu$, while for model B, $\phi = \alpha$ and not $2 - d_{\text{eff}}\nu$. Since the randomness in model B is highly correlated, the Harris criterion is less expected to be valid here as opposed to model A. but it turns out to be so.

In order to construct a general framework for checking the validity of the Harris criterion, we start with the Taylor expansion of the recursion relation of Eq. (86),

$$\varepsilon = g_s(b)(\varepsilon_1 + \varepsilon_2 + \dots \varepsilon_{\mathcal{N}}) + 0(\varepsilon_i^2) + \dots, \quad (\mathcal{N} = 2b) \quad (89)$$

which defines $g_s(b)$. Simple arguments show that $g_s(b)$ determines α whose positivity is guaranteed if $g_s(b) > \sqrt{2}\mathcal{N}^{-1}$. Now, suppose that the bonds are grouped in n sets with N_i bonds in the i th group such that the members of a set have the same randomness. Obviously $\sum N_i = \mathcal{N}$. Starting with a narrow distribution, the relevance of the disorder at the pure transition then requires

$$g_s(b) > (N_1^2 + N_2^2 + \dots + N_n^2)^{-1/2}. \quad (90)$$

Hence, the Harris criterion holds good if either

$$g_s(b) > \max \left[\frac{\sqrt{2}}{\mathcal{N}}, \frac{1}{\sqrt{\sum N_i^2}} \right], \text{ or } g_s(b) < \min \left[\frac{\sqrt{2}}{\mathcal{N}}, \frac{1}{\sqrt{\sum N_i^2}} \right]. \quad (91)$$

For model A, $\mathcal{N} = 2b$ and $N_i = 1 \forall i$, while for model B, $n = 2$ with $N_i = b$.

If the disordered models are classified by \pm according to the sign of α , and I or R according to irrelevance or relevance of disorder, then the Harris criterion suggests the existence of only two classes (+R) and (-I). On the other hand, the above inequalities allow special classes like (+I) and (-R) where the Harris criterion fails. Model A is in the (+I) class for $b > 2 + \sqrt{2}$. Model B is in either the (+R) or (-I) class depending on b . It is possible to construct models that would belong to any of the four possible classes, especially (-R) [46].

11 Summary

The behaviour of a directed polymer in a random medium in 1+1 dimensions seems to be well understood. There is a strong disordered phase at all temperatures for $d < 2$. For $d > 2$ renormalization group analysis shows a phase transition from a low temperature strong disordered phase to a weak disorder, pure-like phase. There are rare configurations with degenerate widely separated ground states, giving a contribution to “overlap”, and strong sample dependent response to an unzipping force.

The RANI model remains less understood compared to the random medium problem. Exact renormalization analysis establish the marginal relevance of the disorder at $d = 1$, indicating a disorder dominated unbinding transition in $d \geq 1$. Several features including a generalization of the Harris criterion for this criticality via relevant disorder and aspects of unzipping have been discussed.

In both cases of random medium and random interaction, many issues still remain open.

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APPENDIXES

A Typical vs. average

Consider a random variable x that takes two values

$$X_1 = e^{\alpha\sqrt{N}} \text{ and } X_2 = e^{\beta N}, \beta > 1, \quad (92a)$$

with probabilities

$$p_1 = 1 - e^{-N}, \text{ and } p_2 = e^{-N}. \quad (92b)$$

In the limit $N \rightarrow \infty$, the average value $[x]_{\text{av}} \rightarrow e^{(\beta-1)N}$ while the typical or most probable value is $x = X_1$ with probability 1. On the other hand $[\ln x]_{\text{av}} \rightarrow \alpha\sqrt{N}$ in the same limit, showing that $[\ln x]_{\text{av}}$ is determined by the typical value of the variable while the moments are controlled by the rare events. Note that this peculiarity disappears if x has a smooth probability distribution in the sense of no special or rare events.

B Pure Polymers

The universality of the ‘‘Gaussian’’ polymer is actually a consequence of the Central limit theorem. Suppose we construct a flexible polymer from bonds with independent probability distribution $\psi(\mathbf{r})$ for a bond vector \mathbf{r} . The end-to-end distance is given by $\mathbf{R} = \sum_i \mathbf{r}_i$ so that the probability density of \mathbf{R} is

$$\begin{aligned} P(\mathbf{R}) &= \int \prod_i d\mathbf{r}_i \psi(r_i) \delta(\sum_i \mathbf{r}_i - \mathbf{R}) \\ &= \int \prod_i d\mathbf{r}_i \psi(r_i) \exp(i\mathbf{k} \cdot \mathbf{r}_i) \exp(-i\mathbf{k} \cdot \mathbf{R}) d\mathbf{k} \end{aligned} \quad (93)$$

$$= \int [\hat{\psi}(\mathbf{k})]^N \exp(-i\mathbf{k} \cdot \mathbf{R}) d\mathbf{k}, \quad (94)$$

where $\hat{\psi}(\mathbf{k})$ is the Fourier transform of $\psi(\mathbf{r})$. For a symmetric distribution with finite variance, $\ln \hat{\psi}(\mathbf{k}) \approx 1 - Ak^2/2\dots$, which on integration over k leads to a Gaussian distribution. This is valid for a lattice model also.

With the probability distribution of Eq. (3), the entropy in a fixed distance ensemble can be written as

$$S(\mathbf{r}) = S(0) - \frac{1}{2} \frac{r^2}{R_0^2}, \text{ which gives } F(\mathbf{r}) = F(0) + \frac{1}{2} \frac{Tr^2}{R_0^2}. \quad (95)$$

This identifies an effective spring constant for the polymer, namely $3T/2R_0^2$. This spring like behaviour is purely an entropic effect.

An important and general point is to be noted. The macroscopic quantity involves an ‘‘external’’ parameter like \mathbf{r} which is scaled by R_0 the characteristic long-length scale size of the polymer. That the long distance behaviour can be described by a single length scale is the basis of ‘‘scaling’’ approach to polymers.

Another approach to scaling is to study the changes in the properties of a polymer as the microscopic variables are scaled. E.g., if we make a scale change, $\mathbf{r} \rightarrow b\mathbf{r}$ and $z \rightarrow b^{1/\nu}z$, the Hamiltonian of Eq. (2) remains invariant if $\nu = 1/2$. Under such a transformation, the size behaves as

$$R_0(N) = b^{-1} R_0(b^{1/\nu} N), \quad (96)$$

so that by choosing $b = N^{-\nu}$, $R_0 \sim N^\nu$, i.e., the size exponent becomes $\nu = 1/2$. In presence of interaction or disorder, ν may not be obtained so simply and then renormalization group methods become useful.

B.1 Scaling approach in presence of a force

If the polymer is now pulled with a force g , keeping the end at $z = 0$ fixed, the polymer would align on the average with the force. The polymer can be thought of as consisting of blobs within each of which a polymer can be considered as free from the force whereas the blobs as unit form a still coarse grained model that shows stretching. This is shown schematically in Fig. 3 and is used in Sec. 8.2.1. This description is called a ‘‘blob picture’’.

This picture essentially depends on the scaling idea that R_0 is the only relevant scale for the macroscopic description. This can definitely be justified at the Gaussian level. The partition function with the unzipping force can be written as

$$Z = \int d\mathbf{r} P(\mathbf{r}, N) \exp(\beta \mathbf{g} \cdot \mathbf{r}), \quad (97)$$

where $P(\mathbf{r}, N)$ is Eq. (3). The Gaussian integral can be done (keeping T explicitly in Eq. (3)) to obtain

$$|\langle \mathbf{r} \rangle| = \frac{R_0^2}{T} g, \quad (98)$$

which is consistent with the idea of an effective spring constant of the polymer derived after Eq. (95).

A scaling approach to derive Eq. (98) would as follows. Let R_g be the characteristic size of the polymer in presence of the force. Then, from a dimensional analysis point of view, this is similar to the zero force size R_0 . From the nature of the force term, g is dimensionally like T divided by a length scale. Only lengthscale in the problem is R_0 . Hence a dimensionally correct form is

$$R_g \sim R_0 f(gR_0/T). \quad (99)$$

Note the absence of any microscopic scale in the above form. All microscopics go in R_0 . For a linear law at small force we require $f(x) \sim x$ ($x \rightarrow 0$) giving back Eq. (98). One may rephrase this by saying that the force has a characteristic size $\xi_g \sim T/g$.

If the polymer is confined in a tube of diameter D then the dimensionless variable is R_0/D . This will appear in the form for change in entropy or in “confinement energy”. This is used in Sec. 4.3.2.

C Self-averaging

Let us build a large system by adding blocks A, B, C, D, ... systematically so that an extensive quantity is a sum over its values on individual blocks. In case this averaging over blocks leads to a very sharp probability distribution, then no further disorder averaging is warranted, i.e., any large sample would show the average behaviour. A quantity with this property is often called *self-averaging*. This may not be the case if the distribution is broad especially in the sense discussed in App. A. A self-averaging quantity has the advantage that one may study one realization of a large enough system without any need of further disorder or sample averaging. For numerical simulations, the statistics of a non-self-averaging quantity cannot be improved by increasing the number of realizations.

To be quantitative, let us choose a quantity M which is extensive meaning $M = Nm$ where m is the “density” or per particle value. This is based on the additive property over subsystems $M = \sum M_i$. For a random system we better write $M = M(N, \{Q\})$, with $\{Q\}$ representing all the random variables. To recover thermodynamics, we want $[M]_{\text{av}}$ to be proportional to N for $N \rightarrow \infty$. Now, if it so happens that for large N

$$M(N, \{Q\}) \rightarrow Nm_d, \quad (100)$$

with m_d independent of the explicit random variables, then M is said to have the self averaging property. Note that no averaging has been done in Eq. (100). One way to guarantee this self-averaging is to have a probability distribution

$$P(M/N) \xrightarrow{N \rightarrow \infty} \delta(m_d). \quad (101)$$

This is equivalent to the statement that the sum over a large number of subsystems gives the average value, something that would be expected in case the central limit theorem (CLT) is applicable. This generally is the case if quantities like M for the sub-blocks are independent and uncorrelated random variables.

For many critical systems CLT may not be applicable and self-averaging is not self-evident. A practical procedure for testing self-averaging behaviour of a quantity X is to study the fluctuations $\sigma_N^2 \equiv [X^2]_{\text{av}} - [X]_{\text{av}}^2$ and then check if

$$R_{X,N} = \frac{\sigma_N^2}{[X]_{\text{av}}^2} \rightarrow 0, \text{ as } N \rightarrow \infty. \quad (102)$$

A quantity is not self-averaging if the corresponding R does not decay to zero. The central limit theorem would suggest $R_{X,N} \sim N^{-1}$, while a decay of R_X slower than this would be termed as “weakly” self-averaging.

We may then classify a quantity X , based on the large N behaviour, as follows:

$$\begin{aligned} R_{X,N} &\sim \text{constant} && \Rightarrow \textit{non} \text{ self-averaging} \\ &\sim N^{-1} && \Rightarrow \textit{strongly} \text{ self-averaging} \\ &\sim N^{-p} \text{ with } p < 1 && \Rightarrow \textit{weakly} \text{ self-averaging.} \end{aligned}$$

Recent renormalization group arguments seem to suggest that if disorder is relevant then at the new (disorder-dominated) critical point thermodynamic quantities are not self-averaging [50]. The arguments leading to this extremely significant prediction of non self-averaging nature of critical quantities can be summarized as follows.

Let $\tilde{t}_i = |T - T_c(i, N)|/T_c$ be a sample dependent reduced temperature where $T_c(i, N)$ is a pseudo-critical temperature of sample i of N sites with T_c as the ensemble averaged critical temperature in the $N \rightarrow \infty$ limit. In terms of this temperature, a critical quantity X is expected to show a sample dependent finite size scaling form

$$X_i(T, N) = N^\rho Q(\tilde{t}_i N^{1/d\zeta}) \quad (103)$$

where ρ characterizes the behaviour of $[X]_{\text{av}}$ at T_c , ζ being the length scale exponent. (E.g. $\rho = \gamma/d\zeta$ when X is the magnetic susceptibility χ). This is plausible because the critical region sets in when the size of the system is comparable to the correlation volume ξ^d which diverges as $|T - T_c|^{-\zeta}$. The RG approach validates this hypothesis of Eq. (103), especially the absence of any extra anomalous dimension in powers of N for R_X . Incidentally, this hypothesis, Eq. (103), excludes rare events that may lead to Griffiths’ singularity. Using this scaling form, the relative variance R_X at the critical point or in the critical region can be determined as

$$R_X \sim [(\delta T_c)^2] N^{2/d\zeta}, \quad (104)$$

where $[(\delta T_c)^2]$ is the sample average variance of the pseudo-critical temperature. A finite size scaling form for R_X can also be written down, but it is not required here.

A random system can have several temperature scales, namely $(T_c(N) - T_c)$ and $(T - T_c) \sim \xi^{-1/\zeta}$, in addition to the shift in the transition temperature itself. For a system with *relevant disorder*, all these scales should behave in the same way so that typical fluctuations in the pseudo-critical temperature is set by the correlation volume (ξ^d). In the finite size scaling limit $\xi^d \sim N$, and then

$$[(\delta T_c)^2] \sim N^{-2/d\zeta} \quad (\text{relevant disorder}) \quad (105)$$

An immediate consequence of this is that R_X approaches a constant as $N \rightarrow \infty$ indicating *complete absence of self-averaging* at the critical point in a random system.

For a pure type critical point (*irrelevant disorder*) $\alpha < 0$ where α is the specific heat exponent (i.e. $c \sim |T - T_c|^{-\alpha}$) of the pure system. In this case the fluctuation in T_c is set by the size, i.e.

$$[(\delta T_c)^2] \sim N^{-1} \quad (\text{irrelevant disorder}) \quad (106)$$

so that, by using the hyperscaling relation $2 - \alpha = d\zeta$, one gets $0 < p = |\alpha/\zeta| < 1$ where $R_x \sim N^{-p}$. Hence all critical quantities in this case are weakly self-averaging. Moreover, it is the *same power law* involving α and ζ , for every critical quantity X no matter what its critical exponent is.

These predictions have been verified for various types relevant and irrelevant disorders [51, 52, 53]. Exception to such non self-averaging behaviour with relevant disorder occurs if the T_c distribution approaches a delta function for large lattices. In such a situation, one gets back strong self-averaging behaviour [54].

D Details of RG for polymers: dimensional regularization

Some details of the renormalization group approach for polymers as done in Sec. 4.2.1 are given here [12]. We consider the problem of two interacting directed polymers and study the second virial coefficient. The second virial coefficient is related to the two-chain partition function with all the ends free. Dimensional regularization is to be used here.

For this appendix we take a simpler form of the Hamiltonian given by Eq. (26) as

$$H_2 = \frac{d}{2} \int_0^N dz \sum_i^2 \left(\frac{\partial \mathbf{r}_i}{\partial z} \right)^2 - v_0 \int_0^N dz \delta(\mathbf{r}_1(z) - \mathbf{r}_2(z)), \quad (107)$$

where v_0 is the bare or starting interaction strength. By introducing an arbitrary length scale L (may be the scale chosen to study the system or in momentum shell approach, this is the cutoff length), we may define the dimensionless variables $u_0 = v_0 L^\epsilon / (2\pi)^{d/2}$ and $\mathbf{N} = N/L^2$, with $\epsilon = 2 - d$. For long distance properties, we want L to be large.

By definition, the second virial coefficient comes from the connected two chain partition function with all the ends free. An expansion in terms of the coupling constant would involve

polymer configurations as shown in Fig. 5. This is incidentally identical to Eq. (27). Each line represents the probability of free polymer going from \mathbf{r}, z to \mathbf{r}', z ,

$$G(\mathbf{r}', z' | \mathbf{r}, z) = \frac{1}{[2\pi(z' - z)]^{d/2}} \exp\left(-\frac{(\mathbf{r}' - \mathbf{r})^2}{2(z' - z)}\right) \quad (108)$$

as in Eq. (4). A crossing of the lines in a diagram represents an interaction at (\mathbf{r}, z) which can take place anywhere requiring an integration over \mathbf{r} and $z \in (0, N)$.

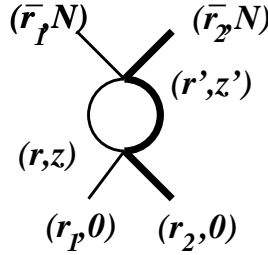


Figure 11: The one loop diagram for two polymers. The labels denote the position and the z of the points. There are integrations over all these free coordinates.

Since G is normalized, the spatial integration over the free end points lead to unity and so the dangling legs of the diagrams do not require any evaluation. One needs to do only the loop integrals. For example, the one loop diagram shown in Fig. 11 corresponds to

$$\int_0^N dz' \int_0^{z'} dz \int d\mathbf{r} \int d\mathbf{r}' G^2(\mathbf{r}' z' | \mathbf{r} z) \times \int d\mathbf{r}_1 G(\mathbf{r} z | \mathbf{r}_1 0) \int d\mathbf{r}_2 G(\mathbf{r} z | \mathbf{r}_2 0) \int d\bar{\mathbf{r}}_1 G(\bar{\mathbf{r}}_1 N | \mathbf{r}' z') \int d\bar{\mathbf{r}}_2 G(\bar{\mathbf{r}}_2 N | \mathbf{r}' z') \quad (109)$$

which after integrations over the end coordinates $\mathbf{r}_1, \mathbf{r}_2, \bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2$ reduces to an integral of the type

$$\int_0^N dz' \int_0^{z'} dz \int d\mathbf{r} \int d\mathbf{r}' G^2(\mathbf{r}' z' | \mathbf{r} z) \quad (110)$$

so that integrations over the space coordinates lead to integrals of the type given by Eqs. 28, 29. Using d as a continuous variable, one can write

$$\int_0^N dz z^{-\Psi} = \frac{N^{1-\Psi}}{1-\Psi} \quad (111)$$

with $\Psi = d/2$. This form is now valid for all d so that the loop integral may be written as, besides other constant factors,

$$2N\mathcal{V}\Delta^2 \frac{N^{1-d/2}}{2-d} \quad (112)$$

with a simple pole at $d = 2$. \mathcal{V} is the total volume. One then identifies $d = 2$ as the special dimension.

Using the above form with the singularity at $d = 2$, we may write the second virial coefficient as

$$A_2 = 2\pi N \mathcal{V} u_0 L^{2-\epsilon} \left[1 + u_0 (1 + (\epsilon/2) \ln N + \dots) \frac{2}{\epsilon} + \dots \right], \quad (113)$$

where a is a constant, and we used $N^{\epsilon/2} = 1 + (\epsilon/2) \ln N + \dots$. The series has problem at $d = 2$. We try to absorb the divergences by redefining all the parameters in hand, in this particular case, only u_0 and N . It is clear the the source of divergence is the region when z, z' are close by. On a bigger scale it is these close by reunions which would contribute to the effective interaction seen. It is therefore natural that these divergences ultimately determine the RG flow of the coupling constant.

Define a *renormalized* coupling constant as

$$u_0 = u + D_1 u^2 + D_2 u^2 + \dots = u Z_u. \quad (114)$$

Substitute in the expression for A_2 , and choose D 's to cancel the poles. Z_u is called a *multiplicative renormalization factor*.

We adopt the minimal subtraction scheme where the D 's are chosen to subtract the poles and only the poles. The calculation is order by order and so, to one loop order, one cannot determine D_2 , which involves u^3 (two loop term).

Choosing $D_1 = -1/\epsilon$, we see that the divergence is removed to $O(1/\epsilon)$. Upto this order, it follows that

$$u_0 = u - \frac{u^2}{\epsilon} + \dots \quad (115)$$

Since this absorbs all the divergences, we do not have to renormalize N . The divergence-free quantity is $A_{2R}(L, u, N)$ (R for renormalized; expressed in terms of u). Now, A_2 should not depend on L because L is put in by hand. This is ensured by demanding that

$$L \frac{dA_2}{dL} = 0, \quad (116)$$

where the factor of L in front has been put in for a dimensionless derivative operator. Written in a long form

$$\left[L \frac{\partial}{\partial L} + \beta(u) \frac{\partial}{\partial u} \right] A_{2R}(L, u, N) = 0, \quad \text{with } \beta(u) = L \frac{\partial u}{\partial L}. \quad (117)$$

Note that $A_{2,R}(u) = A_2(u_0)$, though, in general, additional renormalization factors (multiplicative and/or additive) may be needed.

By definition, $\beta(u)$ tells us how the renormalized u changes with the length scale and is called the beta function in RG. Some more algebra gives

$$\beta(u) \equiv L \frac{\partial u}{\partial L} = \frac{\partial u}{\partial u_0} L \frac{\partial u_0}{\partial L} = \epsilon u_0 \frac{\partial u}{\partial u_0} = \epsilon \left(\frac{\partial \ln u_0}{\partial u} \right)^{-1} \quad (118)$$

so that the variation of the coupling constant with scale L is given by

$$L \frac{\partial u}{\partial L} = \beta(u) \equiv \epsilon u + u^2/(2\pi). \quad (119)$$

The factor of 2π can be absorbed in the definition of u , as have often done. This equation, Eq. 119 is called a renormalization group flow equation with the initial condition $u = u_0$ for some $L = L_0$. It is analytic in ϵ so that various dimensions can be handled with this equation. Initial condition may be taken as $u = u_0$ for some $L = L_0$. In this particular case the beta function is exact to all orders.

The zeros of the beta function are called fixed points which can be of two types, stable or unstable. For this particular case, the fixed points are $u^* = 0$ and $u^* = -\epsilon$. If we start with a very small u , the flow equation $Ldu/dL \approx \epsilon u$ shows a growth of u with L if $\epsilon > 0$, i.e., if $d < 2$. This means u is a relevant variable at the noninteracting point. For the disordered system it translates to relevance of disorder at the pure fixed point.

One gets a nontrivial fixed point at $u^* = -2\pi\epsilon$ where $\beta(u) = 0$ which is an unstable fixed point if $\epsilon < 0$. This unstable fixed point for $d > 2$ represents the disorder induced critical point.

For the disorder problem, $u < 0$ is not meaningful, but for the interaction problem as in the RANI model, full range of u is allowed. The nontrivial fixed point for $d < 2$ is a stable fixed point and it describes the phase of repulsive polymers (fermion like). There is a critical binding-unbinding transition for $d > 2$ for pure attractive short range interaction. The unbinding transition is at zero potential for $d \leq 2$

For the transition one can define a characteristic length ξ so that at scales $> \xi$, the critical effects can be ignored. If we start with an initial value $u_0 = u^* + \Delta_0$, then one may say this crossover happens at some arbitrarily chosen value of $\Delta u = 1$ say where Δu is the renormalized deviation from the fixed point. One may determine this by linearization as

$$L \frac{d\Delta u}{dL} = |\epsilon| \Delta u, \quad (120)$$

which gives $\Delta u = \Delta_0 (L/L_0)^{|\epsilon|}$. If we take $\Delta_0 \ll 1$ as a measure of the deviation from the critical point (like $T - T_c$), then a small deviation grows and goes over either to the stable f.p. for the unbound phase or to $-\infty$ for the bound phase depending on the starting initial sign.

Setting $L = \xi$, we then get

$$\xi = |\Delta_0|^{-\zeta}, \text{ and } \zeta = \frac{1}{|\epsilon|}. \quad (121)$$

Since the beta function is exact, we have obtained the exact correlation length scale exponent for the binding-unbinding critical transition.

At the critical dimension, $d = 2$, the flow equation is

$$L \frac{d\Delta u}{dL} = (\Delta u)^2, \quad (122)$$

which gives $\xi \sim \exp(1/\Delta_0)$. This exponential dependence of ξ on Δ_0 accounts for the divergence of ν at $\epsilon = 0$.

E RG of the KPZ equation: momentum shell technique

We discuss the momentum-shell RG approach to the KPZ equation

$$\frac{\partial F}{\partial z} = \frac{T}{2K} \nabla^2 F - \frac{1}{2K} (\nabla F)^2 + \eta(\mathbf{r}, z). \quad (123)$$

which is Eq. (37). More details on this equation may be found in Ref. [55]. The idea is to get the behaviour of the parameters of the differential equation in the long distance limit. The three parameters are T , K and Δ , of which K remains invariant so that $2\nu = \theta + 1$, Eq. (42), is satisfied. Note that if the nonlinear term (i.e. the force term in the fixed distance ensemble) is absent, the differential equation becomes linear which can be solved exactly[25]. The RG scheme uses this exact solution for an iterative approach to tackle the nonlinear term. In contrast to the RG in the polymer approach of App. D, here the starting point is not a Gaussian polymer but a linear equation which need not represent any polymer.

It is convenient to work in the Fourier space

$$F(\mathbf{x}, z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{k < \Lambda} \frac{d^d \mathbf{k}}{(2\pi)^d} F(\mathbf{k}, \omega) \equiv \int_{\mathbf{q}} \int_{\omega} F(\mathbf{k}, \omega) \quad (124)$$

where Λ is an upper cutoff in k -space (related to a real space short distance cutoff). The long chain, long distance limit corresponds to $\omega, k \rightarrow 0$. The KPZ equation can be written in the following form

$$F(\mathbf{k}, \omega) = G_0(\mathbf{k}, \omega) \eta(\mathbf{k}, \omega) + \frac{1}{2K} G_0(\mathbf{k}, \omega) \int_{\mathbf{q}} \int_{\omega} \mathbf{q} \cdot (\mathbf{k} - \mathbf{q}) F(\mathbf{q}, \omega - \Omega) F(\mathbf{k} - \mathbf{q}, \omega), \quad (125)$$

which suggests an iterative scheme. In this equation

$$G_0(\mathbf{k}, \omega) \equiv \frac{\delta F}{\delta \eta(\mathbf{k}, \omega)} = \frac{1}{\mathcal{D}k^2 - i\omega}, \quad (126)$$

for the linear equation with $\mathcal{D} = T/2k$. The effect of the nonlinear or the second term of the right hand side of Eq. (125) is to change G_0 to G (such that $F = G\eta$). From this G^{-1} , the coefficient of k^2 in the $k, \omega \rightarrow 0$ limit would give the effective temperature of the problem. G may be written as

$$\begin{aligned} G(\mathbf{k}, \omega) &= G_0(\mathbf{k}, \omega) + \frac{2\Delta}{K^2} G_0^2(\mathbf{k}, \omega) \int_{\mathbf{q}} \int_{\omega} \left(\mathbf{q} - \frac{\mathbf{k}}{2} \right) \cdot \left(\mathbf{q} + \frac{\mathbf{k}}{2} \right) \mathbf{k} \cdot \left(\mathbf{q} + \frac{\mathbf{k}}{2} \right) \\ &\quad \times G_0\left(\mathbf{q} - \frac{\mathbf{k}}{2}, \frac{\omega}{2} - \Omega\right) G_0\left(\mathbf{q} + \frac{\mathbf{k}}{2}, \frac{\omega}{2} + \Omega\right) G_0\left(\mathbf{q} + \frac{\mathbf{k}}{2}, -\frac{\omega}{2} - \Omega\right). \end{aligned} \quad (127)$$

For $\omega = 0$ and $k \rightarrow 0$, the above equation simplifies to

$$G^{-1}(\mathbf{k}, 0) = G_0^{-1}(\mathbf{k}, 0) \left(1 - u \frac{d-2}{4d} \int dq q^{d-3} \right), \quad (128)$$

where

$$u = \frac{K\Delta}{T^3} L^{2-d}, \quad (129)$$

is the dimensionless coupling constant in this problem. Note that this is the same as in App. D. The power of q in the integral follows from power counting while the prefactor $(d-2)$ comes from the angular contributions of the dot products. In writing this form, the d -dependent solid angle contribution has been absorbed in the definition of u . The nature of ultraviolet (small distance) divergence (as $\Lambda \rightarrow \infty$) is apparent in the integral of Eq. (128). If in the residual integration in Eq. (128), we perform a thin-shell integration between $\Lambda(1-\delta l) < q < \Lambda$ and set $\Lambda = 1$, the effective temperature for the left-over long wavelength part is given by

$$T^< = T \left(1 - u \frac{d-2}{4d} \delta l \right). \quad (130)$$

On rescaling $x \rightarrow bx$, we have $\mathbf{k} \rightarrow (1-\delta l)\mathbf{k}$ where $b = \exp(\delta l) \approx 1 + \delta l$. The renormalized temperature is then

$$\tilde{T} = b^{(1-2\nu)/\nu} T^< \approx T^< \left(1 + \delta l \frac{1-2\nu}{\nu} \right). \quad (131)$$

Combining Eqs. 131 and 130, one gets ($\delta l \equiv \delta L/L$)

$$L \frac{\partial T}{\partial L} = \left(\frac{1-2\nu}{\nu} + \frac{2-d}{4d} u \right) T. \quad (132)$$

as quoted in Eq. (46). The renormalization of the disorder strength can be obtained from the definition

$$[F^*(\mathbf{k}, \omega) F(\mathbf{k}, \omega)]_{\text{av}} = 2 \tilde{\Delta} G(\mathbf{k}, \omega) G(-\mathbf{k}, -\omega). \quad (133)$$

By using Eqs. 125 and 128 for F and G and following the same iterative procedure as above, the flow equation for Δ can be determined as

$$L \frac{\partial \Delta}{\partial L} = \left(\frac{1-2\theta}{\nu} - d + \frac{1}{4} u \right) \Delta. \quad (134)$$

Combining these two, one gets the flow equation for u

$$L \frac{du}{dL} = (2-d)u + \frac{2d-3}{2d} u^2, \quad (135)$$

as quoted in Eq. (45). The extra $d-2$ factor in the flow equation for T leads to the stable fixed point at $d=1$. If the d -dependent u term in the flow of T in Eq. (132) is ignored, the resulting flow equation for u would be similar to Eq. (118) upto a constant factor for u . As already pointed out the coefficient of the quadratic term in the flow equation can always be scaled to 1 by a redefinition of u . There is a major difference between the RG of App. D and the RG done here. In App. D, $\epsilon = 2-d$ is a small parameter and used as such in various expansions, though ultimately the equations remain valid for a wider range. Here however there is no small parameter and so the approximation cannot be controlled by choosing small ϵ .

F Various flow diagrams

We discuss the various possibilities of the flows for an RG flow equation represented by

$$L \frac{du}{dL} = \epsilon u + c u^2, \quad (136)$$

where $u(L)$ is the running coupling constant at length scale L . The equation has two fixed points (fp), $u^* = 0, -\epsilon/c$. For convenience, let us call the $u = 0$ case as the “free” problem so that the $u^* = 0$ fp corresponds to the free case. This fixed point is to be called the trivial fixed point while the nonzero one is the nontrivial fp.

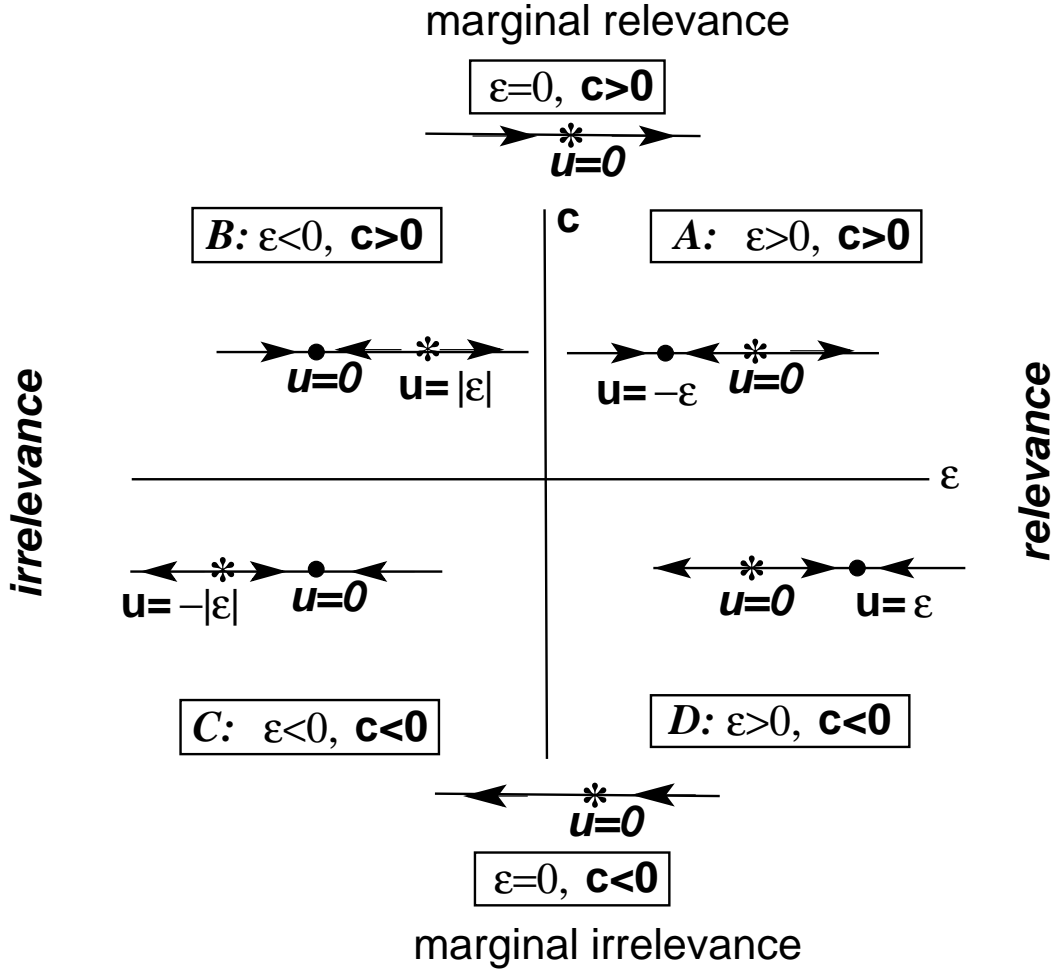


Figure 12: The various types of flow diagrams depending on ϵ and c , the coefficients of the linear and the quadratic terms of the flow equation. Solid bullet (\bullet) represents a stable fixed point while the star ($*$) represents an unstable fixed point.

The behaviour of the coupling constant with length scale is determined by the signs of the two constants ϵ and c , and the initial value $u(L_0)$ at $L = L_0$.

If $\epsilon > 0$, then u is a relevant variable at the free fixed point. while u is irrelevant there if $\epsilon < 0$. Special situations correspond to $\epsilon = 0$ for which u is a marginal variable. The possibilities we need to consider are

1. **A:** $\epsilon > 0, c > 0$. Here the nontrivial fixed point is negative and stable. There is no fixed point for $u > 0$. A bare $u < 0$ would then be equivalent to a state described by the nontrivial fixed point. For the random medium problem, $u > 0$ and so a relevant u flows to large values. The resulting state cannot be described in this approach. If u represents the interaction between two polymers, then $u < 0$ in Sec. 4.2.5 or App. D represents a repulsive interaction while $u > 0$ is for attraction. Hence one gets a stable nontrivial phase with a repulsive interaction in dimensions $\epsilon > 0$ in region A. For one dimension ($d = 1$), one may associate this fixed point with a fermion (or hard core boson)-like behaviour.
2. **B:** $\epsilon < 0, c > 0$. Change in stability of the fixed points. Here the nontrivial fixed point is on the positive side and the trivial or the free fixed point is stable. A system with negative u would behave on a long scale like a free system and so also for small values of $u > 0$. The nontrivial fixed point now represents a critical point so that a phase transition can be obtained by tuning u . Large values of u correspond to a different phase not accessible by this RG flow equation because the flow goes to infinity.
3. **C:** $\epsilon < 0, c < 0$. The situation is similar to Region B except that the phase transition is now at a negative value of u and large negative u phase is not accessible. All positive u values are equivalent to the free case (asymptotically free).
4. **D:** $\epsilon > 0, c < 0$. Here we see the free system is unstable while a positive u case is described by the nontrivial fixed point.
5. $\epsilon = 0$. This is the c -axis, representing the marginal case. The second order leads to a growth of $u > 0$ if $c > 0$. In this case, u is a marginally relevant variable while for $c < 0$ it is a marginally irrelevant variable. There is no fixed point to describe the system. However, the general trend is that if for $\epsilon = 0$, there is a marginally relevant variable, then that variable leads to a critical behaviour (phase transition) for $\epsilon < 0$.

The various possibilities are summarized in Fig. 12.

In all these cases, if the nontrivial fixed point is stable, then it represents a “critical” phase with characteristic exponents, while if it is unstable it represents a critical type phase transition with its own characteristic exponents. The reunion behaviour in Sec. 4.2.3 and the reunion exponents[13] are examples of nontrivial exponents at a stable fixed point. The unstable fixed point will be associated with a diverging length scale with an exponent $\zeta = 1/|\epsilon|$ as in Eq. (32).

Regions A and C are related by $u \rightarrow -u$ but others are distinct. For the polymer problem, as d (dimensionality) changes, the nature of flow in the partition function approach (Sec. 4.2.5 or App. D) goes from region **A** to **B**. One sees a new criticality developing for higher d ($d = 2 - \epsilon$) via a marginally relevant variable. The RG flow for the RANI model (Sec. 9 or App. H) also belongs to this type. For the KPZ approach (Sec. 5.6 or App. E) one goes from region **D** for $d = 1$ to region **B**.

G On Transfer Matrix

The directed nature of the directed polymer problem makes it amenable to a transfer matrix approach. This is another feather in the cap of the directed polymer problem. If we know

the partition function at a point (x, z) , we may construct the partition function for the $z + 1$ th step because it is completely determined by the information available at the z step. The exact form of the transfer matrix would depend on the particular geometry used.

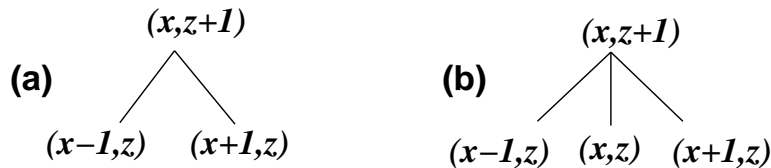


Figure 13: Lines show the possible steps from step z to $z+1$ on a square lattice. Backward steps are not allowed for directed polymers. (1) Lattice oriented by 45 degrees. The x and z axes are along the diagonals of the square lattice. (b) Axes are along the two directions of the square lattice but the polymer can take steps along the diagonals of the unit cell also.

If we consider the geometry shown in Fig. 1b, where the x and z axes are along the diagonals of the square lattice, then the partition function satisfies (see Fig. 13a)

$$Z(x, z + 1) = e^{-\eta(x, z+1)/T} [Z(x - 1, z) + Z(x + 1, z)]. \quad (137)$$

For the standard geometry (Fig. 13b) with x and z axes along the principal directions of the lattice but if we allow diagonal steps, then the partition function is given by

$$Z(x, z + 1) = e^{-\eta(x, z+1)/T} [Z(x, z) + \gamma Z(x - 1, z) + \gamma Z(x + 1, z)], \quad (138)$$

where an extra γ factor has been introduced to provide appropriate elasticity to the polymer. One may set $\gamma = 1$ for a fully flexible polymer. The initial condition is

$$Z(x, 0) = \delta_{x,0}, \quad (139)$$

where $\delta_{a,b}$ is the Kronecker delta. The partition function for a chain of length N starting from $(0, 0)$ can be obtained by iterating this equation. For a given realization of the randomness η , the partition function (and therefore any physical quantity) can be calculated exactly for finite N . For quenched averaging one has to average over various realizations, and this is where the exactness of the approach gets lost.

As the length N of the polymer increases, the span of x also increases linearly so that for $N \rightarrow \infty$ one has to study an infinitely large matrix. This allows the possibility of phase transitions in a seemingly one dimensional problem. For numerical analysis, special care needs to be taken to keep track of the rapid growth of the partition function as the length increases.

For $T \rightarrow 0$, the problem reduces to determination of the ground state energy. One may take the limit $E(x, N) = -\lim_{T \rightarrow 0} T \ln Z(x, N)$, but a direct approach is also possible. For geometry of Fig. 13a, the energy can be obtained from

$$E(x, z + 1) = \min(E(x - 1, z) + \eta(x, z + 1), E(x + 1, z) + \eta(x, z + 1)), \quad (140)$$

so that the globally minimum energy path is

$$E(N) = \min_x E(x, N). \quad (141)$$

Though we are considering the square lattice (in 1 + 1 dimensions), generalization to other lattices and higher dimensions are straightforward. Similarly, one may consider cases with random energies on the bonds instead of vertices.

For the overlap problem of Sec. 6, the partition function would satisfy in 1+1 dimensions

$$Z(x_1, x_2, z + 1) = e^{-(\eta(x_1, z+1) + \eta(x_2, z+1))/T} \times \sum_{p=\pm 1} \sum_{q=\pm 1} Z(x_1 + p, x_2 + q, z) [(1 - \delta_{x_1, x_2}) + e^{-v/T} \delta_{x_1, x_2}], \quad (142)$$

where the last term involving v is the Boltzmann factor for the interaction on contact ($x_1 = x_2$). If v is taken as a random quantity chosen from a predetermined distribution, then the same transfer matrix can be used to treat the RANI model also.

H RG for the RANI model

We show that the flow equation Eq. (74) is exact in the minimal subtraction scheme using dimensional regularization. More details may be found in Ref. [36, 37].

The Hamiltonian needed for $[Z^2]_{\text{av}}^c$ is

$$\mathcal{H} = \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \left(\frac{\partial \mathbf{r}_i(z)}{\partial z} \right)^2 - \bar{r}_0 \int_0^N dz \delta(\mathbf{r}_{12}(z)) \delta(\mathbf{r}_{34}(z)), \quad (143)$$

where we have set the effective two body interaction to zero. For the cumulant, one need only consider the “connected” partition function for this Hamiltonian. It is advantageous to consider the Laplace transform of the N -dependent partition function as

$$\mathcal{Z} = \int_0^\infty dN e^{-sN} [Z^2]_{\text{av}}^c, \quad (144)$$

the Laplace conjugate variable being s .

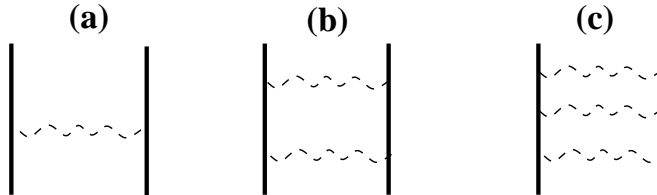


Figure 14: (a) The contributing ladder diagrams for $[Z^2]_{\text{av}}^c$. A thick line corresponds to a pair of chains. A wiggly line stands for an \bar{r}_0 factor in the evaluation of the diagrams. There are one and two loops in diagrams (b) and (c) respectively. Divergences arise from loop integrations.

An expansion in the coupling constant \bar{r}_0 can be arranged like a ladder (“time-ordered” diagrams) as shown in Fig. 14 The individual pairs of chains are represented by thick lines. The horizontal wiggly lines in these diagrams stand for \bar{r}_0 . Such a representation is possible because the δ function in H_2 , Eq. (73), forces the members of a pair to have the same \mathbf{r}, z coordinates. Each chain is described by the free distribution (“propagator”)

$G(\mathbf{r}_f - \mathbf{r}_i, z_f - z_i) = [2\pi(z_f - z_i)]^{-d/2} \exp[-(\mathbf{r}_f - \mathbf{r}_i)^2/2(z_f - z_i)]$ with end points (\mathbf{r}_f, z_f) and (\mathbf{r}_i, z_i) . In conformity with the current usage we use the word ‘‘propagator’’ for the lines. Two chains are therefore described by

$$G^2(\mathbf{r}, z) = (4\pi z)^{-d/2} G(\mathbf{r}, z/2). \quad (145)$$

This G^2 is the propagator for the thick lines. At each wiggly line, connecting four chains (all four having the same chain length z), there are two integrations over the spatial coordinates of the two separate pairs of chains (thick lines). The loops formed out of the wiggly lines are only responsible for the divergence at $d = 1$.

In order to trace the algebraic origin of the singularity, note that, by very nature of the interaction, the spatial integrations associated with the two thick lines are independent of each other. Each section of the thick lines, with z_1, z_2 as the end points, in a loop formed with the wiggly lines, contributes $(z_1 - z_2)^{-d/2}$ from the identity in Eq. (145). Since the interaction demands same z for the two thick lines, the z integrals involve $(z_1 - z_2)^{-d}$ type factors whose Laplace transform would contribute $\Gamma(1-d)$ with pole at $d = 1$. The two independent spatial coordinates which are left out after the successive use of the normalization $\int d\mathbf{r} G(\mathbf{r}, z) = 1$, lead to a \mathcal{V}^2 factor (total volume) for each diagram. The convolution nature of the z integrals, thanks to the time ordering, leads to a simple product of the individual Laplace transforms of the integrands, resulting in a geometric series for \mathcal{Z} .

As an example we consider the two loop diagram of Fig. 14(c). After integration over the free ends, we are left with the following

$$\bar{r}_0^3 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_{\{r, r'\}} G^2(\mathbf{r}_{12}, z_{12}) G^2(\mathbf{r}_{23}, z_{23}) G^2(\mathbf{r}'_{12}, z_{12}) G^2(\mathbf{r}'_{23}, z_{23}). \quad (146)$$

Here the subscripts denote the successive points along the thick lines while the two sets of chains are distinguished by the prime. In the Laplace space this becomes

$$\bar{r}_0^3 \mathcal{V}^2 \Gamma^2(\epsilon') (4\pi)^{-2d} s^{-(2+2\epsilon')}, \quad (\epsilon' = 1 - d). \quad (147)$$

This can be generalized to arbitrary orders since only ladder type diagrams are involved.

Defining the dimensionless coupling constant r_0 through an arbitrary length scale L as $r_0 = \bar{r}_0 L^{2\epsilon} (4\pi)^{-d}$, $\epsilon = 1 - d$, we write the series for \mathcal{Z} to all orders in r_0 as

$$\mathcal{Z}|_{\bar{v}_0=0} = (4\pi)^d \mathcal{V}^2 s^{-2} L^{-2\epsilon'} \left[r_0 + \sum_{n=1}^{\infty} r_0^{n+1} (sL^2)^{-n\epsilon'} \Gamma^n(\epsilon') \right]. \quad (148)$$

It is clear from the above expression that there is a divergence at $d = 1$ at each order (> 1).

A renormalization through minimal subtraction would require absorption of the poles in ϵ' through

$$r_0 = r(1 + a_1 r + a_2 r^2 + \dots). \quad (149)$$

with $a_n = \sum_{p=1}^n a_{n,p} \epsilon^{-p}$ and r as the renormalized coupling constant. In such a scheme, $a_{n,p} (p \neq n)$ terms are required to take care of the subleading divergences.

The geometric series of Eq. (148) guarantees that the removal of the leading poles is sufficient to remove the subleading ones. All the divergences can be absorbed by the choice $a_p = (-\epsilon')^{-p}$ which can be obtained by an explicit order by order calculation.

The β function is therefore exact to all orders in perturbation series and is given by

$$\beta(r) \equiv L \frac{\partial r}{\partial L} = 2(\epsilon' r + r^2). \quad (150)$$

There are two fixed points: (i) $r = 0$ and (ii) $r^* = -\epsilon'$. The bare coupling constant r_0 which originates from $v_0^2 \Delta$, where Δ , the variance of the distribution, is strictly positive, requires a positive r . Therefore, the nontrivial fixed point for $d < 1$ in negative r is unphysical. It however moves to the physical domain for $d > 1$.

Exactly at $d = 1$, $\epsilon = 0$, r grows with length L as

$$r(L) = r(0) \left[1 + 2r(0) \ln \frac{L_0}{L} \right]^{-1}, \quad (151)$$

$r(0)$ being the coupling at length L_0 . Hence, the disorder is marginally relevant. For $d > 1$, there exists an unstable nontrivial fixed point at $r = |\epsilon|$ which separates two distinct regimes of disorder. If we start with a strong enough disorder, on the right side of the fixed point, it increases with length scale, going beyond the perturbative regime. This is the strong disorder phase. On the other hand, the left side of the fixed point is the weak disorder regime, since r flows to zero (the stable fixed point).

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