THE VISIBILITY OF ULTRASONIC WAVES AND ITS PERIODIC VARIATIONS.

BY N. S. NAGENDRA NATH.
(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received August 20, 1936.

1. *Introduction.*

THAT ultrasonic waves travelling in a medium may be observed directly with suitable optical arrangements has been shown by Hiedemann, Bachem, Asbach and Hoesch[1] at Köln. Their method consists in observing through a microscope a beam of light which emerges from a rectangular cell containing supersonic waves which travel perpendicular to the direction of the propagation of the incident light. If the supersonic waves are progressive, a Kerr cell is also needed. Since supersonic waves are present in the medium they create periodic fluctuations in its density and consequently its optical properties. If light passes through such a medium, one should naturally expect periodic changes in the intensity of light on the emerging wave front. The wave-length of the periodic intensity fluctuation will be equal to the wave-length of the supersonic waves. However, in the case of standing waves, the wave-length corresponding to the fluctuations of the *average* intensity of light with respect to time will be half of the wave-length of the supersonic waves. This method has been used to determine the ultrasonic velocities in transparent solids and liquids.

Recently Raman and Nath[2] pointed out, in these *Proceedings*, the theoretical basis of the observability of supersonic waves in general terms based on their general theory of the propagation of light in a medium containing supersonic waves. Later, Pisharoty[3] developed the theory of the visibility of ultrasonic waves basing his investigation on Raman-Naths' preliminary theory. He obtained the very interesting result that even a periodic corrugated beam of light having initially a constant intensity will, during its propagation, develop into a periodic corrugated beam with periodic intensity changes on it. This means that if a microscope be used for observing the emerging beam, periodic intensity changes can be seen on the beam when the microscope is not exactly focussed on the emerging face of the

[1] *Zeits. f. Phy.*, **87**, 734, 738; **88**, 395; **89**, 502; **90**, 322; **96**, 268; **98**, 141.
[2] C. V. Raman and N. S. Nagendra Nath, *Proc. Ind. Acad. Sci.*, 1936, **3**, 459.
[3] P. R. Pisharoty, *Proc. Ind. Acad. Sci.*, 1936, **4**, 27.

cell where there are no intensity changes. He also pointed out that if the microscope be continuously focussed away from the emerging face, any particular form of the intensity grating can be seen exactly at distances which are integral multiples of a definite length. But Pisharoty's paper is marred by some errors.

The purpose of this paper is to develop the general theory of the visibility of ultrasonic waves and to show that the periodic visibility is characteristic of any general periodic supersonic wave. The above fact in a special case was first pointed out by Pisharoty. Special cases have also been worked out here and diagrams are drawn to illustrate the theoretical results.

2. *The Propagation of Light in a Medium filled with Sound Waves.*

The partial differential equation governing the propagation of light in a medium with time-variation and space-variation in its refractive index is

$$\nabla^2 \psi = \left[\frac{\mu(x, y, z, t)}{c}\right]^2 \frac{\partial^2 \psi}{\partial t^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

if the frequency of the time-variation of $\mu(x, y, z, t)$ is very slow compared to the time variation of the wave-function of light. This is indeed so in the problem we are considering for the time-variation of $\mu(x, y, z, t)$ corresponds to the frequency of the sound waves which is negligible compared to the frequency of light.

The dependence of ψ on y can be ignored if we choose the axes of reference such that the x-axis points to the direction of the propagation of the plane sound waves and the z-axis points to the direction of propagation of the incident plane wave of light.

Considering now that $\mu(x, t)$ varies slowly in time compared to $\exp[2\pi i\nu t]$, where ν is the frequency of the incident light, we can write ψ as given by

$$\exp[2\pi i\nu t]\, \phi(x, z, t) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where ϕ varies slowly in time compared to $\exp[2\pi i\nu t]$. On the consideration that $\nu^* << \nu$ where ν^* denotes the frequency of the sound waves we can show that

$$\left| 4\pi\nu \frac{\partial \phi}{\partial t} \right| << | 4\pi^2\nu^2\phi | \text{ and } \left| \frac{\partial^2 \phi}{\partial t^2} \right| << | 4\pi^2\nu^2\phi | \ldots \quad \ldots \quad (3)$$

Substituting (2) in (1) and using (3), we get the equation for ϕ as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2} \{\mu(x, t)\}^2 \phi \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and we may get ψ by the equation (2).

The form of the wave-front in the medium.—If we imagine a translation of the cell by an integral multiple of the wave-length of the sound waves along the x-axis, the experimental conditions will remain unchanged. This symmetry of the experiment demands that ϕ is some general periodic function of x, *i.e.*,

$$\phi\,(x+p\lambda^*,\,z,\,t) = \phi\,(x,\,z,\,t) \qquad \cdots \qquad (5)$$

where p is an integer. Since the experimental conditions also repeat themselves after a time q/ν^* where q is an integer, ϕ is also periodic in time with the period $1/\nu^*$, *i.e.*,

$$\phi\,(x,\,z,\,t+q/\nu^*) = \phi\,(x,\,z,\,t) \qquad \cdots \qquad (6)$$

where q is an integer.

Conditions (5) and (6) enable us to write the double Fourier expansion of ϕ as given by

$$\phi\,(x,\,z,\,t) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{r,s}\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t} \qquad \cdots \qquad (7)$$

Progressive sound waves.—In the case of progressive sound waves travelling along the positive direction of the x-axis, we have the property that the form and position of the wave obtained by a translation of the cell along the x-axis by $\rho\lambda^*$ where ρ is any number is the same as those of the wave which had appeared at a time ρ/ν^* earlier.

This condition can be put analytically as

$$\phi\,(x+\rho\lambda^*,\,z,\,t) = \phi\,(x,\,z,\,t-\rho/\nu^*) \qquad \cdots \qquad (8)$$

Using (8) in (7), we get

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{r,s}\,(z)\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t}\, e^{2\pi ir\rho}$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{r,s}\,(z)\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t}\, e^{-2\pi is\rho} \qquad \cdots \qquad (9)$$

Comparing the Fourier coefficients on each side of (9), we get

$$f_{r,s}\,(z)\, e^{2\pi ir\rho} = f_{r,s}\,(z)\, e^{-2\pi is\rho} \qquad \cdots \qquad (10)$$

where ρ is any number. (10) can only be true if

$$f_{r,s}\,(z) = 0 \quad \text{when } r \neq -s \qquad \cdots \qquad (11)$$

The condition (11) restricts the number of terms in the Fourier expansion of ϕ given by (7), so that

$$\phi\,(x,\,z,\,t) = \sum_{-\infty}^{\infty} f_r\,(z)\, e^{2\pi irx/\lambda^*}\, e^{-2\pi ir\nu^* t} \qquad \cdots \qquad (12)$$

Standing sound waves.—In the case of standing sound waves, we have the property that the form and position of the wave obtained by a translation of the cell along the x-axis by $p\lambda^*/2$ where p is an integer, is the same as those of the wave which would appear at a time $p/2\nu^*$ after or at the same time earlier. By this condition, ϕ obeys the property

$$\phi\left(x + \frac{p\lambda^*}{2}, z, t\right) = \phi\left(x, z, t \pm \frac{p}{2\nu^*}\right) \quad \ldots \quad (13)$$

where p is an integer. Using (13) in (7), we get

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{r,s}(z)\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t}\, e^{\pi irp}$$

$$= \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{r,s}(z)\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t}\, e^{\pm\pi isp} \quad \ldots \quad (14)$$

Comparing the Fourier coefficients on each side of (14), we get

$$f_{r,s}(z)\, e^{\pi irp} = f_{r,s}(z)\, e^{\pi isp} \quad \ldots \quad (15)$$

where p is an integer. (15) can only be true if r and s are both even integers or odd integers. Thus the Fourier expansion of ϕ can be written as given by

$$\phi(x, z, t) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{2r,2s}(z)\, e^{2\pi i 2rx/\lambda^*}\, e^{2\pi i 2s\nu^* t}$$

$$+ \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{2r+1,\, 2s+1}(z)\, e^{2\pi i \overline{2r+1}x/\lambda^*}\, e^{2\pi i \overline{2s+1}\nu^* t} \quad \ldots \quad (16)$$

The wave-front at the emerging face of the cell.—Since the boundary of the emerging face is given by $z = L$, where L is the length of the cell along the z-axis, the wave-front at $z = L$ will be

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{r,s}(L)\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t} \quad \ldots \quad (17)$$

If we only change the origin of the axes of reference to (0, L), we will have to write the above as

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, e^{2\pi irx/\lambda^*}\, e^{2\pi is\nu^* t} \quad \ldots \quad (18)$$

where

$$a_{r,s} = f_{r,s}(0) \quad \ldots \quad (19)$$

3. *Propagation of a Periodic Corrugated Wave in a Homogeneous Medium like Air.*

Let us consider the propagation of light whose initial wave-front at $z = 0$ is given by

$$\exp[2\pi i\nu t]\, \Phi(x, 0, t)$$

where

$$\Phi(x, 0, t) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, e^{2\pi irx/\lambda}\, e^{2\pi is\nu^* t} \quad \ldots \quad (20)$$

The symmetry of the initial wave-front demands that the form of the propagated wave should also be periodic in λ^* along the x-axis and in time with the period $1/\nu^*$. Thus the propagated wave is of the form

$$\Phi = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} g_{r,s}(z)\, e^{2\pi i r x/\lambda^*}\, e^{2\pi i s \nu^* t} \quad \ldots \quad (21)$$

such that

$$g_{r,s}(0) = a_{r,s} \quad \ldots \quad (22)$$

satisfying the boundary conditions. Φ satisfies the equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2}\Phi \quad \ldots \quad (23)$$

taking the refrative index of the medium as unity. Substituting (21) in (23) and comparing the Fourier coefficients on each side of (23), we get

$$\frac{d^2 g_{r,s}}{dz^2} + \frac{4\pi^2}{\lambda^2}\left(1 - \frac{r^2\lambda^2}{\lambda^{*2}}\right) g_{r,s} = 0 \quad \ldots \quad (24)$$

Since we are only interested in the propagation of light to the right of the x-axis, the solution for $g_{r,s}$ is given by

$$g_{r,s}(z) = a_{r,s}\, e^{-\frac{2\pi i z}{\lambda}\sqrt{\left(1 - \frac{r^2\lambda^2}{\lambda^{*2}}\right)}} \quad \ldots \quad (25)$$

so that the propagated wave ψ is of the form given by

$$\Psi = e^{2\pi i \nu t}\,\Phi$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} a_{r,s}\, e^{2\pi i\left[(\nu + s\nu^*)t - \frac{z\cos\theta_r - x\sin\theta_r}{\lambda}\right]} \quad \ldots \quad (26)$$

where

$$\sin\theta_r = \frac{r\lambda}{\lambda^*} \quad \ldots \quad (27)$$

Each of the component terms in (26) represents a wave of constant amplitude $a_{r,s}$ propagated in the direction whose z- and x-direction cosines are $\cos\theta_r$ and $-\sin\theta_r$ respectively. From the analysis of the wave given in (26) it should not be inferred that a superposition of a set of plane waves of constant amplitudes would also lead to a wave of constant amplitude. Indeed such a wave possesses, in general, an amplitude grating on it, *i.e.*, the amplitude on the wave-front will be a general periodic function of x. The intensity on the wave-front given by (26) is given by

$$|\Phi|^2 = \Phi\Phi^{\dagger} = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} g_{r,s}\, g^{\dagger}_{u,v}\, e^{2\pi i(r-u)x/\lambda^*}\, e^{2\pi i(s-v)\nu^* t} \quad (28)$$

Putting $r - u = l$ and $s - v = m$, we get

$$|\Phi|^2 = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} g_{r,s}\, g^{\dagger}_{r-l,\,s-m}\, e^{2\pi i l x/\lambda^*}\, e^{2\pi i m \nu^* t}$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} B_{l,m}\, e^{2\pi i l x/\lambda^*}\, e^{2\pi i m \nu^* t} \quad \ldots \quad (29)$$

where

$$B_{l,m} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} g_{r,s}\, g^{\dagger}_{r-l,\, s-m}$$

$$= \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, a^{\dagger}_{r-l,\, s-m}\, e^{-\frac{2\pi i z}{\lambda}\left[\sqrt{\left(1-\frac{r^2\lambda^2}{\lambda^{*2}}\right)}-\sqrt{\left(1-\frac{(r-l)^2\lambda^2}{\lambda^{*2}}\right)}\right]} \quad (30)$$

Let us suppose that λ/λ^* is small such that higher powers of it than λ^2/λ^{*2} are negligible in the index of the exponential term. This will be so under the experimental conditions. The first term having an influence in the exponential term is of the order $\lambda z/\lambda^{*2}$ and the next term is of the order $\lambda^3 z/\lambda^{*4}$. The influence of the second term is practically zero under the experimental conditions employed. With this consideration we can write $B_{l,m}$ as given by

$$B_{l,m} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, a_{r-l,\, s-m}\, e^{\frac{\pi i z}{\lambda}\left[\frac{r^2\lambda^2}{\lambda^{*2}} - \frac{(r-l)^2\lambda^2}{\lambda^{*2}}\right]}$$

$$= \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, a^{\dagger}_{r-l,\, s-m}\, e^{-\pi i \lambda z l\,(l-2r)/\lambda^{*2}} \quad \cdot\cdot \quad \cdot\cdot \quad (31)$$

By the help of (31) we can write $\Phi\Phi^{\dagger}$ as given by

$$\boxed{\Phi\,\Phi^{\dagger} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, a^{\dagger}_{r-l,\, s-m}\, e^{2\pi i l x/\lambda^*}\, e^{2\pi i m v^* t}\, e^{-\pi i \lambda z l\,(l-2r)/\lambda^{*2}}} \quad (32)$$

(32) shows that the intensity grating on the wave-front along the x-axis with the period λ^* repeats itself during its propagation with a definite period along the z-axis. This can be easily seen if one substitutes for z

$$z + \frac{2k\lambda^{*2}}{\lambda} \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad (33)$$

where k is an integer. From this it follows that

$$|\,\Phi\,(x, z, t)\,|^2 = \left|\Phi\,\left(x, z + \frac{2k\lambda^{*2}}{\lambda}, t\right)\right| \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad (34)$$

So, we can say that the intensity grating on the wave-front at z is exactly the same as that on the one at $z + \frac{2k\lambda^{*2}}{\lambda}$. That is, the intensity grating on the wave-front repeats itself along the z-axis with the period $2\lambda^{*2}/\lambda$.†

† For $\lambda^* = 0{\cdot}1$ cm., and $\lambda = 5 \times 10^{-5}$ cm., the period is 400 cm. while for $\lambda^* = 0{\cdot}01$ cm., and for the same wave-length of light, the period is 4 cm.

Progressive corrugation.—We will now consider as a particular case of the above, that of a progressive corrugation on the wave-front at the boundary. According to (11)

$$a_{r,s} = 0 \qquad \text{for } r \neq -s \quad \ldots \quad (35)$$

Thus

$$|\Phi|^2 = \sum_{-\infty}^{\infty} B_l \, e^{2\pi i l x/\lambda^*} \, e^{-2\pi i l \nu^* t} \quad \ldots \quad (36)$$

where

$$B_l = \sum_{-\infty}^{\infty} a_r \, a^\dagger_{r-l} \, e^{-\pi i z l (l-2r)\lambda/\lambda^{*2}} \quad \ldots \quad (37)$$

The average intensity with respect to time on the wave-front is

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi \, \Phi^\dagger d\epsilon \quad \ldots \quad (38)$$

where $\epsilon = 2\pi\nu^* t$. This is of course constant on the wave-front given by B_0 which is $\Sigma a_r \, a^\dagger_r$. The intensity on the wave-front, at any instant, is given by (36) and (37) which shows that the intensity grating on the wave-front at z is the same as the grating on the wave-fronts at $z + 2k\lambda^{*2}/\lambda$ where k is a positive integer. Conditions may be such that any particular form of the grating may have a lower period than $2\lambda^{*2}/\lambda$. For example, suppose the initial values a's are such that

$$\sum_{r=-\infty}^{\infty} a_r \, a^\dagger_{r-l} = 0 \qquad l \neq 0 \quad \ldots \quad (39)$$

From (37) and (39) one can see that

$$B_l(x, 0, t) = \sum_{r=-\infty}^{\infty} a_r \, a^\dagger_{r-l} = 0 \qquad l \neq 0 \quad \ldots \quad (40)$$

and

$$B_0(x, 0, t) = \Sigma \, a_r \, a^\dagger_r \quad \ldots \quad (41)$$

Thus

$$|\Phi(x, 0, t)|^2 = \sum_{-\infty}^{\infty} a_r \, a^\dagger_r \quad \ldots \quad (42)$$

which means that the intensity is constant on the wave-front at $z = 0$. The constant intensity on the wave-front has a lower period in z than $2\lambda^{*2}/\lambda$ for

$$B_l(x, z, t) = e^{-\pi i z l^2 \lambda/\lambda^{*2}} \, \Sigma \, a_r \, a^\dagger_{r-l} \, e^{2\pi i z \lambda l r/\lambda^{*2}}$$

so that

$$B_l\left(x, \frac{k\lambda^{*2}}{\lambda}, t\right) = e^{-\pi i k l^2} \, \Sigma \, a_r \, a^\dagger_{r-l} = 0 \qquad l \neq 0 \quad \ldots \quad (43)$$

where k is an integer. Thus we can say that if the propagated wave has at any z a constant intensity on the wave-front, it will have the same on all those wave-fronts at distances differing from z by an integral multiple of λ^{*2}/λ.

Stationary corrugation.—In this case

$$\Phi = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} g_{r,s}(z)\, e^{2\pi i r x/\lambda^*}\, e^{2\pi i s\nu^* t} \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

where r *and* s *are both even integers or odd integers* and

$$g_{r,s}(z) = a_{r,s}\, e^{-\frac{2\pi i z}{\lambda}\sqrt{\left(1 - \frac{r^2\lambda^2}{\lambda^{*2}}\right)}} \quad \ldots \quad \ldots \quad \ldots \quad (45)$$

The intensity on the wave-front is given by

$$\Phi\,\Phi^\dagger = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} g_{r,s}\, g^\dagger_{r-l,\ s-m}\, e^{2\pi i l x/\lambda^*}\, e^{2\pi i m \nu^* t} \quad \ldots \quad (46)$$

where r *and* s *or* l *and* m *are both even integers or odd integers.* The *average intensity* on the wave-front is

$$\mathrm{I}(x, z) = \frac{1}{2\pi} \int_0^{2\pi} \Phi\,\Phi^\dagger\, d\epsilon \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (47)$$

where $\epsilon = 2\pi\nu^* t$

From (47) and (46) we find

$$\mathrm{I}(x, z) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} g_{r,s}\, g^\dagger_{r-l,s}\, e^{2\pi i l x/\lambda^*} \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

where *l should be an even integer.* This is so for the right-hand side expression in (48) is the constant term in (46) for $m = 0$, which is an even integer so that l corresponding to $m = 0$ is also an even integer. In view of this consideration, we can change l to $2m$ where the m may be any integer. Hence

$$\mathrm{I}(x, z) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} g_{r,s}\, g^\dagger_{r-2m,s}\, e^{4\pi i m x/\lambda^*}$$

$$= \sum_{-\infty}^{\infty} \mathrm{B}_{2m}\, e^{2\pi i 2 m x/\lambda^*} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

where

$$\mathrm{B}_{2m} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, a^\dagger_{r-2m,s}\, e^{-4\pi i \lambda z m(m-r)\lambda/\lambda^{*2}} \quad \ldots \quad \ldots \quad (50)$$

From (50), one can easily see that

$$\mathrm{B}_{2m}(x, z) = \mathrm{B}_{2m}(x, z + k\,\lambda^{*2}/2\lambda) \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

where k is an integer. This means that the *average intensity* grating on the wave-front at z repeats itself at all distances differing from the original one by an integral multiple of the distance $\lambda^{*2}/2\lambda$.

4. *Special Cases.*

We will consider in this section two special cases. The wave-front at the boundary is considered to have no intensity variations on it.

A special case of progressive fluctuation.—Let

$$a_r = J_r(v) \quad \ldots \quad (52)$$

where v is a constant and $J_r(v)$ is the Bessel function of the rth order. The wave at the boundary is

$$\Phi(x, 0, t) = \sum_{-\infty}^{\infty} J_r(v)\, e^{2\pi i r x/\lambda^*}\, e^{-2\pi i r \nu^* t} \quad \ldots \quad (53)$$

It can be easily seen that the wave (53) has no intensity grating on it for

$$|\Phi(x, 0, t)|^2 = \Sigma\, B^0_l\, e^{2\pi i l x/\lambda}\, e^{-2\pi i l \nu^* t} \quad \ldots \quad (54)$$

where

$$B^0_l = \sum_{r=-\infty}^{\infty} J_r(v)\, J_{r-l}(v)$$

$$= \sum_{r=-\infty}^{\infty} J_r(v)\, J_{l-r}(-v)$$

$$= J_l(0) = 0 \qquad \text{if } l \neq 0$$

$$= 1 \qquad \text{if } l = 0 \quad \ldots \quad (55)$$

Hence

$$|\Phi(x, 0, t)|^2 = 1 \quad \ldots \quad (56)$$

which shows that the intensity is constant on the wave-front at $z = 0$.

Putting $z = h\lambda^{*2}/\lambda$, we get*

$$\Phi(x, h\lambda^{*2}/\lambda, t) = \sum_{-\infty}^{\infty} J_r(v)\, e^{2\pi i r x/\lambda^*}\, e^{-2\pi i r \nu^* t}\, e^{\pi i r^2 h}$$

$$= \sum_{-\infty}^{\infty} J_{2r}(v)\, e^{2\pi i 2 r x/\lambda^*}\, e^{-2\pi i 2 r \nu^* t}\, e^{4\pi i r^2 h}$$

$$+ \sum_{-\infty}^{\infty} J_{2r+1}(v)\, e^{2\pi i \overline{2r+1} x/\lambda^*}\, e^{-2\pi i \overline{2r+1} \nu^* t}\, e^{\pi i (2r+1)^2 h} \quad (57)$$

(*i*) Suppose $h = \frac{1}{4}$. Then

$$\Phi(x, \lambda^{*2}/4\lambda, t) = \sum_{-\infty}^{\infty} J_{2r}(v)\, e^{2\pi i 2 r x/\lambda^*}\, e^{\pi i r^2}\, e^{-2\pi i 2 r \nu^* t}$$

$$+ \sum_{-\infty}^{\infty} J_{2r+1}(v)\, e^{2\pi i \overline{2r+1} x/\lambda^*}\, e^{\pi i (r^2 + r + \frac{1}{4})}\, e^{-2\pi i \overline{2r+1} \nu^* t}$$

$$= 2 \sum_{0}^{\infty}{}' (-)^r J_{2r}(v) \cos[2\pi 2r (x/\lambda^* - \nu^* t)]$$

$$+ 2i e^{\pi i/4} \sum_{0}^{\infty} J_{2r+1}(v) \sin[2\pi (2r+1) \{x/\lambda^* - \nu^* t\}]$$

$$= \cos(v \cos 2\pi \overline{x/\lambda^* - \nu^* t}) + i e^{\pi i/4} \sin(v \sin 2\pi \overline{x/\lambda^* - \nu^* t}).$$

---

* The constant factor exp $[2\pi i z/\lambda]$ is omitted in Φ which does not give intensity changes.

Hence

$$| \Phi\ (x,\ \lambda^{*2}/4\lambda,\ 0)\ |^2 =$$

$$\cos^2 (v \cos bx) + \sin^2 (v \sin bx) - \sqrt{2} \cos (v \cos bx) \sin (v \sin bx) \quad (58)$$

(*ii*) Also.

$$| \Phi\ (x,\ \lambda^{*2}/4\lambda,\ 0)\ |^2 = | \Phi\ (x,\ 3\lambda^{*2}/4\lambda,\ 0)\ |^2 \quad .. \quad .. \quad (59)$$

(*iii*) Suppose $h = \frac{1}{2}$. Then

$$| \Phi\ (x,\ \lambda^{*2}/2\lambda,\ 0)\ |^2 = 1 - \sin (2v \sin bx) \quad .. \quad .. \quad (60)$$

(*iv*) $| \Phi\ (x,\ 3\lambda^{*2}/2\lambda,\ 0)\ |^2 = 1 + \sin (2v \sin bx)$ .. .. (61)

This example clearly shows that a wave initially having a constant intensity on it will develop itself, during its propagation, into one having an intensity grating on it. The constant intensity on the wave-front will be of course repeating with the period λ^{*2}/λ along the z-axis.

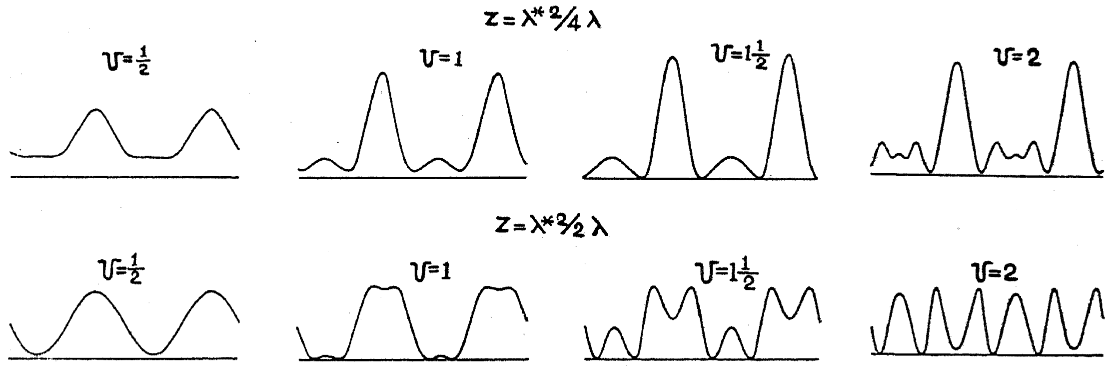

FIG. 1.

A special case of standing fluctuation.—The form for Φ in the case of a standing fluctuation* is

$$\Phi = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} a_{r,s}\, e^{2\pi i r x/\lambda^*}\, e^{2\pi i s \nu^* t}\, e^{\pi i z r^2 \lambda/\lambda^{*2}} \quad .. \quad .. \quad (62)$$

where r *and* s *are both even integers or odd integers.* Let us write

$$b_r\ (t) = \sum_{s=-\infty}^{\infty} a_{r,s}\, e^{2\pi i s \nu^* t} \quad .. \quad .. \quad .. \quad .. \quad (63)$$

Let us choose b's such that each of them is given by[4]

$$b_r\ (t) = \mathrm{J}_r\ (v \sin 2\pi\nu^* t) \quad .. \quad .. \quad .. \quad .. \quad (64)$$

where v is some constant.

Thus

$$\Phi = \sum_{-\infty}^{\infty} \mathrm{J}_r\ (v \sin \epsilon)\ e^{2\pi i r x/\lambda^*}\ e^{\pi i z r^2 \lambda/\lambda^{*2}} \quad .. \quad .. \quad .. \quad (65)$$

* The constant factor $\exp\ (-2\pi i z/\lambda)$ is omitted as it does not create any intensity changes.

[4] C. V. Raman and N. S. Nagendra Nath, *Proc. Ind. Acad. Sci.*, 1936, 3, 75.

So

$$| \Phi |^2 = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} J_r (v \sin \epsilon)\ J_{r-l} (v \sin \epsilon)\ e^{2\pi i l x / \lambda^*}\ e^{-\pi i z l \lambda (l - 2r) / \lambda^{*2}}$$

The *average intensity with respect to time* is given by

$$I (x, z) = \frac{1}{2\pi} \int_0^{2\pi} | \Phi |^2 \, d\epsilon$$

$$= \frac{1}{2\pi} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \int_0^{2\pi} J_r (v \sin \epsilon)\ J_{r-l} (v \sin \epsilon)\ d\epsilon\ e^{2\pi i l x / \lambda^*} \times e^{-\pi i z l \lambda (l-2r) / \lambda^{*2}} \quad ..$$

One can easily find that all integrals in (67) in which l is odd vanish.

$$I (x, z) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} C_{r,2l}\ e^{\pi 4 i l x / \lambda^*}\ e^{-4\pi i z l (l-r) \lambda / \lambda^{*2}} \quad .. \quad ..$$

where

$$C_{r,2l} = \int_0^{2\pi} J_r (v \sin \epsilon)\ J_{r-2l} (v \sin \epsilon)\ d\epsilon \quad .. \quad .. \quad ..$$

(68) shows that the average intensity grating is periodic along the x- with the period $\lambda^*/2$ and that any particular form of the intensity gra will repeat itself along the z-axis with the period $\lambda^{*2}/2\lambda$.

(i) The intensity is constant on the wave-front at the boundary $z =$ (

$$I (x, 0) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} C_{r,2l}\ e^{4\pi i l x / \lambda^*} \quad .. \quad .. \quad ..$$

where

$$\sum_r C_{r,2l} = \int_0^{2\pi} \sum_{r=-\infty}^{\infty} J_r (v \sin \epsilon)\ J_{r-2l} (v \sin \epsilon)\ d\epsilon$$

$$= \int_0^{2\pi} \sum_{r=-\infty}^{\infty} J_r (v \sin \epsilon)\ J_{2l-r} (-v \sin \epsilon)\ d\epsilon$$

$$= 0 \text{ if } l \neq 0$$

$$= 2\pi \text{ if } l = 0 \quad .. \quad .. \quad .. \quad .. \quad ($$

so that

$$I (x, 0) = 1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad ($$

which shows that the intensity is constant on the wave-front at the bounda

(ii) Suppose $z = \lambda^{*2}/4\lambda$. Then

$$I (x, \lambda^{*2}/4\lambda) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} B_{2l}\ e^{4\pi i l x / \lambda^*} \quad .. \quad .. \quad .. \quad ($$

where

$$B_{2l} = e^{-\pi i l^2} \sum_{r=-\infty}^{\infty} C_{r,2l}\ e^{\pi i l r}$$

(*a*) Suppose l is even. Then

$$B_{4m} = \Sigma\, C_{r,4m} = \int_0^{2\pi} \sum_{-\infty}^{\infty} J_r(v \sin\epsilon)\, J_{4m-r}(-v\sin\epsilon)\, d\epsilon$$

$$= 0 \text{ if } m \neq 0$$

$$= 2\pi \text{ if } m = 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (74)$$

(*b*) Suppose l is odd. Then

$$B_{2(2m+1)} = \Sigma(-)^{r+1}\, C_{r,2(2m+1)}$$

$$= -\int_0^{2\pi} \sum_{r=-\infty}^{\infty} J_r(v\sin\epsilon)\, J_{2(2m+1)-r}(v\sin\epsilon)$$

$$= -\int_0^{2\pi} J_{2(2m+1)}(2v\sin\epsilon)\, d\epsilon$$

$$= -4\int_0^{\pi/2} J_{2(2m+1)}(2\,v\sin\epsilon)\, d\epsilon$$

$$= -2\pi\, J_{(2m+1)}{}^2(v) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (75)$$

Thus

$$I(x, \lambda^{*2}/4\lambda) = 1 - \sum_{l=-\infty}^{\infty} J_{(2l+1)}{}^2(v)\, e^{4\pi i(2l+1)x/\lambda^*}$$

$$= 1 - 2\sum_{l=0}^{\infty} J_{(2l+1)}{}^2(v)\cos[4\pi(2l+1)x/\lambda^*]$$

This example clearly shows quantitatively the development of an intensity grating on a wave which had a standing corrugation on it but had constant intensity on it initially. Fig. 2 shows diagrammatically the intensity gratings for some values of v.

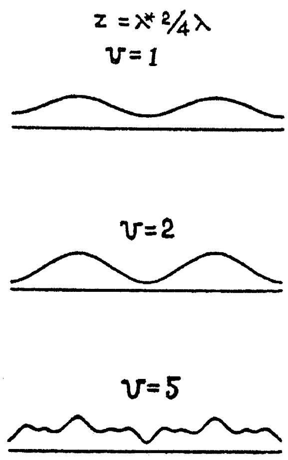

FIG. 2.

These figures give an interpretation of the pictures of the inte gratings obtained by Bär,[5] who has obtained a multiplicity of the fri under suitable experimental conditions.

5. *Summary.*

The general theory of the visibility of ultrasonic waves has been c loped. The theory shows that the intensity grating on a *general per corrugated* (*both amplitude and phase*) wave-front of light will repeat : on all those at distances from the original one by integral multiples 2λ where λ^* is the wave-length of the corrugation and λ is the wave-lengt light. Special cases have been worked out to illustrate the theoretical res These theoretical results await experimental confirmation.

The author is grateful to Prof. C. V. Raman for the discussions on subject.

[5] R. Bär, *Helv. Phy. Acta.*, 1935, **8**, 591 ; 1936, **9**, 265.