THE VIBRATIONS OF AN INFINITE LINEAR LATTICE CONSISTING OF TWO TYPES OF PARTICLES

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Received September 27, 1948

THE vibrations of a finite linear lattice of equal masses were first investigated by Lagrange,[1] assuming the end particles to be at rest and those of an infinite linear lattice by Hamilton.[2] The former work is well quoted in connection with the vibrations of strings and Fourier series, while Hamilton's work has not been properly appreciated.[3]

In this paper the vibrations of an infinite linear lattice consisting of two types of particles have been investigated. We consider a lattice consisting of two types of particles of masses m_1 and m_2 alternating one another at equal distances and in stable equilibrium. At time $t = 0$, one of the particles is given an arbitrary but small displacement a along the lattice and all the others are at rest in their equilibrium configurations. We assume that the forces acting on a particle arise due to changes in the positions of all the particles and are proportional to them. Later in the paper, we shall make certain simplifying assumptions in order to find out the asymptotic nature of the vibrations. We denote the displacements of the particles at the instant of time t by $x_m(t)$ where m is any integer ranging from $-\infty$ to $+\infty$. The initial conditions are given by $x_m(0) = 0$ for all m except $m = 0$, $x_0(0) = a$ and $\dot{x}_m(0) = 0$ for all m.

The equations of motion of the $2r$th and $(2r+1)$th particles are

$$- m_1\ddot{x}_{2r} = B_0 x_{2r} + \sum_{s=1}^{\infty} B_{2s}(x_{2r+2s} + x_{2r-2s})$$
$$+ \sum_{s=0}^{\infty} B_{2s+1}(x_{2r+2s+1} + x_{2r-2s-1}),$$
$$- m_2\ddot{x}_{2r+1} = C_0 x_{2r+1} + \sum_{s=1}^{\infty} C_{2s}(x_{2r+2s+1} + x_{2r-2s+1})$$
$$+ \sum_{s=0}^{\infty} B_{2s+1}(x_{2r+2s+2} + x_{2r-2s}). \tag{1}$$

We seek solutions of the type

$$x_{2r} = f(\phi)\, e^{i(\omega t - r\phi)}, \quad x_{2r+1} = g(\phi)\, e^{i(\omega t - r\phi)} \tag{2}$$

where ω, f and g are functions of the real parameter ϕ.

Substituting (2) in the equations of motion (1) and removing common factors, we get

$$\left[B_0 + 2\sum_{1}^{\infty} B_{2s} \cos s\phi - \omega^2 m_1\right] f(\phi) + 2\left[\sum_{0}^{\infty} B_{2s+1}\, e^{\frac{i\phi}{2}} \cos\left(s+\tfrac{1}{2}\right)\phi\right] g(\phi) = 0,$$

$$2\left[\sum_{0}^{\infty} B_{2s+1}\, e^{-\frac{i\phi}{2}} \cos\left(s+\tfrac{1}{2}\right)\phi\right] f(\phi) + \left[C_0 + 2\sum_{1}^{\infty} C_{2s} \cos s\phi - \omega^2 m_2\right] g(\phi) = 0.$$

Eliminating $f(\phi)$ and $g(\phi)$ in (3), we find that ω satisfies the equation

$$m_1 m_2 \omega^4 - \omega^2\left[m_1 C_0 + m_2 B_0 + 2\sum_{1}^{\infty}(m_1 C_{2s} + m_2 B_{2s}) \cos s\phi\right] + \left[C_0 + 2\sum_{1}^{\infty} C_{2s} \cos s\phi\right]\left[B_0 + 2\sum_{1}^{\infty} B_{2s} \cos s\phi\right] - 4\left[\sum_{0}^{\infty} B_{2s+1} \cos\left(s+\tfrac{1}{2}\right)\phi\right]^2 = 0. \quad (4)$$

The equation gives four values of ω for each ϕ, which we shall denote by ω_1, ω_2, $-\omega_1$, $-\omega_2$ and the corresponding values of $f(\phi)$ and $g(\phi)$ will be denoted by f_1, f_2, f_3, f_4 and g_1, g_2, g_3, g_4. The general solutions can be built up by the superposition of the solutions of the type (2) within the range of ϕ between 0 and 2π. Since the period of ω is 2π, there is no use in repeating the values of ω over and over again by assigning to ϕ, values outside the range. The general solutions of (1) can now be written as

$$x_{2r} = \frac{1}{4\pi}\int_0^{2\pi} \{f_1(\phi)\, e^{i\omega_1 t} + f_2(\phi)\, e^{i\omega_2 t} + f_3(\phi)\, e^{-i\omega_1 t} + f_4(\phi)\, e^{-i\omega_2 t}\}\, e^{-ir\phi}\, d\phi,$$

$$x_{2r+1} = \frac{1}{4\pi}\int_0^{2\pi} \{g_1(\phi)\, e^{i\omega_1 t} + g_2(\phi)\, e^{i\omega_2 t} + g_3(\phi)\, e^{-i\omega_1 t} + g_4(\phi)\, e^{-i\omega_2 t}\}\, e^{-ir\phi}\, d\phi. \quad (5)$$

From equations (3), we get

$$f_1(\phi) = \lambda(\phi)\, g_1(\phi),\ f_2(\phi) = \mu(\phi)\, g_2(\phi),\ f_3(\phi) = \lambda(\phi)\, g_3(\phi),\ f_4(\phi) = \mu(\phi)\, g_4(\phi) \quad (6)$$

where
$$\lambda(\phi) = \frac{m_2\omega_1^2 - \left[C_0 + 2\sum_{1}^{\infty} C_{2s} \cos s\phi\right]}{2\sum_{0}^{\infty} B_{2s+1}\, e^{-\frac{i\phi}{2}} \cos\left(s+\tfrac{1}{2}\right)\phi}$$
and
$$\mu(\phi) = \frac{m_2\omega_2^2 - \left[C_0 + 2\sum_{1}^{\infty} C_{2s} \cos s\phi\right]}{2\sum_{0}^{\infty} B_{2s+1}\, e^{-\frac{i\phi}{2}} \cos\left(s+\tfrac{1}{2}\right)\phi}.$$

The second equation in (5) can now be written as

$$x_{2r+1} = \frac{1}{4\pi}\int_0^{2\pi}\left\{\frac{f_1(\phi)\, e^{i\omega_1 t} + f_3(\phi)\, e^{-i\omega_1 t}}{\lambda(\phi)} + \frac{f_2(\phi)\, e^{i\omega_2 t} + f_4(\phi)\, e^{-i\omega_2 t}}{\mu(\phi)}\right\} e^{-ir\phi}\, d\phi. \tag{7}$$

The boundary conditions are given by $x_0(0) = a$, $x_m(0) = 0$ for all m except $m = 0$ and $\dot{x}_m(0) = 0$ for all m. Hence from the expressions for x_{2r} and x_{2r+1}, we get

$$\int_0^{2\pi} (f_1 + f_2 + f_3 + f_4)\, d\phi = 4\pi a,$$

$$\int_0^{2\pi} (f_1 + f_2 + f_3 + f_4)\, e^{-ir\phi}\, d\phi = 0, \quad \text{for all } r \text{ except } r = 0,$$

$$\int_0^{2\pi} \left(\frac{f_1 + f_3}{\lambda} + \frac{f_2 + f_4}{\mu}\right) e^{-ir\phi}\, d\phi = 0,$$

$$\int_0^{2\pi} \omega_1 (f_1 - f_3) + \omega_2 (f_2 - f_4)\, e^{-ir\phi}\, d\phi = 0,$$

$$\int_0^{2\pi} \left\{\frac{\omega_1}{\lambda}(f_1 - f_3) + \frac{\omega_2}{\mu}(f_2 - f_4)\right\} e^{-ir\phi}\, d\phi = 0, \tag{8}$$

the last three equations being true for all integral values of r.

Thus,

$$\begin{aligned} &f_1 + f_2 + f_3 + f_4 = 2a, \\ &\mu(f_1 + f_3) + \lambda(f_2 + f_4) = 0, \\ &\omega_1 (f_1 - f_3) + \omega_2 (f_2 - f_4) = 0, \\ &\omega_1 \mu (f_1 - f_3) + \omega_2 \lambda (f_2 - f_4) = 0. \end{aligned} \tag{9}$$

From these equations it follows that

$$f_1 = \frac{a\lambda}{\lambda - \mu},\ f_2 = \frac{a\mu}{\mu - \lambda},\ f_3 = f_1,\ f_4 = f_2. \tag{10}$$

The solutions under the specified boundary conditions, can now be written as

$$x_{2r} = \frac{a}{2\pi}\int_0^{2\pi} \frac{m_2\omega_1^2 - \left[C_0 + 2\sum_1^{\infty} C_{2s}\cos s\phi\right]}{m_2(\omega_1^2 - \omega_2^2)}\, e^{-ir\phi}\cos\omega_1 t\, d\phi$$

$$+ \frac{a}{2\pi}\int_0^{2\pi} \frac{m_2\omega_2^2 - \left[C_0 + 2\sum_1^{\infty} C_{2s}\cos s\phi\right]}{m_2(\omega_2^2 - \omega_1^2)}\, e^{-ir\phi}\cos\omega_2 t\, d\phi,$$

and

$$x_{2r+1} = \frac{a}{\pi} \int_0^{2\pi} \frac{\sum\limits_s^{\infty} B_{2s+1} e^{-\frac{i\phi}{2}} \cos (s + \frac{1}{2}) \phi}{m_2 (\omega_1^2 - \omega_2^2)} (\cos \omega_1 t - \cos \omega_2 t) e^{-ir\phi} d\phi. \tag{11}$$

We now make the simplifying assumption that the forces acting on a particle arise only due to changes in the neighbouring distances. This case is obtained by putting $B_{2s} = C_{2s} = 0$ for all s, except B_0 and C_0 and all $B_{2s+1} = 0$ except B_1. We shall put $2a = \frac{B_0}{m_1}$, $2b = \frac{C_0}{m_2}$, $am_1 = bm_2 = - B_1$ and $z = e^{i\phi}$, so that ω_1, ω_2, $-\omega_1$ and $-\omega_2$ are now the roots of

$$z\omega^4 - 2z\omega^2 (a + b) - ab (z - 1)^2 = 0, \tag{12}$$

and the solutions are given by

$$x_{2r} = \frac{a}{2\pi i} \int_C \frac{(\omega_1^2 - 2b) \cos \omega_1 t}{(\omega_1^2 - \omega_2^2) z^{r+1}} dz + \frac{a}{2\pi i} \int_C \frac{(\omega_2^2 - 2b) \cos \omega_2 t}{(\omega_2^2 - \omega_1^2) z^{r+1}} dz,$$

and

$$x_{2r+1} = \frac{a}{2\pi i} \int_C \frac{b (1 + z) (\cos \omega_2 t - \cos \omega_1 t)}{(\omega_1^2 - \omega_2^2) z^{r+2}} dz, \tag{13}$$

which can be expressed in the form

$$x_{2r} = \frac{1}{4\pi i} \oint \frac{f(z) e^{i\omega t}}{z^{r+1}} dz, \quad x_{2r+1} = \frac{1}{4\pi i} \oint \frac{g(z) e^{i\omega t}}{z^{r+1}} dz, \tag{14}$$

where ω is a four-valued function given by the equation (12) and *the path of the integral is a closed curve on the four-sheeted Riemann surface for* $\omega (z)$.

The Riemann surface for $\omega (z)$ is given by

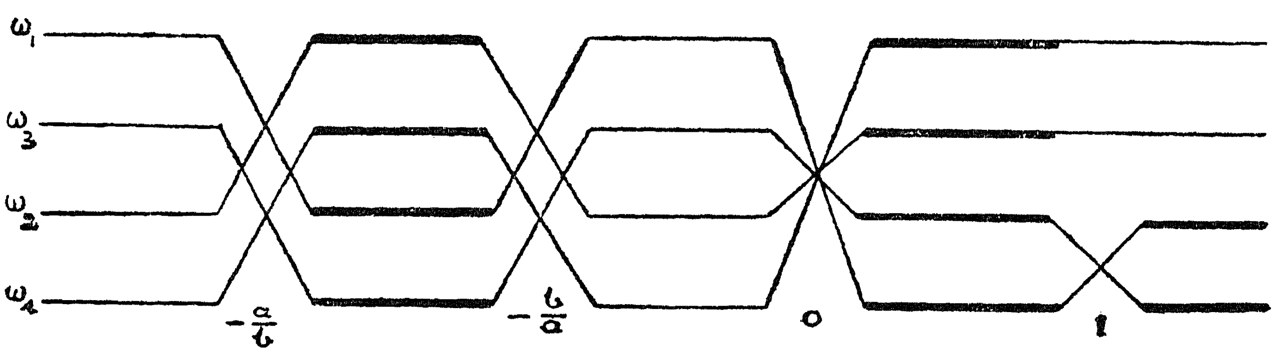

FIG. 1

The thick lines are cuts along which the surfaces are connected for the passage from one plane to the other. $\omega (z)$ is a holomorphic function of z on this four-sheeted surface.

We shall find the asymptotic nature of the solutions by the method of steepest descents. We select the contour of integration in (13) in such a way that it coincides with the paths of the steepest descents.

The equations (13) can be written in the form

$$x_{2r} = \frac{a}{8\pi i}\oint \frac{e^{i\omega_1 t} + e^{-i\omega_1 t} + e^{i\omega_2 t} + e^{-i\omega_2 t}}{z^{r+1}}\, dz$$

$$+ \frac{a(a-b)}{8\pi i}\oint \frac{e^{i\omega_1 t} + e^{-i\omega_1 t} - e^{i\omega_2 t} - e^{-i\omega_2 t}}{z^{r+1}\,\sigma}\, dz,$$

and

$$x_{2r+1} = \frac{a}{8\pi i}\oint \frac{b(1+z)(e^{i\omega_2 t} + e^{-i\omega_2 t} - e^{i\omega_1 t} - e^{-i\omega_1 t})}{z^{r+2}\,\sigma}\, dz, \tag{15}$$

where $2\sigma = \omega_1{}^2 - \omega_2{}^2$ and $\omega_1{}^2 = a + b + \sigma$, $\omega_2{}^2 = a + b - \sigma$.

The above expressions are the sum of terms of the form

$$\mathrm{P} = \oint \mathrm{R}(z)\, e^{\omega(z)t}\, dz. \tag{16}$$

The saddle points are given by $\frac{d\omega}{dz} = 0$. If z_0 be a saddle point then the asymptotic value of P is given by[4]

$$\frac{\mathrm{R}(z_0)\, e^{\omega(z_0)t}\sqrt{2\pi}\, e^{i\theta}}{|\, t\omega''(z_0)\,|^{\frac{1}{2}}} \tag{17}$$

where θ denotes the angle that the path of the steepest descents makes with the real axis and its value is to be found in each case by the usual method.

We shall first assume that $a > b$. The saddle points are given by $\omega_1{}' = 0$, $\omega_2{}' = 0$, that is by $z = \pm 1$. At the point $z = 1$, $\omega_1(z)$ has a development in powers of $(z-1) = z'$ of the type $\mathrm{A} + \mathrm{B}z'^2 + \ldots$ where z' is small and $\mathrm{B} > 0$; therefore, the behaviour of $i\omega_1$ at $z = 1$ is like that of iz'^2. Putting $z' = \xi + i\eta$, we have $i\omega_1(z) = \phi + i\psi = i(\xi^2 - \eta^2) - 2\xi\eta$. Now $\psi = 0$ gives the lines of steepest descents and they are therefore given by $\xi = \pm\eta$, (ξ, η) being the set of rectangular axes at $z = 1$ as origin, the ξ-axis coinciding with the real axis. Also $\phi = -2\xi\eta$ and ϕ has to decrease as we proceed along the lines of steepest descents. ϕ will decrease if both ξ and η be positive or if both of them be negative. Taking the counter-clockwise direction as the path of integration we get $\theta = \frac{1}{4}\pi$. Similarly at $z = -1$ we get $\theta = \frac{3}{4}\pi$. For $i\omega_2(z)$ we get at $z = 1$, $\theta = \frac{3}{4}\pi$ and at $z = -1$, $\theta = \frac{5}{4}\pi$.

After substituting in (17) the values of θ, $\mathrm{R}(z_0)$, etc., we get after some reduction,

$$x_{2r} = a\left(\frac{a}{b}\right)^{\frac{1}{4}}\left(\frac{2}{\pi\vartheta_1 t}\right)^{\frac{1}{2}}\cos\left(\vartheta_1 t - \frac{\pi}{4}\right)$$

$$+(-1)^r a\left\{\frac{|a-b|}{b}\right\}^{\frac{1}{4}}\left(\frac{2}{\pi\vartheta_2 t}\right)^{\frac{1}{2}}\cos\left(\vartheta_2 t + \frac{\pi}{4}\right) \qquad (18)$$

where $\vartheta_1 = \sqrt{2a+2b}$, $\vartheta_2 = \sqrt{2a}$.

Similarly the asymptotic value of x_{2r+1} is given by

$$x_{2r+1} = -a\left(\frac{b}{a}\right)^{\frac{1}{4}}\left(\frac{2}{\pi\vartheta_1 t}\right)^{\frac{1}{2}}\cos\left(\vartheta_1 t - \frac{\pi}{4}\right) \qquad (19)$$

In case $b > a$, the phase $\frac{\pi}{4}$ in the second term of (18) will be changed to $-\frac{\pi}{4}$.

The normal vibrations of the lattice according to Raman are given below:—

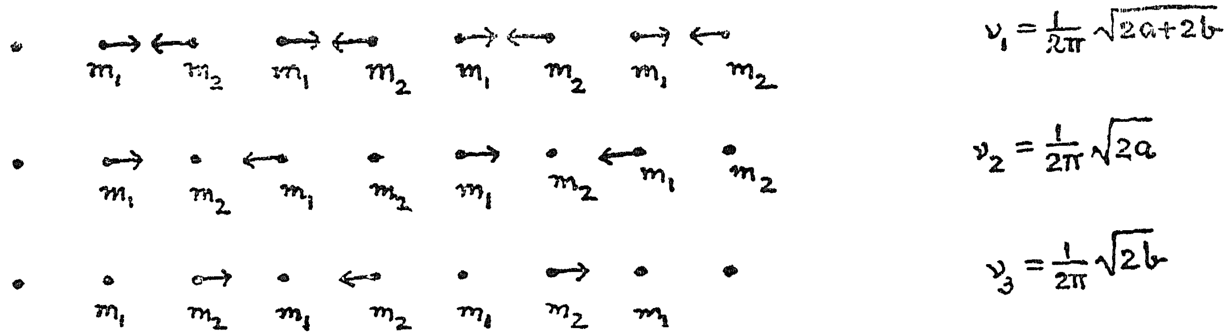

FIG. 2

If the lattice at the time $t = 0$ is disturbed as follows

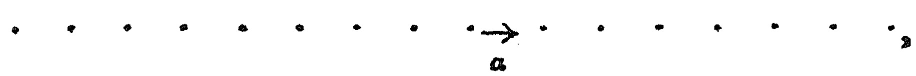

FIG. 3

the amplitudes of the particles for sufficiently large t are asymptotically given by (18) and (19) which show that the solutions tend to a time dependent superposition of normal solutions with frequencies ν_1 and ν_2. If however a particle with mass m_2 had been displaced, the asymptotic nature would have been a time dependent superposition of vibrations with frequencies ν_1 and ν_3. If either both m_1 and m_2 or an arbitrary finite set of particles of the lattice had been disturbed, we can say by the principle of superposition, *that the asymptotic nature of the vibration would tend to a time dependent superposition of the normal vibrations according to Raman.*

If $a = b$, the vibrations with frequencies ν_2 and ν_3 cease to be normal vibrations according to Raman. There is no discontinuity in the result of the above paragraph as $a \rightarrow b$ from that when $a = b$, for the controlling factors in the contributions of oscillations with frequencies ν_2 and ν_3 depend on $| a - b |$. *It may also be noted that when $a = b$, the oscillations ν_2 and ν_3 cease to be saddle point oscillations.* In this case x_{2r} and x_{2r+1} take the form[5] $J_{4r}(\sqrt{2a}t)$ and $J_{4r+2}(\sqrt{2a}t)$ corresponding to Hamilton's solutions.

Summary

The solutions for the displacements of an infinite linear lattice consisting of two types of particles under some initial disturbance have been obtained and their asymptotic nature has been investigated by the method of "steepest descents" which shows that the displacements are asymptotically time dependent superposition of the normal vibrations according to Raman.

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I take this opportunity of recording here my indebtedness and gratitude to my Professor, Sir C. V. Raman, on the occasion of his sixtieth birthday.

N. S. Nagendra Nath.

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