THE NEUTRINO THEORY OF LIGHT—II.

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1. Introduction.

When we published our report[1] on Jordan's[2] neutrino theory of light in these *Proceedings*, we assumed a standpoint differing in some respects from Jordan's[3] and Kronig's[4] initial papers. Meanwhile they have published new representations[5] of the theory which are in complete agreement with our report. The point is that the theory as presented on the basis of the Dirac theory of holes in our report which is followed recently in Jordan's[6] last paper, deals with two kinds of neutrinos—neutrinos and anti-neutrinos—but neglects the spin.

But there is no doubt that neutrinos have a spin. There is the general theorem that particles with an integral angular momentum must obey the Bose–Einstein statistics and that particles with a half-integral angular momentum must obey the Fermi-Dirac statistics.[7] Empirical evidence for the existence of the spin is afforded by the theory of β-decay* ; if electron, proton and neutron have each of them the spin $\frac{1}{2}$, then the process

$$\text{neutron} \rightarrow \text{proton} + \text{electron} + \text{neutrino}$$

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1 M. Born and N. S. Nagendra Nath, *Proc. Ind. Acad. Sci.*, 1936, **3**, 318. (Referred to as I)
2 P. Jordan, *Zeits. f. Phy.*, 1935, **93**, 464.
3 P. Jordan, *Zeits. f. Phy.*, 1936, **98**, 709 and 759; 1936, **99**, 109.
4 R. de L. Kronig, *Physica*, 1935, **2**, 491, 854, 965.
5 P. Jordan and R. de L. Kronig, *Zeits. f. Phy.*, 1936, **100**, 569.
6 P. Jordan, *Zeits. f. Phy.*, 1936, **102**, 243.
7 P. Jordan, *Anschauliche Quantenmechanik* (Julius Springer), 1936, p. 244.

* This argument implies the identity between Jordan's neutrino which is the fundamental particle in his theory of light, and Pauli's neutrino whose existence has been assumed to account for β-decay. Experimental evidence on β-decay seems to show that Pauli's neutrino has practically no charge and no rest mass. Thus, the common characteristics between Jordan's neutrino and Pauli's neutrino in that they have no rest mass and no electrical charge, constitute strong arguments to believe that Jordan's neutrino and Pauli's neutrino are identical. An experimental proof for this identification might result by studying a possible influence of light field on β-decay.

shows that the neutrino must also have the spin $\frac{1}{2}$. Another argument is that the spin of the neutrino is necessary to account for the polarisation of light. Indeed the theory which neglects the spin (as in I and in Jordan's papers) leads to half the value given by Planck's formula which can easily be understood by the consideration that the internal degree of freedom of the neutrinos (spin) and of the photons (polarisation) is not incorporated in the formalism.

One of us[8] has shown how the spin can be easily introduced by adding a corresponding index to the operators representing neutrinos $a_{\kappa,i}$ where $i = \mathrm{R}, \mathrm{L}$, characterising the two spin states. A photon state is then represented by a *pair* of operators; namely,

$$b_{k,\rho} = \frac{-1}{\sqrt{|2k|}} \sum_{\kappa=-\infty}^{\infty} \sum_{i} a_{\kappa,i}\, a_{\kappa}{}^{\dagger}{}_{-k,i}$$

$$b_{k,\lambda} = \frac{-1}{\sqrt{|2k|}} \sum_{\kappa=-\infty}^{\infty} \sum_{i,j}{}' a_{\kappa,i}\, a_{\kappa}{}^{\dagger}{}_{-k,j}$$

where the dash over the (i, j) summation indicates that $i \neq j$ and ρ and λ characterise the two polarisation states of light. Though this method leads to correct results, for instance Planck's formula, it is not very satisfactory from the formal standpoint. We shall present here a new form of this theory in which each physical quantity belongs to only *one* operator. The fundamental matrices representing them have now of course four rows and columns instead of those of two in the earlier representations. This procedure is in better agreement with the general definitions of quantum mechanics and leads to very simple and elegant results.

2. *The Operators of the Neutrino Field.*

The neutrino field is described by two sets of infinite numbers of non-commuting operators a_κ and γ_κ which are enumerated by half-integral positive numbers, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The half-integral numbers are chosen primarily for the sake of convenience. The operators with negative indices are defined by the relations

$$\kappa > 0, \quad a_{-\kappa} = \gamma_{\kappa}{}^{\dagger}, \quad \gamma_{-\kappa} = a_{\kappa}{}^{\dagger}; \tag{2.1}$$

where $\dagger$ means the adjoint operator. Indeed, one may see by (2.1) that there is no necessity for the introduction of two sets of operators but we retain them, following Jordan, for the sake of symmetry between neutrinos and anti-neutrinos whose relations with the above operators will be defined subsequently. The meaning of these operators can be understood with the help of the Correspondence Principle as the Fourier coefficients of the wave

[8] N. S. Nagendra Nath, *Proc. Ind. Acad. Sci.*, 1936, 3, 448.

functions which will be in the case of one dimension

$$\begin{aligned}\psi\,(t - x/c) &= \sum_{\kappa=-\infty}^{\infty} a_\kappa \exp\,[2\pi i\nu_1\kappa\,(t - x/c)],\\ \chi\,(t - x/c) &= \sum_{\kappa=-\infty}^{\infty} \gamma_\kappa \exp\,[2\pi i\nu_1\kappa\,(t - x/c)] = \psi^\dagger,\end{aligned} \tag{2.2}$$

where ν_1 corresponds to the fundamental frequency of the " Hohlraum ".

We postulate the following commutation rules for the operators :

$$\begin{aligned}a_\kappa\, a_\mu + a_\mu\, a_\kappa &= 0,\\ \gamma_\kappa\,\gamma_\mu + \gamma_\mu\,\gamma_\kappa &= 0,\\ \gamma_\kappa\, a_\mu + a_\mu\,\gamma_\kappa &= \delta_{\mu,-\kappa},\end{aligned} \tag{2.3}$$

where $\delta_{\mu,-\kappa}$ is the Dirac operator which is zero if $\mu \neq -\kappa$ and is the unit matrix if $\mu = -\kappa$. The operators describing the number of neutrinos and anti-neutrinos of energy κ (the unit of energy being $h\nu_1$) are defined only for $\kappa > 0$ by

$$\begin{aligned}\mathrm{N}_\kappa^{(+)} &= a_\kappa^\dagger\, a_\kappa = 1 - a_\kappa\, a_\kappa^\dagger,\\ \mathrm{N}_\kappa^{(-)} &= \gamma_\kappa^\dagger\,\gamma_\kappa = 1 - \gamma_\kappa\,\gamma_\kappa^\dagger.\end{aligned} \tag{2.4}$$

The relations

$$\begin{aligned}a_\kappa^\dagger\, a_\kappa &= 1 - a_\kappa\, a_\kappa^\dagger,\\ \gamma_\kappa^\dagger\,\gamma_\kappa &= 1 - \gamma_\kappa\,\gamma_\kappa^\dagger,\end{aligned} \tag{2.5}$$

are true in virtue of (2.3) and (2.1).

3. *The Fundamental Operators and Their Properties.*

In I, we started with the matrices

$$a = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad a^\dagger = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \text{ and } s = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{3.1}$$

which have the following properties :

$$\begin{aligned}&a^2 = 0, \qquad a^{\dagger 2} = 0, \qquad s^2 = 1,\\ &a^\dagger a + a\,a^\dagger = 1, \quad as + sa = 0, \quad a^\dagger s + sa^\dagger = 0.\end{aligned} \tag{3.2}$$

We may note that the above matrices are related to Pauli's spin matrices by the equations

$$\sigma_x = a^\dagger + a,\ \sigma_y = i\,(a^\dagger - a) \text{ and } \sigma_z = s. \tag{3.3}$$

We build the fundamental matrices of the present theory by the above matrices of the old one and with the notion of the direct product of the matrices defined in I. The fundamental matrices are here defined as

$$\begin{aligned}\mathrm{A} &= a \times 1\\ \mathrm{S} &= s \times 1\end{aligned} \tag{3.4}$$

Corresponding to the rule of the direct product of matrices $\mathrm{A}_{n_1n_2}$; $_{m_1m_2} = a_{n_1m_1}\, 1_{n_2m_2}$ and re-enumerating the combinations of the suffixes 11, 12, 21, 22,

as 1, 2, 3, 4 we find

$$A = a \times 1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^\dagger = a^\dagger \times 1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$S = s \times 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.5}$$

We then have

$$AS = as \times 1 = -a \times 1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$SA = sa \times 1 = a \times 1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

from which we conclude

$$AS + SA = 0. \tag{3.7}$$

It is easy to see that

$$\begin{aligned} A^\dagger S + SA^\dagger &= 0, \\ A^2 = AA &= 0, \\ A^{\dagger 2} = A^\dagger A^\dagger &= 0, \\ S^2 = SS &= 1. \end{aligned} \tag{3.8}$$

The products $A^\dagger A$ and $AA^\dagger$ are given by

$$A^\dagger A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$AA^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

which show that the matrix $A^\dagger A$ has the eigen-values 0, 0, 1 and 1 while $AA^\dagger$ has the same eigen-values in the reverse order. It is also to be seen that

$$A^\dagger A + AA^\dagger = 1. \tag{3.10}$$

The relations between the fundamental operators and Dirac's matrices.—We note that our matrices A and S can be expressed in terms of Dirac's[9] matrices.

[9] P. A. M. Dirac, *Quantum Mechanics*, page 255.

The Dirac matrices are

$$\rho_1 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0&0&-i&0\\ 0&0&0&-i\\ i&0&0&0\\ 0&i&0&0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix} \tag{3.11}$$

$$\sigma_x = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0&-i&0&0\\ i&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix},$$

$$\alpha_x = \rho_1 \sigma_x, \qquad \alpha_y = \rho_1 \sigma_y, \qquad \alpha_z = \rho_1 \sigma_z.$$

It is now easy to see that

$$\begin{aligned} \mathrm{A} &= \tfrac{1}{2}(\rho_1 + i\rho_2),\\ \mathrm{A}^\dagger &= \tfrac{1}{2}(\rho_1 - i\rho_2),\\ \mathrm{S} &= \rho_3 \end{aligned} \tag{3.12}$$

or

$$\begin{aligned} \rho_1 &= \mathrm{A}^\dagger + \mathrm{A}\\ \rho_2 &= i(\mathrm{A}^\dagger - \mathrm{A})\\ \rho_3 &= \mathrm{S} \end{aligned} \tag{3.13}$$

closely resembling the relations (3.3) which exist between Pauli's matrices and the two-dimensional matrices of the old theory. We also note that we may choose for the development of the present theory $\frac{1}{2}(\alpha_x + i\alpha_y)$, $\frac{1}{2}(\alpha_x - i\alpha_y)$ and σ_z as the fundamental operators which may be expressed in terms of a, $a^\dagger$ and s.

Replacing the enumeration, 1, 2, 3, 4 by 0, $\bar{0}$, 1 and $\bar{1}$ the matrix elements of the fundamental matrices which do not vanish are given by

$$\begin{aligned} &\mathrm{A}(0,1) = 1, \qquad \mathrm{S}(0,0) = 1,\\ &\mathrm{A}(\bar{0},\bar{1}) = 1, \qquad \mathrm{S}(\bar{0},\bar{0}) = 1,\\ &\mathrm{A}^\dagger(1,0) = 1, \qquad \mathrm{S}(1,1) = -1,\\ &\mathrm{A}^\dagger(\bar{1},\bar{0}) = 1, \qquad \mathrm{S}(\bar{1},\bar{1}) = -1. \end{aligned} \tag{3.14}$$

4. The Jordan-Wigner Representations of the Operators.

As in I, we can write down the Jordan-Wigner representations of the neutrino operators by the following scheme:

$$\begin{array}{lcccccccccccc} & \frac{1}{2} & -\frac{1}{2} & & & -(\kappa-1) & & \kappa & -\kappa & (\kappa+1) & & \\ \alpha_\kappa \mp \gamma_{-\kappa}^\dagger = & \mathrm{S} & \times \mathrm{S} & \times \cdots & \times & \mathrm{S} & \times & \mathrm{A} & \times 1 & \times 1 & \times \cdots & \\ \alpha_{-\kappa} = \gamma_\kappa^\dagger = & \mathrm{S} & \times \mathrm{S} & \times \cdots & \times & \mathrm{S} & \times & \mathrm{S} & \times \mathrm{A}^\dagger & \times 1 & \times \cdots & \\ \kappa > 0 & & & & & & & & & & & (4.1)\\ \gamma_\kappa = \alpha_{-\kappa}^\dagger = & \mathrm{S} & \times \mathrm{S} & \times \cdots & \times & \mathrm{S} & \times & \mathrm{S} & \times \mathrm{A} & \times 1 & \times \cdots & \\ \gamma_{-\kappa} = \alpha_\kappa^\dagger = & \mathrm{S} & \times \mathrm{S} & \times \cdots & \times & \mathrm{S} & \times & \mathrm{A}^\dagger & \times 1 & \times 1 & \times \cdots & \end{array}$$

Each matrix has only two types of non-vanishing elements which are given by the following scheme in which t_κ is $0, \bar{0}, 1, \bar{1}$. They are

$$\begin{aligned}
&a_\kappa\left(t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, \overset{0}{\bar{0}}, t_{-\kappa}, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, \overset{1}{\bar{1}}, t_{-\kappa}, --\right) \\
&\qquad = (-1)^{p_{\frac{1}{2}}+p_{-\frac{1}{2}}+\cdots+p_{-(\kappa-1)}} \\
&a_{-\kappa}\left(t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, \overset{1}{\bar{1}}, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, \overset{0}{\bar{0}}, --\right) \\
&\qquad = (-1)^{p_{\frac{1}{2}}+p_{-\frac{1}{2}}+\cdots+p_\kappa} \\
&\kappa > 0 \qquad\qquad (4.2) \\
&\gamma_\kappa\left(t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, \overset{0}{\bar{0}}, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, \overset{1}{\bar{1}}, --\right) \\
&\qquad = (-1)^{p_{\frac{1}{2}}+p_{-\frac{1}{2}}+\cdots+p_\kappa} \\
&\gamma_{-\kappa}\left(t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, \overset{1}{\bar{1}}, t_{-\kappa}, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, \overset{0}{\bar{0}}, t_{-\kappa}, --\right) \\
&\qquad = (-1)^{p_{\frac{1}{2}}+p_{-\frac{1}{2}}+\cdots+p_{-(\kappa-1)}}
\end{aligned}$$

where

$$\begin{aligned}
p_\kappa &= 0 \quad \text{if } t_\kappa = 0 \text{ or } \bar{0}, \\
p_\kappa &= 1 \quad \text{if } t_\kappa = 1 \text{ or } \bar{1}.
\end{aligned}$$

Using the definition of the direct product of the matrices and the equations (3.7), (3.8) and (3.10), it can easily be verified that the representations of the fundamental operators given in (4.1) satisfy the commutation rules (2.3).

5. *A New System of Operators.*

As in I, we introduce a new system of operators defined by

$$\begin{aligned}
a_\kappa &= \frac{a_\kappa + \gamma_\kappa}{\sqrt{2}}, \\
c_\kappa &= \frac{a_\kappa - \gamma_\kappa}{i\sqrt{2}},
\end{aligned} \qquad (5.1)$$

so that

$$\begin{aligned}
a_\kappa &= \frac{a_\kappa + ic_\kappa}{\sqrt{2}}, \\
\gamma_\kappa &= \frac{a_\kappa - ic_\kappa}{\sqrt{2}}.
\end{aligned} \qquad (5.2)$$

One sees from (5.1) and (2.1) that

$$\begin{aligned}
a_{-\kappa} &= a_\kappa^\dagger, \\
c_{-\kappa} &= c_\kappa^\dagger.
\end{aligned} \qquad (5.3)$$

The commutation rules satisfied by a's and c's are found by means of (2.3) and (5.1). They are

$$\begin{aligned} a_\kappa a_\mu + a_\mu a_\kappa &= \delta_{\mu,-\kappa}, \\ c_\kappa c_\mu + c_\mu c_\kappa &= \delta_{\mu,-\kappa}, \\ a_\kappa c_\mu + c_\mu a_\kappa &= 0, \end{aligned} \tag{5.4}$$

for $\kappa, \mu = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \cdots$. Introducing the operators

$$\begin{aligned} \mathrm{L}_\kappa &= a_\kappa^\dagger a_\kappa = a_{-\kappa} a_\kappa, \\ \mathrm{N}_\kappa &= c^\dagger_\kappa c_\kappa = c_{-\kappa} c_\kappa, \end{aligned} \qquad \kappa > 0 \tag{5.5}$$

one has as in I

$$\begin{aligned} \mathrm{L}_\kappa + \mathrm{N}_\kappa &= \mathrm{N}_\kappa^{(+)} + \mathrm{N}_\kappa^{(-)}, \\ \mathrm{L}_\kappa - \mathrm{N}_\kappa &= a_\kappa^\dagger \gamma_\kappa + \gamma_\kappa^\dagger a_\kappa, \end{aligned} \qquad \kappa > 0 \tag{5.6}$$

so that the expectation value

$$\bar{\mathrm{L}}_\kappa - \bar{\mathrm{N}}_\kappa = 0, \qquad \kappa > 0 \tag{5.7}$$

from which we can conclude that $\sum_{\kappa > 0} \mathrm{L}_\kappa$ and $\sum_{\kappa > 0} \mathrm{N}_\kappa$ are convergent if we assume that all states above a certain state K are unoccupied, *i.e.*,

$$\mathrm{N}_\kappa^{(+)} = 0, \ \mathrm{N}_\kappa^{(-)} = 0 \text{ for } \kappa > \mathrm{K}. \tag{5.8}$$

6. The Photon Operators and Their Commutation Rules.

As in I, the photon operator is defined by

$$b_k = \frac{i}{\sqrt{|k|}} \sum_{\kappa=-\infty}^{\infty} a_\kappa c_{k-\kappa}, \tag{6.1}$$

for $k = \pm 1, \pm 2, \pm 3, \cdots$.

One may note that

$$b_k^\dagger = b_{-k}. \tag{6.2}$$

It can be demonstrated with the necessary assumptions pointed out in I that

$$\begin{aligned} b_k b_j - b_j b_k &= 0 \quad \text{if } k + j \neq 0, \\ b_k b_{-k} - b_{-k} b_k &= 1 \quad \text{if } k > 0. \end{aligned} \tag{6.3}$$

The various other forms of b_k are

$$b_k = -\frac{1}{\sqrt{|k|}} \sum_{\kappa=-\infty}^{\infty} a_\kappa \gamma_{k-\kappa},$$

$$b_k = -\frac{1}{\sqrt{|k|}} \sum_{\kappa=-\infty}^{\infty} a_\kappa a^\dagger_{\kappa-k},$$

$$b_k = -\frac{1}{\sqrt{|k|}} \left\{ \sum_{\kappa=\frac{1}{2}}^{k-\frac{1}{2}} a_\kappa \gamma_{k-\kappa} + \sum_{\kappa=\frac{1}{2}}^{\infty} (a_{k+\kappa} a_\kappa^\dagger - \gamma_{k+\kappa} \gamma_\kappa^\dagger) \right\}. \tag{6.4}$$

7. *The Operator B and Its Properties.*

If we define the operator B as

$$\mathrm{B} = i \sum_{-\infty}^{\infty} a_\kappa \, c_{-\kappa}, \tag{7.1}$$

it can be found as in I that

$$\begin{aligned} \mathrm{B} &= \sum_{\frac{1}{2}}^{\infty} (a_\kappa^\dagger \, a_\kappa - \gamma_\kappa^\dagger \, \gamma_\kappa), \\ &= \sum_{\frac{1}{2}}^{\infty} (\mathrm{N}_\kappa^{(+)} - \mathrm{N}_\kappa^{(-)}), \end{aligned} \tag{7.2}$$

and that it has whole number eigen-values. As in I, we note that B commutes with all b_k, *i.e.*,

$$\mathrm{B}\, b_k - b_k \, \mathrm{B} = 0. \tag{7.3}$$

8. *Photons and Their Statistical Equilibrium with Matter.*

For a state with given numbers of neutrinos and anti-neutrinos the operator representing the number of photons in the kth energy state is not a diagonal matrix. But we can calculate its average value or expectation value—the diagonal element of P_k :

$$\begin{aligned} \mathrm{P}_k \, (t_{1/2}, t_{-1/2}, t_{3/2}, t_{-3/2}, - - -) &= (b_k^\dagger \, b_k) \, (t_{1/2}, t_{-1/2}, - - - -) \\ &= (b_k^\dagger \, b_k) \, (- -, t, - -) \\ &= \sum_{--, t', --,} | \, b_k \, (- -, t', - - ; - -, t, - -) \, |^2. \end{aligned} \tag{8.1}$$

We find from (6.4) that

$$\begin{aligned} &| \, b_k \, (- -, t', - - ; \; - -, t, - -,) \, | = \\ &\quad \frac{1}{\sqrt{|\, k \,|}} \Big\{ \Big| \sum_{\frac{1}{2}}^{k - \frac{1}{2}} (a_\kappa \, \gamma_{k-\kappa}) \, (- -, t', - - ; - -, t, - -) \\ &\qquad\qquad + - - + - - \Big| \Big\}, \\ &= \frac{1}{\sqrt{|\, k \,|}} \Big\{ \Big| \sum_{\frac{1}{2}}^{k - \frac{1}{2}} a_\kappa \, (- -, t', - -; - -, t'', - -) \, \gamma_{k-\kappa} \, (- -, t'', - - ; \\ &\qquad\qquad - -, t, - -) + - - + - - \Big| \Big\}. \end{aligned} \tag{8.2}$$

We know from (4.2) that the non-vanishing elements of a_κ are those for which $t'_\kappa \rightarrow t''_\kappa$ is $0 \rightarrow 1$ or $\bar{0} \rightarrow \bar{1}$ whereas other t's are unchanged. The elements of $\gamma_{k-\kappa}$ which do not vanish are those for which $t''_{k-\kappa} \rightarrow t_{k-\kappa}$ is $0 \rightarrow 1$ or $\bar{0} \rightarrow \bar{1}$, other t's unchanged. Thus the elements of $a_\kappa \, \gamma_{k-\kappa}$ do not vanish if all t''s are equal to the corresponding t'''s and t's *except*

$$\begin{aligned} t''_\kappa = t_\kappa &= 1 \text{ or } \bar{1}, \\ t'_\kappa &= 0 \text{ or } \bar{0}\,; \end{aligned} \tag{8.3}$$

and

$$t'_{k-\kappa} = t''_{k-\kappa} = 0 \text{ or } \bar{0},$$
$$t_{k-\kappa} = 1 \text{ or } \bar{1}.$$

Thus the elements of $a_\kappa\ \gamma_{k-\kappa}$ do not vanish only when $(t_\kappa, t_{k-\kappa})$ assumes the values (1, 1), $(\bar{1}, \bar{1})$, $(1, \bar{1})$ and $(\bar{1}, 1)$ respectively. *In the first two combinations there is no change in the spin states between the absorbed and the emitted neutrinos while in the remaining two combinations the spin states are opposite.** If t_κ is 1 the probabilities that $t_{k-\kappa}$ is 1 or $\bar{1}$ are equal. From this we conclude that the contribution to $\bar{P}_k$ by the matrix $a_\kappa\ \gamma_{k-\kappa}$ is

$$\frac{1}{2k} N_\kappa^{(+)} N_{k-\kappa}^{(-)} \tag{8.4}$$

As in I, $\bar{P}_k$ can be written as

$$\bar{P}_k = \frac{1}{2k} \sum_{\frac{1}{2}}^{k-\frac{1}{2}} N_\kappa^{(+)} N_{k-\kappa}^{(-)} + \frac{1}{2k} \sum_{\frac{1}{2}}^{\infty} \{N_{k+\kappa}^{(+)} (1 - N_\kappa^{(+)}) + N_{k+\kappa}^{(-)} (1 - N_\kappa^{(-)})\}$$

One may note, however, that the maximum value of N_κ is *two* instead of *one* in the old theory. In considering the statistical equilibrium between matter and radiation, we assume that the average number of particles in the (1, 1) state of $a_\kappa^\dagger\ a_\kappa$ given by (4.1) is equal to the number of particles in the $(\bar{1}, \bar{1})$ state of $a_\kappa^\dagger\ a_\kappa$. That is,

$$\begin{aligned} N_\kappa^{(+)} (1, 1) &= N_\kappa^{(+)} (\bar{1}, \bar{1}), \\ N_\kappa^{(-)} (1, 1) &= N_\kappa^{(-)} (\bar{1}, \bar{1}). \end{aligned} \tag{8.6}$$

Replacing k in (8.5) by l so that there may not be any confusion between it and the Boltzmann constant, we can show as in I that

$$\begin{aligned} N_l^{(+)} (1, 1) &= N_l^{(+)} (\bar{1}, \bar{1}) = \frac{ay}{1 + ay} \\ N_l^{(-)} (1, 1) &= N_l^{(-)} (\bar{1}, \bar{1}) = \frac{y/a}{1 + y/a} \end{aligned} \tag{8.7}$$

where $y = \exp(-\beta l)$ and $\beta = h\nu_1/k\mathrm{T}$, k being the Boltzmann constant. Putting

and

$$\begin{aligned} \kappa &= l\ \omega, \\ \nu_1 l &= \nu, \end{aligned} \tag{8.8}$$

we have

$$\exp(-\beta\kappa) = y^\omega \tag{8.9}$$

and

$$d\kappa = l\ d\omega. \tag{8.10}$$

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* These considerations are the same as put forward earlier by the help of two dimensional matrix representations. Indeed, $b_{k,\rho}$ corresponded to the case of the same spin between the emitted and absorbed particles while $b_{k,\lambda}$ corresponded to the case of opposite spin between them (reference 8).

Replacing the sum by integrals

$$\overline{P(\nu)} = 2\int_0^1 \frac{ay^\omega}{1+ay^\omega}\cdot\frac{{}^1/_a\, y^{1-\omega}}{1+{}^1/_a\, y^{1-\omega}}\,d\omega + \int_0^\infty \frac{ay^{1+\omega}}{(1+ay^{1+\omega})}\cdot\frac{1}{(1+ay^\omega)}\,d\omega$$
$$+\int_0^\infty \frac{{}^1/_a\, y^{1+\omega}}{1+{}^1/_a\, y^{1+\omega}}\cdot\frac{1}{1+{}^1/_a\, y^{1+\omega}}\,d\omega.$$

If we substitute $(1-z)/z$ for ay^ω in the first integral, for $ay^{1+\omega}$ in the second integral and ${}^1/_a\, y^{1+\omega}$ in the third integral, we get

$$P(\nu) = \frac{2y}{1-y} \tag{8.12}$$

As in I, we can also show that

$$P(\nu) = 2\int_{-\infty}^{\infty} \frac{ay^\omega}{(1+ay^\omega)}\left(1-\frac{ay^{\omega-1}}{1+ay^{\omega-1}}\right)d\omega$$
$$= \tfrac{1}{2}\sum_{-\infty}^{\infty} N_\kappa\,(1-N_{\kappa-l}) \text{ where } \nu = l\nu_1. \tag{8.13}$$

Since $y = \exp(-h\nu/kT)$, we get

$$P(\nu) = \frac{2}{\exp(h\nu/kT)-1}. \tag{8.14}$$

The factor 2 in (8.14) distinguishes this formula from the corresponding one in I. It expresses the fact that there are two independent states of different polarisation for a photon with a given frequency. The presence of this factor is necessary to get Planck's formula with the correct factor. Assuming that (8.14) holds also in the case of three dimensions which case has been treated by Jordan and Kronig, one gets Planck's formula by multiplying (8.14) by the individual energy $h\nu$ of the photons having a frequency between ν and $\nu+d\nu$ and by their number $4\pi\nu^2 d\nu/c^3$,

$$\frac{8\pi\nu^2 d\nu}{c^3}\,\frac{h\nu}{\exp(h\nu/kT)-1}. \tag{8.15}$$

10. The Relation between the Energy of Neutrinos and Photons.

Just as in I, the energy of the neutrino field

$$E = \sum_{\kappa=\frac{1}{2}}^{\infty} \kappa\,(L_\kappa + N_\kappa) = \sum_{\kappa=\frac{1}{2}}^{\infty} \kappa\,(N_\kappa^{(+)} + N_\kappa^{(-)}) \tag{9.1}$$

and the energy of the photon field

$$W = \sum_{k=1}^{\infty} k\,P_k = \sum_{k=1}^{\infty} k\,b_k^\dagger\,b_k \tag{9.2}$$

have the following relation

$$E - W = B^2/2. \tag{9.3}$$

The proof of this relation due to Kronig, is contained in I which holds here also.