THE DIFFRACTION OF LIGHT BY HIGH FREQUENCY SOUND WAVES: GENERALISED THEORY.

The Asymmetry of the Diffraction Phenomena at Oblique Incidence.

BY N. S. NAGENDRA NATH.

(From the Department of Physics, Indian Institute of Science, Bangalore.)

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1. *Introduction.*

IN a series of five papers[1] published in these *Proceedings*, the theory of the diffraction of light by high frequency sound waves has been developed. There have been however two stages in the development of the theory. The first one had a restriction in the theory in order to simplify the treatment of the problem and bring out its essential features without unnecessary complications. In the second one, the above restriction in the theory has been removed and the theory of the phenomenon under general conditions has been developed. The general theory includes the preliminary one as a special case.

Preliminary Theory.—In Parts I, II and III, the essential idea is that the optical effects are due to the corrugated form of the emerging wave-front and that the corrugations due to the density fluctuations could be simply calculated by the phase changes accompanying the traversing beam ignoring the amplitude changes. The exact condition under which this restriction could be realised in practice is indicated in the papers. The theory accounts for the appearance of a large number of diffraction orders and also the wandering of the intensity of light amongst them as the length of the cell, the supersonic intensity and the wave-length of the incident light are changed. All these results have been strikingly confirmed quantitatively by Bär[2] who actually realised in practice the restriction we had imposed in our preliminary theory. The intensity variations and the symmetry of the diffraction effects in the case of oblique incidence have been also confirmed by Bär.[2] In Part III of the theory, we investigated the Doppler effects in the diffraction orders when the supersonic wave is either a progressive one or a standing

[1] C. V. Raman and N. S. Nagendra Nath, *Proc. Ind. Acad. Sci.*, 1935, **2**, 406 and 413; 1936, **3**, 75, 119 and 459.

[2] R. Bär, *Helv. Phy. Acta.*, 1936. Bär realised it by diminishing the frequency of the supersonic waves.

one. The results in the case of a standing wave are really interesting. We obtained the result that any order would cohere partly with any other order counted in an even sequence from it while it would not cohere with the remaining ones lying on an odd sequence from it. This result is in remarkable agreement with the same result obtained by Bär[3] experimentally.

Generalised Theory.—In Parts IV and V, the restriction of the preliminary theory was removed by considering the partial differential equation governing the propagation of light in a quasi-homogeneous medium. The results regarding the coherence phenomena amongst the diffraction orders were found to be true even if the supersonic wave be a general periodic progressive one or a standing one. We then considered the cases of a simple periodic progressive wave and a standing wave to investigate the amplitudes of the various diffraction orders. A difference—differential equation was obtained whose solutions correspond to the amplitudes of the diffraction orders. This equation enabled us to show that, in the case of oblique incidence, the diffraction pattern will be, in general, asymmetric which agrees with the results of Debye and Sears, Lucas and Biquard, Bär and Parthasarathy. The purpose of this paper is to solve the difference—differential equation occurring in the theory by the series method and offer an explanation for the quantitative experimental results obtained by Parthasarathy[5] in the case of the oblique incidence.

In this connection we desire to make some remarks regarding Brillouin's theory. The idea of characteristic reflection in these experiments does not seem very appropriate in view of the fact that the wave-length of the periodic fluctuation of the density is large compared with the wave-length of the light. The concept of reflection does not explain the presence of other orders and the *non-sharpness* of the maximum intensity of the reflected order at the characteristic obliquity of light to the sound waves. Thus, the use of the general word *'propagation'* is certainly preferable to the word *'reflection' for we know only and are here concerned with the equation governing the propagation of light in the medium.* In Brillouin's rigorous theory[4] of the diffraction phenomenon, he starts from the well-known partial differential equation governing the propagation of light in a quasi-homogeneous medium as we have also done. Thus the basis of Brillouin's rigorous theory and our general theory are the same. But the developments of the theory are different. His fundamental idea is that the emerging wave-front will be equivalent

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3 R. Bär, *Helv. Phy. Acta.*, 1935, **8**, 591.

4 L. Brillouin, *Act. Sci. et Ind.*, 1933, **59**.

5 S. Parthasarathy, *Proc. Ind. Acad. Sci.*, 1936, **3**, 594.

to a set of plane waves travelling in the same direction of the incident light but with an amplitude grating on each one of them given by a multiple of a Mathieu Function. This analysis though perfect leads to complicated difficulties for, to find the diffraction effects in any particular direction, one will have to find the effects due to all the analysed waves. On the other hand, we have analysed the emerging corrugated wave into a set of plane waves inclined to one another at the characteristic diffracted angles. To find the diffraction effects in any particular direction, one has only to consider the plane wave travelling in that direction. Our analysis led in the preliminary theory to results which have been later beautifully confirmed by Bär[2] and, in the general theory, has answered satisfactorily the occurrence of the coherence phenomena and the asymmetry in the intensity of the diffraction orders in the case of oblique incidence.

2. Propagation of Light in a Quasi-Homogeneous Medium.

In Parts IV and V, the wave-function governing the propagation of light in a medium was assumed to satisfy the partial differential equation

$$\nabla^2 \psi = \left[\frac{\mu(x, y, z, t)}{c}\right]^2 \frac{\partial^2\psi}{\partial t^2}$$

or $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (1)

$$\nabla^2 \psi = \frac{k}{c^2} \frac{\partial^2\psi}{\partial t^2}$$

when the frequency of the time variation of $\mu(x, y, z, t)$ is very slow compared to that of the wave-function of light. This would be so in the case of a medium filled with sound waves, for the frequency of the time variation of $\mu(x, y, z, t)$ corresponds to the frequency of the sound waves which is negligible compared to the frequency of the sound waves.

We desire to point out that the equation (1) can be derived on the basis of the electromagnetic equations if we assume that the medium is non-magnetic and transparent and that ν^*, the frequency of sound waves, is small compared to ν, the frequency of the incident light.

The Maxwell equations† for the propagation of electromagnetic waves in the medium are

$$\text{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \qquad \text{div}\,\vec{H} = 0,$$
$$\text{rot}\,\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \qquad \text{div}\,\vec{D} = 0, \qquad (2)$$

where

$$\vec{D} = k\,\vec{E}$$

† Frenkel's *Elektrodynamik*, 1928, page 236.

k being the dielectric constant of the medium as a function of x, y, z, and t. Eliminating $\vec{H}$, one obtains

$$\frac{1}{c^2}\frac{\partial^2 \vec{D}}{\partial t^2} = \nabla^2 \vec{E} + \frac{\nabla k}{k} \text{ rot } \vec{E} + \left(\frac{\nabla k}{k} \nabla\right)\vec{E} + (\vec{E}\,\nabla)\frac{\nabla k}{k} \quad .. \quad (3)$$

Since $\nu^* << \nu$

$$|\nabla k|\,\lambda << 1.$$

This reduces (3) to

$$\frac{1}{c^2}\frac{\partial^2 \vec{D}}{\partial t^2} = \nabla^2 \vec{E}$$

or

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(k\,\vec{E}) = \nabla^2 \vec{E} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4)$$

Using again the assumption that the frequency of the time variation of k is small compared to that of $\vec{E}$, the equation (4) reduces to

$$\frac{k}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \nabla^2 \vec{E} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

One can find some discussions of this equation in *Frenkel's "Electrodynamik,"* (*Zweiter Band*).

3. *Generalised Theory of the Phenomenon.*

The equation governing the propagation of light in the medium we are considering is

$$\nabla^2 \psi = \left[\frac{\mu\,(X, Y, Z, t)}{c}\right]^2 \frac{\partial^2 \psi}{\partial t^2} \quad .. \quad .. \quad .. \quad (6)$$

We choose the axes of reference such that the Z-axis points to the direction of the propagation of the incident light and the X-axis is contained in a plane perpendicular to the sound waves containing the direction of the propagation of the incident light. This choice of the axes of reference enables us to ignore the dependence of ψ on Y and write the differential equation as

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Z^2} = \left[\frac{\mu\,(X, Z, t)}{c}\right]^2 \frac{\partial^2 \psi}{\partial t^2} \quad .. \quad .. \quad .. \quad (7)$$

If $\mu\,(X, Z, t)$ did not depend on time, ψ would have had the only time factor $\exp\,[2\pi i\nu t]$. Considering the actual case where $\mu\,(X, Z, t)$ depends on time, we can write ψ as given by

$$\psi = \exp\,[2\pi i\nu t]\;\Phi\,(X, Z, t) \quad .. \quad .. \quad .. \quad .. \quad (8)$$

where Φ varies slowly in time compared to $\exp\,[2\pi i\nu t]$ for $\nu^* << \nu$. It can be seen easily that

$$\left|4\pi\nu\frac{\partial\Phi}{\partial t}\right| << |4\pi^2\nu^2\Phi| \text{ and } \left|\frac{\partial^2\Phi}{\partial t^2}\right| << |4\pi^2\nu^2\Phi|.$$

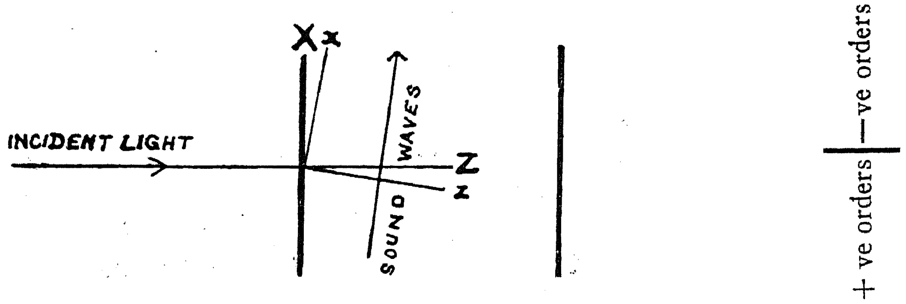
Thus we can reduce the differential equation to

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Z^2} = - \frac{4\pi^2}{\lambda^2} [\mu (X, Z, t)]^2 \Phi \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

and obtain ψ by the equation

$$\psi = \exp [2\pi i \nu t] \; \Phi.$$

The following figure gives the schematic representation of the phenomenon we are considering. The angle between the sound wave-fronts and



the direction of propagation of the incident light is ϕ. The x-axis points to the direction of propagation of the sound waves and the z-axis is contained in the plane containing the X-axis and the Z-axis. Thus $\cos \phi$ and $\sin \phi$ are the z- and the x- direction cosines of the direction of propagation of the incident light.

Even if we consider a general periodic sound disturbance with the period λ^* along the x-axis, the symmetry of the experiment with respect to the wave-fronts demands

$$\Phi (X, Z, t) = \Phi (X + p\lambda^* \sec \phi, Z, t) \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where p is an integer. This is true for, if we imagine a translation of the sound waves by a distance $p\lambda^* \sec \phi$ along the X-axis, the experimental conditions are the same as before. As the sound wave-fronts repeat themselves with the ferquency ν^*, Φ should be periodic in time with frequency ν^*, *i.e.*,

$$\Phi (X, Z, t) = \Phi \left(X, Z, t + \frac{q}{\nu^*}\right) \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where q is an integer. The conditions (10) and (11) enable us to write down the double Fourier expansion of Φ as given by

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{rs} (Z) \, e^{2\pi i r X \cos \phi / \lambda^*} \, e^{2\pi i s \nu^* t} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Progressive Sound Waves.—In the case of progressive sound waves travelling along the positive direction of the x-axis, we have the property that

$$\Phi (X + \rho\lambda^* \sec \phi, Z, t) = \Phi (X, Z, t - \rho/\nu^*) \quad \ldots \quad \ldots \quad (13)$$

where ρ is any number. This condition enables us to write (12) as

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{rs}(\mathrm{Z})\, e^{2\pi i r \mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i s\nu^* t}\, e^{2\pi i r\rho}$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{rs}(\mathrm{Z})\, e^{2\pi i r \mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i s\nu^* t}\, e^{-2\pi i s\rho} \quad \ldots \quad \ldots \quad (14)$$

Comparing the Fourier coefficients on each side of (14), we get

$$f_{rs}(\mathrm{Z})\, e^{2\pi i r\rho} = f_{rs}(\mathrm{Z})\, e^{-2\pi i s\rho} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where ρ is any number. This could only be true if

$$f_{rs}(\mathrm{Z}) = 0 \qquad \text{when } r \neq -s \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

The condition (16) restricts the number of terms in the Fourier expansion of Φ so that it can be written as given by

$$\Phi = \sum_{-\infty}^{\infty} f_r(\mathrm{Z})\, e^{2\pi i r \mathrm{X}\cos\phi/\lambda^*}\, e^{-2\pi i r\nu^* t} \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

so that ψ is given by

$$\psi = \Sigma f_r(\mathrm{Z})\, e^{2\pi i r \mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i(\nu - r\nu^*)t} \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

If one considers the diffraction effects of ψ given by (18) it will be fairly obvious that the rth order diffraction component will be inclined at an angle $\sin^{-1}(-r\lambda\cos\phi/\lambda^*)$ with the incident beam of light, will have the frequency $\nu - r\nu^*$ and the relative intensity $|f_r(\mathrm{Z})|^2$

Standing Sound Waves.—In this case

$$\Phi\left(\mathrm{X} + \frac{p\lambda^*\sec\phi}{2}, \mathrm{Z}, t\right) = \Phi\left(\mathrm{X}, \mathrm{Z}, t \pm \frac{p}{2\nu^*}\right) \quad \ldots \quad \ldots \quad (19)$$

where p is an integer. Using (19) in (12) we get

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{rs}(\mathrm{Z})\, e^{2\pi i r \mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i s\nu^* t}\, e^{\pi i r p}$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{rs}(\mathrm{Z})\, e^{2\pi i r \mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i s\nu^* t}\, e^{\pm\pi i s p} \quad \ldots \quad \ldots \quad (20)$$

Comparing the Fourier coefficients in the above, we get

$$f_{rs}(\mathrm{Z})\; e^{\pi i r p} = f_{rs}(\mathrm{Z})\, e^{\pm\pi i s p} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where p is an integer. (21) could only be true if all $f_{rs}(\mathrm{Z})$ are zero except those in which r and s are both odd or both even. So the Fourier expansion for Φ in this case is

$$\Phi = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r,2s}(\mathrm{Z})\, e^{2\pi i 2r\mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i 2s\nu^* t}$$

$$+ \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f_{2r+1,2s+1}(\mathrm{Z})\, e^{2\pi i\overline{2r+1}\mathrm{X}\cos\phi/\lambda^*}\, e^{2\pi i\overline{2s+1}\nu^* t} \quad \ldots \quad (22)$$

Thus

$$\psi = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{2r,2s}(Z)\, e^{2\pi i 2r X \cos\phi/\lambda^*}\, e^{2\pi i(\nu + 2s\nu^*)t}$$
$$+ \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{2r+1,2s+1}(Z)\, e^{2\pi i \overline{2r+1} X \cos\phi/\lambda^*}\, e^{2\pi i(\nu + \overline{2s+1}\nu^*)t} \quad \ldots \quad (23)$$

If one considers the diffraction effects of ψ given by (23), it is fairly obvious that the diffraction orders could be divided into two groups, one containing the even ones and the other odd ones; any even order would contain radiations with frequencies ν, $\nu \pm 2\nu^*$,— —, $\nu \pm 2r\nu^*$,— — and any odd order would contain radiations with frequencies $\nu \pm \nu^*$, $\nu \pm 3\nu^*$, — —, $\nu \pm \overline{2r+1}\,\nu^*$, — —. It should be remembered that the above results are valid for sound waves which are *general periodic* and either progressive as in the preceding case or standing as in the present case.

General Periodic Sound Waves.—In the case of sound waves which are neither progressive nor standing we should expect, from (12), any order to contain radiations with frequencies $\nu + r\nu^*$ where r is an integer both positive and negative. So, any two orders, *in general*, cohere partly for they contain radiations with the same wave-lengths. It is worthwhile to show that this is true by an experiment, taking care that the wave is neither progressive nor standing.

4. *The Case when the Disturbance in the Medium is Progressive and Simple Harmonic.*

If we suppose that the variation in the refractive index of the medium is simple harmonic and progressive along the x-axis, it can be represented as

$$\mu(x, t) - \mu_0 = \mu \sin 2\pi(\nu^* t - x/\lambda^*) \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where μ_0 is the constant refractive index of the medium when the sound waves are not present and μ is the amplitude of the variation of the refractive index when the sound waves are present. It can be written as

$$-\frac{\mu}{2i}\left\{e^{i(bx-\epsilon)} - e^{-i(bx-\epsilon)}\right\}$$

or

$$-\frac{\mu}{2i}\left\{e^{i(\overline{bX\cos\phi + Z\sin\phi} - \epsilon)} - e^{-i(\overline{bX\cos\phi + Z\sin\phi} - \epsilon)}\right\} \quad \ldots \quad (25)$$

where $\epsilon = 2\pi\nu^* t$ and $b = 2\pi/\lambda^*$.

We know that Φ satisfies the equation

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Z^2} = -\frac{4\pi^2}{\lambda^2}\{\mu(X, Z, t)\}^2\Phi \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

with the representation for it as

$$\sum_{-\infty}^{\infty} f_r\,(Z)\, e^{2\pi i r X \cos\phi/\lambda^*}\, e^{-2\pi i r \nu^* t} \qquad \ldots \qquad \ldots \qquad \ldots \quad (27)$$

Substituting the Fourier series (27) and the expression (25) for μ (X, Z, t) in the equation (26) and neglecting the second order term with the coefficient μ^2, we get by comparing the coefficients

$$\frac{d^2 f_r}{d\,Z^2} - \frac{4\pi^2 r^2 \cos^2\phi}{\lambda^{*2}} f_r - A f_r =$$

$$\frac{B}{2i}\left\{f_{r-1}\, e^{ibZ\sin\phi} - f_{r+1}\, e^{-ibZ\sin\phi}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \quad (28)$$

where $A = -4\pi^2\mu_0{}^2/\lambda^2$ and $B = 8\pi^2\mu_0\mu/\lambda^2$. Putting $f_r(Z) = \exp\,(-iu\mu_0 Z)\ \Phi_r(Z)$ where $u = 2\pi/\lambda$ and putting $Z = (2\pi\mu)^{-1}\lambda\xi$, we obtain

$$\mu^2 \frac{d^2\Phi_r}{d\,\xi^2} - 2i\mu_0\mu\,\frac{d\,\Phi_r}{d\xi} - \frac{r^2\lambda^2\cos^2\phi}{\lambda^{*2}}\,\Phi_r =$$

$$-\ \mu_0\mu i\,\{\Phi_{r-1}\, e^{ia\xi\sin\phi} - \Phi_{r+1}\, e^{-ia\xi\sin\phi}\} \qquad \ldots \qquad \ldots \quad (29)$$

where $a = \lambda/\mu\lambda^*$.

As μ is very small compared to μ_0, we may consider the equation

$$2\,\frac{d\Phi_r}{d\xi} - (\Phi_{r-1}\, e^{ia\xi\sin\phi} - \Phi_{r+1}\, e^{-ia\xi\sin\phi}) = \frac{ir^2\lambda^2\cos^2\phi}{\mu_0\mu\lambda^{*2}}\,\Phi_r \qquad \ldots \quad (30)$$

Denoting $\dfrac{\lambda^2\cos^2\phi}{\mu_0\mu\,\lambda^{*2}}$ by ρ and putting $\Phi_r = \Psi_r\, e^{ira\xi\sin\phi}$ $\qquad \ldots \quad (31)$

we get

$$2\,\frac{d\,\Psi_r}{d\xi} - \Psi_{r-1} + \Psi_{r+1} = i\,(r^2\rho - 2ra\sin\phi)\,\Psi_r \qquad \ldots \quad (32)$$

The boundary conditions of the problem are

$$\Psi_r\,(0) = 0, \quad r \neq 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (33)$$

and $\quad \Psi_0\,(0) = 1.$

The solution of the equation for Ψ's seems to be not quite easy in terms of the well-known functions. We have therefore attempted here to solve the equation, a little more generalised, by the series method.

5. *Solution of the Difference—Differential Equation by the Series Method.*

Consider the following equation†

$$2\,\frac{d\,\Psi_r}{dx} - \Psi_{r-1} + \Psi_{r+1} = c_r\,\Psi_r \qquad \ldots \qquad \ldots \qquad \ldots \quad (34)$$

† Throughout this section, x is used for ξ of the previous section for convenience.

where r is an integer ranging from $-\infty$ to ∞ and c_r is a constant depending only on r. The boundary conditions of the problem are

$$\Psi_r(0) = 0, \; r \neq 0 \quad \text{and} \quad \Psi_0(0) = 1 \qquad \ldots \quad (35)$$

Let us write the series representations for Ψ's as given by

$$\Psi_n = \frac{x^n}{2^n\, n!} \sum_{r=0}^{\infty} A_{n,r}\, x^r \qquad \text{for } n \geqslant 0 \quad \ldots \quad (36)$$

and

$$\Psi_n = \frac{x^{|n|}}{2^{|n|}\, |n|!} \sum_{r=0}^{\infty} A_{n,r}\, x^r \qquad \text{for } n < 0 \quad \ldots \quad (37)$$

These representations satisfy the boundary conditions (35) if

$$A_{0,0} = 1 \qquad \ldots \quad (38)$$

Case I. Let $n \geqslant 1$. Then

$$\Psi_n = \frac{x^n}{2^n n!} \sum_0^{\infty} A_{n,r}\, x^r$$

$$\frac{d\Psi_n}{dx} = \frac{1}{2^n n!} \sum_0^{\infty} (n+r)\, A_{n,r}\, x^{(n+r-1)}$$

$$\Psi_{n-1} = \frac{1}{2^{n-1}\overline{n-1}!} \sum_0^{\infty} A_{n-1,r}\, x^{n+r-1} \qquad \ldots \quad (39)$$

and

$$\Psi_{n+1} = \frac{1}{2^{n+1}(n+1)!} \sum_0^{\infty} A_{n+1,r}\, x^{n+r+1}.$$

Substituting (39) in (34) and comparing the coefficients on both sides of the equation, we get the difference equation

$$(n+r+1)\, A_{n,r+1} - n\, A_{n-1,r+1} + \frac{1}{4(n+1)} A_{n+1,r-1} = \frac{c_n}{2} A_{n,r} \qquad \ldots \quad (40)$$

where n ranges from 1 to ∞ and r ranges from 0 to ∞.

Case II. Let $n = 0$. Then

$$\frac{d\Psi_0}{dx} = \sum_0^{\infty} r\, A_{0,r}\, x^{r-1}$$

$$\Psi_{-1} = \tfrac{1}{2} \sum_0^{\infty} A_{-1,r}\, x^{r+1} \qquad \ldots \quad (41)$$

$$\Psi_1 = \tfrac{1}{2} \sum_0^{\infty} A_{1,r}\, x^{r+1}$$

Substituting (41) in (34) and comparing the coefficients, we get

$$2\,(r+1)\,A_{0,r+1} - \tfrac{1}{2}\,A_{-1,r-1} + \tfrac{1}{2}\,A_{1,r-1} = 0 \qquad \ldots \quad (42)$$

Case III.—Let $n \leqslant -1$. Then

$$\Psi_n = \frac{x^m}{2^m m!}\,\Sigma\,A_{n,r}\,x^r \text{ where } n = -m,$$

$$\frac{d\,\Psi_n}{dx} = \frac{1}{2^m m!}\,\Sigma\,(m+r)\,A_{n,r}\,x^{m+r-1}$$

$$\Psi_{n-1} = \frac{1}{2^{(m+1)}(m+1)!}\,\Sigma\,A_{n-1,r}\,x^{m+r+1} \qquad \ldots \quad (43)$$

$$\Psi_{n+1} = \frac{1}{2^{(m-1)}(m-1)!}\,\Sigma\,A_{n+1,r}\,x^{m+r-1}$$

Substituting (43) in (34), and comparing the coefficients, we get

$$(-n+r+1)\,A_{n,r+1} + \frac{1}{4(n-1)}\,A_{n-1,r-1} - n\,A_{n+1,r+1} = \frac{c_n}{2}A_{n,r} \quad (44)$$

Thus we have the following three difference equations to determine the co-efficients of the terms in the power series (36) and (37).

$$\left.\begin{array}{l} n \geqslant 1,\ (n+r+1)\,A_{n,r+1} - n\,A_{n-1,r+1} + \dfrac{1}{4(n+1)}\,A_{n+1,r-1} = \dfrac{c_n}{2}\,A_{n,r} \\ n = 0,\ 2(r+1)\,A_{0,r+1} - \tfrac{1}{2}\,A_{-1,r-1} + \tfrac{1}{2}\,A_{1,r-1} = 0 \\ n \leqslant -1,\ (-n+r+1)\,A_{n,r+1} - n\,A_{n+1,r+1} + \dfrac{1}{4(n-1)}\,A_{n-1,r-1} = \dfrac{c_n}{2}\,A_{n,r} \end{array}\right\} \ldots \quad (45)$$

Considering the first equation it can be seen that if we write c_{-n} for c_n, $(n \geqslant 1)$ and write $(-)^n\,A_{-n,r}$ for $A_{n,r}$, $(n \geqslant 1)$, we get the third equation. So it is only necessary to solve the first two equations of (45).

(A) Suppose $r = -1$. Then

$$n\,A_{n,0} - n\,A_{n-1,0} = 0,\ n \geqslant 1$$

$$A_{0,0} = 1 \qquad \text{(boundary condition)}$$

Thus

$$A_{n,0} = 1, \qquad n \geqslant 0 \qquad \ldots \quad (46)$$

and

$$A_{n,0} = (-)^n, \qquad n \leqslant -1$$

(B) Suppose $r = 0$. Then

$$(n+1)\,A_{n,1} - n\,A_{n-1,1} = \frac{c_n}{2}\,A_{n,0}, \qquad n \geqslant 1,$$

$$2\,A_{0,1} = 0.$$

(i) Let $n = 1$. Then

$$2\,A_{1,1} - A_{0,1} = \frac{c_1}{2}\,A_{1,0} = \frac{c_1}{2},$$

$$\therefore \quad A_{1,1} = c_1/4.$$

(ii) Let $n = 2$. Then

$$3\,A_{2,1} - 2\,A_{1,1} = \frac{c_2}{2}\,A_{2,0} = \frac{c_2}{2},$$

$$3\,A_{2,1} = \frac{c_2}{2} + 2\,A_{1,1}$$

$$= \frac{c_1 + c_2}{2}.$$

$$\therefore \quad A_{2,1} = \frac{c_1 + c_2}{6}.$$

(iii) Let $n = 3$. Then

$$4\,A_{3,1} - 3\,A_{2,1} = \frac{c_3}{2}.$$

$$\therefore \quad A_{3,1} = \frac{c_1 + c_2 + c_3}{8}.$$

(iv) Substituting $n = 4, 5$, we get $A_{4,1}$ and $A_{5,1}$.

Thus

$$A_{0,1} = 0$$

$$\left.\begin{array}{ll}
A_{1,1} = \dfrac{c_1}{2\cdot 2}, & A_{-1,1} = -\dfrac{c_{-1}}{2\cdot 2}, \\[2ex]
A_{2,1} = \dfrac{c_1 + c_2}{2\cdot 3}, & A_{-2,1} = \dfrac{c_{-1} + c_{-2}}{2\cdot 3}, \\[2ex]
A_{3,1} = \dfrac{c_1 + c_2 + c_3}{2\cdot 4}, & A_{-3,1} = -\dfrac{c_{-1} + c_{-2} + c_{-3}}{2\cdot 4}, \\[2ex]
A_{4,1} = \dfrac{c_1 + c_2 + c_3 + c_4}{2\cdot 5}, & A_{-4,1} = \dfrac{c_{-1} + c_{-2} + c_{-3} + c_{-4}}{2\cdot 5}, \\[2ex]
A_{5,1} = \dfrac{c_1 + c_2 + c_3 + c_4 + c_5}{2\cdot 6}, & A_{-5,1} = -\dfrac{c_{-1} + c_{-2} + c_{-3} + c_{-4} + c_{-5}}{2\cdot 6},
\end{array}\right\} \quad \ldots \quad (47)$$

(B) We will now calculate the coefficients $A_{n,2}$.

For this purpose we put $r = 1$ in (45) and obtain

$$(n + 2)\,A_{n,2} - n\,A_{n-1,2} + \frac{1}{4(n+1)}\,A_{n+1,0} = \frac{c_n}{2}\,A_{n,1} \quad \ldots \quad (48)$$

for $n \geqslant 1$ and

$$4\,A_{0,2} - \tfrac{1}{2}\,A_{-1,0} + \tfrac{1}{2}\,A_{1,0} = 0$$

(i) From (48) and using (46), we get

$$4\,A_{0,2} = -\tfrac{1}{2}\,A_{1,0} + \tfrac{1}{2}\,A_{-1,0} = -1$$

Thus $A_{0,2} = -\frac{1}{4}$.

(ii) Let $n = 1$. Then

$$3\, A_{1,2} - A_{0,2} + \frac{1}{4\cdot 2} A_{2,0} = \frac{c_1}{2} A_{1,1},$$

$$3\, A_{1,2} = -\frac{1}{4\cdot 2} - \tfrac{1}{4} + \frac{c_1^2}{2\cdot 2\cdot 2}$$

$$A_{1,2} = -\frac{1}{4\cdot 2} + \frac{c_1^2}{2\cdot 2\cdot 2\cdot 3}$$

(iii) Substituting $n = 2, 3, 4$ successively in (48), we get

$$A_{2,2} = -\frac{1}{4\cdot 3} + \frac{c_1^2 + c_1c_2 + c_2^2}{2\cdot 2\cdot 3\cdot 4}$$

$$A_{3,2} = -\frac{1}{4\cdot 4} + \frac{c_1^2 + c_2^2 + c_3^2 + c_1c_2 + c_2c_3 + c_3c_1}{2\cdot 2\cdot 4\cdot 5}$$

$$A_{4,2} = -\frac{1}{4\cdot 5} + \frac{c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_1c_2 + c_2c_3 + c_3c_1 + c_1c_4 + c_2c_4 + c_3c_4}{2\cdot 2\cdot 5\cdot 6}$$

Thus

$$A_{-1,2} = \frac{1}{4\cdot 2} - \frac{c_{-1}^2}{2\cdot 2\cdot 2\cdot 3}$$

$$A_{-2,2} = -\frac{1}{4\cdot 3} + \frac{c_{-1}^2 + c_{-1}c_{-2} + c_{-2}^2}{2\cdot 2\cdot 3\cdot 4}$$

$$A_{-3,2} = \frac{1}{4\cdot 4} - \frac{c_{-1}^2 + c_{-2}^2 + c_{-3}^2 + c_{-1}c_{-2} + c_{-2}c_{-3} + c_{-3}c_{-1}}{2\cdot 2\cdot 4\cdot 5}$$

$$A_{-4,2} = -\frac{1}{4\cdot 5} + \frac{c_{-1}^2 + c_{-2}^2 + c_{-3}^2 + c_{-4}^2 + c_{-1}c_{-2} + c_{-2}c_{-3} + c_{-3}c_{-1} + c_{-1}c_{-4} + c_{-2}c_{-4} + c_{-3}c_{-4}}{2\cdot 2\cdot 5\cdot 6}$$

(C) We will now obtain the coefficients $A_{n,3}$ by putting $r = 2$ in (45). We get

$$(n + 3)\, A_{n,3} - n\, A_{n-1,3} + \frac{1}{4(n+1)} A_{n+1,1} = \frac{c_n}{2} A_{n,2} \quad \ldots \quad (49)$$

for $n \geqslant 1$ and

$$6\, A_{0,3} - \tfrac{1}{2} A_{-1,1} + \tfrac{1}{2} A_{1,1} = 0$$

(i) From (49), we get

$$A_{0,3} = -\frac{c_1 + c_{-1}}{48}.$$

(ii) Putting $n = 1, 2, 3$ in (49) successively, we get

$$A_{1,3} = \frac{1}{192}\{c_1^3 - (c_{-1} + 5\, c_1 + c_2)\}$$

$$A_{2,3} = \frac{1}{480}\{(c_1^3 + c_2^3 + c_1^2c_2 + c_1c_2^2) - (c_{-1} + 6c_1 + 6c_2 + c_3)\}$$

$$A_{3,3} = \frac{1}{960}\{(c_1^3 + c_2^3 + c_3^3 + c_1^2c_2 + c_1c_2^2 + c_2^2c_3 + c_2c_3^2 + c_3^2c_1 + c_3c_1^2 + c_1c_2c_3) - (c_{-1} + 7c_1 + 7c_2 + 7c_3 + c_4)\}$$

$$A_{4,3} = \frac{1}{1680}\{(c_1^3 + c_2^3 + c_3^3 + c_4^3 + c_1^2c_2 + c_1c_2^2 + c_1^2c_3 + c_1c_3^2 + c_1^2c_4 + c_1c_4^2 + c_2^2c_3 + c_2c_3^2 + c_2^2c_4 + c_2c_4^2 + c_3^2c_4 + c_3c_4^2 + c_1c_2c_3 + c_1c_2c_4 + c_4c_1c_3 + c_4c_2c_3) - (c_{-1} + 8c_1 + 8c_2 + 8c_3 + 8c_4 + c_5)\}.$$

The coefficients $A_{-1,3}$, $A_{-2,3}$, $A_{-3,3}$ and $A_{-4,3}$ are obtained from the above by changing c_n to c_{-n} and $A_{n,r}$ to $(-)^n A_{-n,r}$.

(D) To calculate the coefficients $A_{n,4}$, we put $r = 3$ in (45) and get

$$(n+4)\,A_{n,4} - n\,A_{n-1,4} + \frac{1}{4(n+1)}\,A_{n+1,2} = \frac{c_n}{2}\,A_{n,3} \qquad .. \quad (50)$$

for $n \geqslant 1$ and

$$8\,A_{0,4} - \tfrac{1}{2}\,A_{-1,2} + \tfrac{1}{2}\,A_{1,2} = 0$$

Following the same procedure as in the above we get

$$A_{0,4} = -\frac{1}{384}(c_1^2 + c_{-1}^2) + \frac{1}{64}$$

$$A_{1,4} = \frac{1}{1920}\{c_1^4 - (c_{-1}^2 + 7c_1^2 + c_2^2 + 2c_1c_2 + c_1c_{-1})\} + \frac{1}{192}$$

$$A_{2,4} = \frac{1}{5760}\{(c_1^4 + c_2^4 + c_1^3c_2 + c_1c_2^3 + c_1^2c_2^2) - (c_{-1}^2 + 8c_1^2 + 8c_2^2 + c_3^2 + 9c_1c_2 + c_1c_{-1} + c_1c_3 + 2c_2c_3 + c_2c_{-1})\} + \frac{1}{384}$$

$$A_{3,4} = \frac{1}{13440}\{(c_1^4 + c_2^4 + c_3^4 + c_1^2c_2^2 + --- + c_1^3c_2 + --- + c_1c_2c_3^2 + --) - (c_{-1}^2 + 9c_1^2 + 9c_2^2 + 9c_3^2 + c_4^2 + 10c_1c_2 + 10c_2c_3 + 9c_1c_3 + c_1c_{-1} + c_2c_{-1} + c_1c_4 + c_2c_4 + 2c_3c_4 + c_3c_{-1})\} + \frac{1}{640}$$

Similarly we have found

$$A_{0,5} = \frac{1}{3840}\{-(c_1^3 + c_{-1}^3) + 6c_1 + 6c_{-1} + c_{-2} + c_2\}$$

$$A_{0,6} = \frac{1}{46080} \times \{-(c_1^4 + c_{-1}^4) + (8c_1^2 + 8c_{-1}^2 + c_2^2 + c_{-2}^2 + 2c_1c_2 + 2c_{-1}c_{-2} + 2c_1c_{-1})\} - \frac{1}{2304}$$

So we can write the following series for Ψ's.

$$\Psi_0 = 1 - \frac{x^2}{4} - \frac{1}{48}(c_1 + c_{-1})\, x^3 + \frac{1}{64}\left(1 - \frac{c_1^2 + c_{-1}^2}{6}\right) x^4 + \frac{1}{3840}\{-(c_1^3 + c_{-1}^3) + 6c_1 + 6c_{-1} + c_{-2} + c_2\}\, x^5$$
$$+ \frac{x^6}{2304}\left\{-1 - \frac{c_1^4 + c_{-1}^4}{20} + \frac{8c_1^2 + 8c_{-1}^2 + c_2^2 + c_{-2}^2 + 2c_1c_2 + 2c_{-1}c_{-2} + 2c_1c_{-1}}{20}\right\} + ---$$

$$\Psi_1 = \frac{x}{2}\left[1 + \frac{c_1}{4}x + \tfrac{1}{8}\left(\frac{c_1^2}{3} - 1\right)x^2 + \frac{1}{192}(c_1^3 - c_{-1} - 5c_1 - c_2)\, x^3 + \frac{1}{1920}(c_1^4 - \{c_{-1}^2 + 7c_1^2 + c_2^2 + 2c_1c_2 + c_1c_{-1}\} + 10)\, x^4 + --\right]$$

$$\Psi_{-1} = -\frac{x}{2}\left[1 + \frac{c_{-1}}{4}x + \tfrac{1}{8}\left(\frac{c_{-1}^2}{3} - 1\right)x^2 + \frac{1}{192}(c_{-1}^3 - c_1 - 5c_{-1} - c_{-2})\, x^3 + \frac{1}{1920}(c_{-1}^4 - \{c_1^2 + 7c_{-1}^2 + c_{-2}^2 + 2c_{-1}c_{-2} + c_{-1}c_1\} + 10)\, x^4 + --\right]$$

$$\Psi_2 = \frac{x^2}{8}\left\{1 + \frac{c_1 + c_2}{6}x + \frac{1}{12}\left(-1 + \frac{c_1^2 + c_1c_2 + c_2^2}{4}\right)x^2 + --\right\} \quad \ldots (51)$$

$$\Psi_{-2} = \frac{x^2}{8}\left\{1 + \frac{c_{-1} + c_{-2}}{6}x + \frac{1}{12}\left(-1 + \frac{c_{-1}^2 + c_{-1}c_{-2} + c_{-2}^2}{4}\right)x^2 + --\right\}$$

$$\Psi_3 = \frac{x^3}{48}\left\{1 + \frac{c_1 + c_2 + c_3}{8}x + --\right\}$$

$$\Psi_{-3} = -\frac{x^3}{48}\left\{1 + \frac{c_{-1} + c_{-2} + c_{-3}}{8}x + --\right\}$$

We will now apply these calculations to obtain the intensity expressions in the theory of the phenomenon we are concerned.

6. *Applications of the above Calculations to the Theory of the Phenomenon.*

In the problem we are concerned, x has to be written as ξ and c's will have the following special values as given by (32).

$$\left.\begin{array}{ll} c_1 = i(\rho - 2a\sin\phi)\,; & c_{-1} = i(\rho + 2a\sin\phi) \\ c_2 = 4i(\rho - a\sin\phi)\,; & c_{-2} = 4i(\rho + a\sin\phi) \\ c_3 = 3i(3\rho - 2a\sin\phi)\,; & c_{-3} = 3i(3\rho + 2a\sin\phi) \end{array}\right\} \quad \ldots (52)$$

where

$$\rho = \frac{\lambda^2 \cos^2\phi}{\mu_0\mu\, \lambda^{*2}}$$

and

$$a = \frac{\lambda}{\mu \, \lambda^*}.$$

Let us write $a \sin \phi$ as $\alpha\rho$. Then

$$\alpha = \frac{\mu_0 \, \lambda^*}{\lambda} \tan \phi \quad \ldots \quad (53)$$

Thus

$$\left.\begin{aligned} c_1 &= i\rho\,(1 - 2\alpha)\,; & c_{-1} &= i\rho\,(1 + 2\alpha) \\ c_2 &= 4i\rho\,(1 - \alpha)\,; & c_{-2} &= 4i\rho\,(1 + \alpha) \\ c_3 &= 3i\rho\,(3 - 2\alpha)\,; & c_{-3} &= 3i\rho\,(3 + 2\alpha) \end{aligned}\right\} \quad \ldots \quad (54)$$

Substituting the above values of c's in (51), we get for the amplitude functions

$$\begin{aligned} \Psi_0 = 1 - \frac{\xi^2}{4} - \frac{i\rho}{24}\xi^3 + \frac{1}{64}\left[1 + \frac{(1 + 4\,\alpha^2)\,\rho^2}{3}\right]\xi^4 \\ + \frac{i}{1920}\{10\rho + \rho^3\,(1 + 12\alpha^2)\}\,\xi^5 - \left\{\frac{1}{2304} + \frac{\rho^2\,(11 + 20\alpha^2)}{7680}\right. \\ \left. + \frac{\rho^4\,(1 + 24\alpha^2 + 16\alpha^4)}{23040}\right\}\xi^6 + - - - \end{aligned}$$

$$\begin{aligned} \Psi_1 = \frac{\xi}{2}\Bigg[1 + \frac{i\rho\,(1 - 2\alpha)}{4}\,\xi - \frac{\xi^2}{8}\left(1 + \frac{\rho^2\,(1 - 2\alpha)^2}{3}\right) \\ + \frac{i\,\xi^3}{192}\{-\,\rho^3\,(1 - 2\alpha)^3 - 10\rho + 12\alpha\rho\} \\ + \xi^4\left\{\frac{\rho^4\,(1 - 2\alpha)^4}{1920} + \frac{\rho^2\,(33 - 80\alpha + 60\alpha^2)}{1920} + \frac{1}{192}\right\} \\ + - - - \Bigg] \end{aligned}$$

$$\begin{aligned} \Psi_{-1} = -\frac{\xi}{2}\Bigg[1 + \frac{i\rho\,(1 + 2\alpha)}{4}\,\xi - \frac{\xi^2}{8}\left(1 + \frac{\rho^2\,(1 + 2\alpha)^2}{3}\right) \\ + \frac{i\xi^3}{192}\{-\,\rho^3\,(1 + 2\alpha)^3 - 10\rho - 12\alpha\rho\} \\ + \xi^4\left\{\frac{\rho^4\,(1 + 2\alpha)^4}{1920} + \frac{\rho^2\,(33 + 80\alpha + 60\alpha^2)}{1920} + \frac{1}{192}\right\} \\ + - - - \Bigg] \quad \ldots \quad (55) \end{aligned}$$

$$\Psi_2 = \frac{\xi^2}{8}\left[1 + \frac{i\rho\,(5 - 6\alpha)}{6}\,\xi - \frac{1}{12}\left(1 + \frac{\rho^2\,(21 - 48\alpha + 28\alpha^2)}{4}\right)\xi^2 + - - \right]$$

$$\Psi_{-2} = \frac{\xi^2}{8}\left[1 + \frac{i\rho\,(5 + 6\alpha)}{6}\,\xi - \frac{1}{12}\left(1 + \frac{\rho^2\,(21 + 48\alpha + 28\alpha^2)}{4}\right)\xi^2 + - - \right]$$

The relative intensity expressions for the various diffraction orders can be found by finding the series for $\Psi_r \Psi_r^\dagger$ where † denotes the conjugate expression. Let the intensity of the rth order be denoted by I_r. Then

$$I_0 = |\Psi_0|^2 = 1 - \frac{\xi^2}{2} + \frac{\xi^4}{32}\left(3 + \frac{\rho^2(1+4a^2)}{3}\right) - \xi^6\left[\frac{5}{576} + \frac{\rho^4(1+24a^2+16a^4)}{11520} + \frac{7\rho^2}{1280} + \frac{\rho^2 a^2}{64} - \frac{\rho^2}{576}\right] + - - -$$

$$I_1 = |\Psi_1|^2 = \frac{\xi^2}{4} - \frac{\xi^4}{16}\left[1 + \frac{\rho^2(1-2a)^2}{12}\right] + \frac{\xi^6}{192}\left[\frac{5}{4} + \frac{\rho^2(9-20a+20a^2)}{10} + \frac{\rho^4(1-2a)^4}{120}\right] + - -$$

$$I_{-1} = |\Psi_{-1}|^2 = \frac{\xi^2}{4} - \frac{\xi^4}{16}\left[1 + \frac{\rho^2(1+2a)^2}{12}\right] + \frac{\xi^6}{192}\left[\frac{5}{4} + \frac{\rho^2(9+20a+20a^2)}{10} + \frac{\rho^4(1+2a)^4}{120}\right] + - - \quad (56)$$

$$I_2 = \frac{\xi^4}{64} - \frac{\xi^6}{384}\left[1 + \frac{\rho^2}{12}(12a^2 - 24a + 13)\right] + - - -$$

$$I_{-2} = \frac{\xi^4}{64} - \frac{\xi^6}{384}\left[1 + \frac{\rho^2}{12}(12a^2 + 24a + 13)\right] + - - -$$

$$I_3 = \frac{\xi^6}{2304} + - - -$$

One can now clearly see the asymmetry between I_1, and I_{-1} and I_2 and I_{-2}. The asymmetry between I_3 and I_{-3} consists in the higher terms than the one term written.†

7. *Discussion of Experimental Data.*

(*a*) *The case when ρ is negligible and a is zero.*—The various I's are the squares of the Bessel functions if $\rho = 0$. This case corresponds to that treated in Part I. Thus the restrictions in Part I amount to ρ being small. This could be so when the wave-length of sound is so large that ρ becomes a magnitude which has not much influence in the intensity expressions. This case has been achieved experimentally by Bär who finds perfect quantitative agreement with the theory.

(*b*) *The case when ρ is not negligible.*—When a is not zero (*i.e.*, oblique incidence) the theory shows that there will be no symmetry in the diffraction pattern, *i.e.*, the intensity of the rth order will not be equal to that of the $-r$th order. This fact is in agreement wit the experimental results of Debye and Sears, Lucas and Biquard, Bär and Parthasarathy.

† One may also verify from the above that
$I_0 + I_1 + I_{-1} + I_2 + I_{-2} + I_3 + I_{-3} + - - = 1.$

Parthasarathy has found that in the case of oblique incidence the intensity is distributed more towards that side which favours the 'reflection' of the incident light if the sound waves had acted as mirrors. Theoretically the expressions obtained in (56) seem to show that this would be so if we restrict our attention to the *first two terms* in each of the intensity expressions. If we make such a restriction, the expressions (56) show that the intensity of any positive order is greater than the corresponding negative order. The positive orders are situated towards that side which favour reflection of the incident light if the supersonic waves had acted as mirrors. However, it is certainly very necessary to carry our calculations of the intensity expressions for a greater number of terms and then alone we can definitely say which side would be more intense under any definite experimental conditions.

Recently, Parthasarathy made an investigation of the diffraction phenomenon by keeping all the parameters constant except the angle of incidence of light to the sound waves which he changed continuously. He has recorded that the first order attains its maximum intensity when the incident angle to the sound waves corresponds nearly to the Bragg reflection angle which is $\lambda/2\lambda^*$. Similarly he has recorded that the intensity of the second order attains its maximum when the incident angle is about λ/λ^*. If we restrict our attention *to the first two terms* of the intensity expressions (56), it is easy to see that the first order would attain maximum intensity when $a = \frac{1}{2}$ or $\phi = \lambda/2\mu_0\lambda^*$. Also it can be seen that the second order attains its maximum intensity when $\phi = \lambda/\mu_0\lambda^*$. It is to be remembered that μ_0 is the refractive index of the medium. If the path of the incident light is not normal to the face of the cell, then the maximum intensity of the first order will appear when the angle between the path of the incident light (which travels in air) and the sound wave-fronts is $\lambda/2\lambda^*$ and the maximum intensity for the second order will appear when the angle between the path of the incident light and the sound waves is λ/λ^*.

It may seem certainly very necessary to further the calculations of the intensity expressions to higher terms and see whether there are actually maxima of the intensity of the various orders at definite angles as is claimed. *The intensity expression obtained here for the first order itself shows that it may be so.* If however the first order would attain maximum intensity at a definite angle independent of the other parameters ξ and ρ, then there should be a solution for the equation $\frac{dI_1}{da} = 0$ independent of ξ and ρ. If $\frac{dI_1}{da} = 0$,*

* ρ has the factor $\text{Cos}^2\,\phi$ and so it depends on ϕ. But if ϕ is to vary in a small range, the dependence can be ignored. ϕ is really very small in experiments.

$$- \frac{\xi^4\rho^2}{16 \times 12} 2(1 - 2\alpha) \times -2 + \frac{\xi^6}{4} \left\{ \frac{4\rho^4 (1-2\alpha)^3 \times -2}{5760} \right.$$

$$\left. + \frac{\rho^2(-20 + 40\alpha)}{480} \right\} + - - = 0 \quad .. \quad (57)$$

Equating the coefficients of the independent terms to 0 we get

$$\alpha = \tfrac{1}{2}$$

$$\text{or} \quad \phi = \lambda / 2\mu_0 \lambda^* \quad .. \quad .. \quad .. \quad .. \quad .. \quad (58)$$

The condition $\frac{dI_1}{d\alpha} = 0$ is only a necessary condition for the occurrence of the maximum. We should then see for a maximum that $\frac{d^2I_1}{d\alpha^2}$ is less than zero. These considerations point out that the maximum of the first order may appear when $\alpha = \frac{1}{2}$, possibly under some conditions.

If we consider only the first two terms for the intensity expression of I_2, it can be seen that I_2 becomes maximum when

$$\alpha = 1$$

$$\text{or} \quad \phi = \lambda / \mu_0 \lambda^* \quad .. \quad .. \quad .. \quad .. \quad .. \quad (59)$$

We are fully aware that the further development of the intensity expressions is greatly needed to interpret the experimental results, but the series method which has been tried here to evaluate them leads to the following result in the theory. *This work seems to clearly show that the maxima of the orders may appear at unique angles, possibly under some conditions.*

8. *Expressions for Amplitudes in the Case of Normal Incidence.*

If we put $\alpha = 0$, in the fourth section, we can obtain the series for the amplitude functions in the case of normal incidence. But we have worked out this case separately and we give in the following our final results without giving the details of calculation as they are similar to those in the case of oblique incidence.

$$\Psi_0 = 1 - \frac{\xi^2}{4} - \frac{i\rho\xi^3}{24} + \frac{1}{64}\left(1 + \frac{\rho^2}{3}\right)\xi^4$$

$$+ \frac{i\rho}{192}\left(1 + \frac{\rho^2}{10}\right)\xi^5 - \left(\frac{\rho^4}{23040} + \frac{11\rho^2}{7680} + \frac{1}{2304}\right)\xi^6$$

$$- i\left(\frac{\rho^5}{322560} + \frac{\rho^3}{2688} + \frac{\rho}{4608}\right)\xi^7$$

$$+ \left(\frac{\rho^6}{5160960} + \frac{463\rho^4}{5160960} + \frac{17\rho^2}{184320} + \frac{1}{147456}\right)\xi^8 + - -$$

$$\begin{aligned}
\Psi_1 = \frac{\xi}{2}\Big[1 &+ \frac{i\rho}{4}\xi - \frac{1}{24}(\rho^2+3)\xi^2 - \frac{i\rho}{192}(\rho^2+10)\xi^3 \\
&+ \left(\frac{\rho^4}{1920} + \frac{11\rho^2}{640} + \frac{1}{192}\right)\xi^4 + i\left(\frac{\rho^5}{23040} + \frac{\rho^3}{192} + \frac{7\rho}{2304}\right)\xi^5 \\
&- \left(\frac{\rho^6}{322560} + \frac{463\rho^4}{322560} + \frac{17\rho^2}{11520} + \frac{1}{9216}\right)\xi^6 \\
&- i\left(\frac{\rho^7}{5160960} + \frac{61\rho^5}{172032} + \frac{901\rho^3}{1290240} + \frac{\rho}{12288}\right)\xi^7 \\
&+ \left(\frac{\rho^8}{92897280} + \frac{2431\rho^6}{30965760} + \frac{14779\rho^4}{46448640} + \frac{115\rho^2}{2211840} + \frac{1}{737280}\right)\xi^8 \\
&+ - - \Big]
\end{aligned}$$

$$\begin{aligned}
\Psi_2 = \frac{\xi^2}{8}\Big[1 &+ \frac{5i\rho}{6}\xi - \tfrac{1}{4}\left(\frac{7\rho^2}{4} + \tfrac{1}{3}\right)\xi^2 - i\left(\frac{17\rho^3}{96} + \frac{\rho}{12}\right)\xi^3 \\
&+ \frac{341\rho^4}{5760} + \frac{17\rho^2}{288} + \frac{1}{384}\Big)\xi^4 \\
&+ i\left(\frac{13\rho^5}{768} + \frac{361\rho^3}{10080} + \frac{7\rho}{2304}\right)\xi^5 \\
&- \left(\frac{5461\rho^6}{1290240} + \frac{6257\rho^4}{322560} + \frac{247\rho^2}{92160} + \frac{1}{23040}\right)\xi^6 - - - \Big]
\end{aligned}$$

$$\begin{aligned}
\Psi_3 = \frac{\xi^3}{48}\Big[1 &+ \frac{7i\rho}{4}\xi - \frac{1}{80}(147\rho^2+5)\xi^2 - i\left(\frac{22}{15}\rho^3 + \frac{23\rho}{192}\right)\xi^3 \\
&+ \left(\frac{1859\rho^4}{1920} + \frac{143\rho^2}{960} + \frac{1}{640}\right)\xi^4 \\
&+ i\left(\frac{2821\rho^5}{5120} + \frac{5391\rho^3}{35840} + \frac{5\rho}{1536}\right)\xi^5 + - - \Big]
\end{aligned}$$

$$\begin{aligned}
\Psi_4 = \frac{\xi^4}{384}\Big[1 &+ 3i\rho\xi - \left(\frac{209\rho^2}{40} + \frac{1}{20}\right)\xi^2 - i\left(\frac{143\rho^3}{21} + \frac{19\rho}{20}\right)\xi^3 \\
&+ \left(\frac{65351\rho^4}{8960} + \frac{71\rho^2}{448} + \frac{1}{960}\right)\xi^4 + i\left(\frac{108511\rho^5}{16128} + \frac{40499\rho^3}{120960} + \frac{\rho}{288}\right)\xi^5 \\
&+ - - \Big] \qquad (60)
\end{aligned}$$

$$\begin{aligned}
\Psi_5 = \frac{\xi^5}{3840}\Big[1 &+ \frac{55i\rho\xi}{12} - \left(\frac{143\rho^2}{12} + \frac{1}{24}\right)\xi^2 - i\left(\frac{30745\rho^3}{1344} + \frac{19\rho}{96}\right)\xi^3 \\
&+ \left(\frac{174889\rho^4}{24192} + \frac{11419\rho^2}{24192} + \frac{1}{1344}\right)\xi^4 + - - \Big]
\end{aligned}$$

$$\begin{aligned}
\Psi_6 = \frac{\xi^6}{46080}\Big[1 &+ \frac{13i\rho\xi}{2} - \left(\frac{2639\rho^2}{112} + \frac{1}{28}\right)\xi^2 - i\left(\frac{125749\rho^3}{2016} + \frac{5\rho}{21}\right)\xi^3 \\
&+ \left(\frac{125749\rho^4}{1120} + \frac{34768\rho^2}{40320} + \frac{1}{1792}\right)\xi^4 + - - \Big]
\end{aligned}$$

$$\Psi_1 = \frac{\xi}{2}\Big[1 + \frac{i\rho}{4}\xi - \frac{1}{24}(\rho^2+3)\xi^2 - \frac{i\rho}{192}(\rho^2+10)\xi^3$$
$$+\left(\frac{\rho^4}{1920}+\frac{11\rho^2}{640}+\frac{1}{192}\right)\xi^4 + i\left(\frac{\rho^5}{23040}+\frac{\rho^3}{192}+\frac{7\rho}{2304}\right)\xi^5$$
$$-\left(\frac{\rho^6}{322560}+\frac{463\rho^4}{322560}+\frac{17\rho^2}{11520}+\frac{1}{9216}\right)\xi^6$$
$$-i\left(\frac{\rho^7}{5160960}+\frac{61\rho^5}{172032}+\frac{901\rho^3}{1290240}+\frac{\rho}{12288}\right)\xi^7$$
$$+\left(\frac{\rho^8}{92897280}+\frac{2431\rho^6}{30965760}+\frac{14779\rho^4}{46448640}+\frac{115\rho^2}{2211840}+\frac{1}{737280}\right)\xi^8$$
$$+ - - \Big]$$

$$\Psi_2 = \frac{\xi^2}{8}\Big[1 + \frac{5i\rho}{6}\xi - \tfrac{1}{4}\left(\frac{7\rho^2}{4}+\tfrac{1}{3}\right)\xi^2 - i\left(\frac{17\rho^3}{96}+\frac{\rho}{12}\right)\xi^3$$
$$+\frac{341\rho^4}{5760}+\frac{17\rho^2}{288}+\frac{1}{384}\Big)\xi^4$$
$$+i\left(\frac{13\rho^5}{768}+\frac{361\rho^3}{10080}+\frac{7\rho}{2304}\right)\xi^5$$
$$-\left(\frac{5461\rho^6}{1290240}+\frac{6257\rho^4}{322560}+\frac{247\rho^2}{92160}+\frac{1}{23040}\right)\xi^6 - - - \Big]$$

$$\Psi_3 = \frac{\xi^3}{48}\Big[1 + \frac{7i\rho}{4}\xi - \frac{1}{80}(147\rho^2+5)\xi^2 - i\left(\frac{22}{15}\rho^3+\frac{23\rho}{192}\right)\xi^3$$
$$+\left(\frac{1859\rho^4}{1920}+\frac{143\rho^2}{960}+\frac{1}{640}\right)\xi^4$$
$$+i\left(\frac{2821\rho^5}{5120}+\frac{5391\rho^3}{35840}+\frac{5\rho}{1536}\right)\xi^5 + - - \Big]$$

$$\Psi_4 = \frac{\xi^4}{384}\Big[1 + 3i\rho\xi - \left(\frac{209\rho^2}{40}+\frac{1}{20}\right)\xi^2 - i\left(\frac{143\rho^3}{21}+\frac{19\rho}{20}\right)\xi^3$$
$$+\left(\frac{65351\rho^4}{8960}+\frac{71\rho^2}{448}+\frac{1}{960}\right)\xi^4 + i\left(\frac{108511\rho^5}{16128}+\frac{40499\rho^3}{120960}+\frac{\rho}{288}\right)\xi^5$$
$$+ - - \Big] \qquad (60)$$

$$\Psi_5 = \frac{\xi^5}{3840}\Big[1 + \frac{55i\rho\xi}{12} - \left(\frac{143\rho^2}{12}+\frac{1}{24}\right)\xi^2 - i\left(\frac{30745\rho^3}{1344}+\frac{19\rho}{96}\right)\xi^3$$
$$+\left(\frac{174889\rho^4}{24192}+\frac{11419\rho^2}{24192}+\frac{1}{1344}\right)\xi^4 + - - \Big]$$

$$\Psi_6 = \frac{\xi^6}{46080}\Big[1 + \frac{13i\rho\xi}{2} - \left(\frac{2639\rho^2}{112}+\frac{1}{28}\right)\xi^2 - i\left(\frac{125749\rho^3}{2016}+\frac{5\rho}{21}\right)\xi^3$$
$$+\left(\frac{125749\rho^4}{1120}+\frac{34768\rho^2}{40320}+\frac{1}{1792}\right)\xi^4 + - - \Big]$$

$$\Psi_7 = \frac{\xi^7}{645120}\left[1 + \frac{35i\rho\xi}{4} - \left(\frac{6069\rho^2}{144} + \frac{1}{32}\right)\xi^2 + - -\right]$$

The above series for the Ψ's are rather slowly convergent and it is rather difficult to find their values for any pair of values ρ and ξ.

If we put $\rho = 0$ in the above, the series then represent the well-known expansions of the Bessel Functions. In this exterme case we can determine the relative values of the amplitudes simply as a function of ξ. But when ρ is not zero the calculation is not simple. If we know μ, the amplitude of the fluctuation of the refractive index, we can determine the values of ξ and ρ by the following relations

$$\xi = \frac{2\pi\mu Z}{\lambda},$$

$$\rho = \frac{\lambda^2}{\mu_0\mu\ \lambda^{*2}}.$$

Only when the series given above converge rapidly for determined ξ and ρ, they will be of use for practical work.

9. A Possible Method of Determining the Amplitude of the Fluctuation of the Refractive Index at High Supersonic Frequencies.

The method outlined in the following is an indirect one and applicable only under heavy restrictions.

(1) The supersonic frequency should be high.

(2) The length of the cell should be decreased to such an extent that there should be good reason to believe that $\frac{2\pi\mu L}{\lambda}$ is small.

The diffraction pattern in the case of normal incidence has to be studied first and the relative intensity of the central order to the first (or *minus* first) order is to be determined. Let it be Δ_1. Then the cell has to be rotated to such an extent that the first order attains its maximum intensity. Experimental conditions ought to be such that the obliquity will be nearly $\lambda/2\lambda^*$. Let now the ratio of the first order to the *minus* first order be Δ_2. If L denotes the length of the cell, $\xi = \frac{2\pi\mu L}{\lambda}$. Then

$$\frac{1 - \frac{\xi^2}{2} + \frac{\xi^4}{32}\left(3 + \frac{\rho^2}{3}\right) - -}{\frac{\xi^2}{4} - \frac{\xi^4}{16}\left[1 + \frac{\rho^2}{12}\right] - -} = \Delta_1$$

$$\frac{\frac{\xi^2}{4} - \frac{\xi^4}{16} + - -}{\frac{\xi^2}{4} - \frac{\xi^4}{16}\left(1 + \frac{\rho^2}{3}\right)} = \Delta_2 \qquad \text{(61)}$$

Since $\rho\xi = 2\pi\lambda L/\mu_0 \lambda^{*2}$, the vaules of ρ and ξ can be determined by each of the above equations. First of all the two sets of values are to be consistent. If so, one can then determine μ, the amplitude of the fluctuation of the refractive index by knowing the formulæ for ρ and ξ.

The restrictions in the above method are *rather very heavy* so that it may be difficult to arrange the experimental conditions in the required manner. The method also depends on the assumption that the *supersonic wave in the medium is a simple harmonic one. Therefore the above is only offered as a suggestion to be tested out experimentally.*

10. Summary.

A general review of the theory of the diffraction of light by high frequency sound waves developed in these *Proceedings* by Raman and Nath is presented. The solution of the difference—differential equation due to them given in Parts IV and V of their papers is attempted here by the series method. The results obtained in this paper offer an explanation for the experimental results due to Bär and Parthasarathy who have studied the phenomenon at oblique incidence of light to the sound waves. That the idea of 'propagation' of light in a quasi-homogeneous medium is fundamental and preferable to the undefined idea 'reflection' in such a medium, is pointed out.

The author is grateful to Professor C. V. Raman for many discussions they had since the theory was initiated by them.

Note added in proof.

A report of an interesting investigation of the theory of this phenomenon by Wannier and Estermann of Geneva based on the partial differential equation governing the propagation of light in a quasi-homogeneous medium has appeared in the recent issue of *Helvetica Physica Acta*. If we write the Fourier expansion for $f(\alpha, z)$ in Brillouin's pamphlet (equation 48) or if we write $\Sigma f_r(k_r) \exp(ik_r z)$ for f_r in Raman-Naths' paper, (Part IV) equation 9, we get the equation obtained by Wannier and Estermann. The comparison of their calculated results for $\Theta = 1$ with those of Raman and Nath in their *preliminary theory* requires however a justification. Bär has definitely found experimentally the region of the perfect quantitative applicability of the preliminary theory. One can however calculate ρ and ξ of this paper on Wannier-Estermann's choice of special experimental conditions and substitute them in the expressions (56) obtained in this paper, and find the region of the applicability of the preliminary theory for Wannier-Estermann's choice of the experimental conditions. Even if the length of the cell is not negligible but if the wave-length of sound is large (10^{-1} cm.), the preliminary theory will be applicable as has been experimentally established by Bär.