PARTIAL POLARISATION, PARTIAL COHERENCE AND THEIR SPECTRAL DESCRIPTION FOR POLYCHROMATIC LIGHT—PART I

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§ 1. INTRODUCTION

IN elementary physical discussions of the subject of partially coherent interference or partial polarisation it is usually assumed that the radiation used is practically monochromatic. This is also true of the treatment in a previous paper (Pancharatnam, 1956 *b*) which contained the generalisation of the Fresnel-Arago interference laws (for dealing with the addition of two completely polarised beams to yield a partially polarised beam, and similar problems). Partial coherence between the two beams is there readily understood on the basis that the 'amplitude' and 'absolute phase' of the disturbance in each beam, while remaining sensibly constant over a duration long compared with the period of the light wave, yet fluctuate rapidly from the macroscopic standpoint. Similarly a partially polarised beam is visualised as one in which the disturbance can be characterised by an instantaneous elliptic vibration whose form (as specified by its ellipticity and azimuth) alters rapidly from the macroscopic standpoint: this picture forms in fact the basis for two equivalent methods of introducing the Stokes parameters characterising the state of a partially polarised beam (Chandrasekhar, 1949; Pancharatnam, 1956 *b*). A little consideration, however, reveals that the above descriptions of the light disturbance presuppose that the light used, while not being precisely monochromatic, is all the same practically so. Hence the extension of the results of our previous work (Pancharatnam, 1956 *a*, 1956 *b*) concerning the generalisation of the Fresnel-Arago laws can be taken up only in Part II, since this requires first a discussion of the basic topics mentioned in the title of the paper.

The case of polychromatic radiation has also been considered theoretically in the literature, particularly by Wolf (Born and Wolf, 1959; Wolf, 1956). However it is necessary to remember that here the response of the detector, *e.g.*, the eye, varies over the spectral range covered by the disturbance and hence the mean square of the total electric intensity is *not* to

be regarded as the primary measurable quantity. For this and other reasons (indicated particularly in Part II, § 2) we present the basic theory of partial polarisation and partial coherence keeping the spectral representation of all relevant properties not merely in the forefront but also as the aim of our theoretical discussion: we recognise from the start that the light disturbance is nothing but a superposition of monochromatic constituents covering a spectrum of frequencies—an elementary idea into which some subtleties have to be incorporated as discussed in § 2. The method followed is an extension of that outlined for quasi-monochromatic light elsewhere (Pancharatnam, 1961, 1962).

In Part I is discussed how the state of partial polarisation of a single polychromatic beam may be described as a function of frequency. An interesting consequence of the discussion is that—contrary to what is often supposed—the usual description of the state of a quasi-monochromatic beam by four Stokes parameters is really inadequate for characterising its optical state. In Part II we take up the subject of the mutual partial coherence of two polychromatic beams and the spectral description of their partial coherence—a topic which we do not require for discussing partial polarisation.

§ 2. Statistical Properties of the Light Disturbance

Since in most elementary problems in wave optics one usually represents the light disturbance in terms of a periodic simple harmonic vibration of definite frequency, it appears natural to use the same concept at least as the basis for the more complete description of the light disturbance which becomes necessary for understanding the phenomena of partial polarisation and partial coherence. We therefore assume that (within a duration of observation T) each component of any of the field vectors characterising the light disturbance can be regarded as the sum of a number of monochromatic constituents of frequencies ν_n thus:

$$f(\vec{r}, t) \sum_{n=1}^{\infty} f_n(\vec{r}) \cos [2\pi\nu_n t - \phi_n(\vec{r})] \qquad (1)$$

$$\nu_n = n/\mathrm{T}; \quad 0 \leq t < \mathrm{T}.$$

The field described by any one monochromatic constituent of frequency ν_n is of the type that could be calculated from electromagnetic wave-theory—if the amplitude and phase of that monochromatic constituent in the vicinity of each of the sources of radiation were definitely known. Actually the above Fourier series is taken to represent the disturbance only for a duration

T, which we suppose to be the duration of observation. It is also convenient to regard T as very large compared with the reciprocal of the spectral resolving power of any instrument used, as also with the time taken by light to travel a distance of the order of the linear dimensions of the experimental set-up. Alternatively we could let $T \to \infty$ at an appropriate stage in the discussion.

It is now realised (Blanc-Lapierre and Dumontet, 1955; Born and Wolf, 1959) that the light disturbance regarded directly as a function of time has the statistical properties of noise (Rice, 1954); but it appears worthwhile to enumerate some of these statistical properties when the spectral description (1) is used, and to indicate how they are necessary for understanding the phenomena associated with partial polarisation and partial coherence which will be discussed in later sections.

Before proceeding further we introduce complex notation: each monochromatic constituent in Equation (1) is the real part of the corresponding complex periodic function which may be uniquely associated with it. In order to fix our ideas and concentrate on essentials we consider specifically the case of a beam travelling along the z-axis. The components E_x, E_y of the electric vector at any particular point in the beam may now be written in one of the alternative forms:

$$E_x(t) = \sum_{n=1}^{\infty} U_n \exp. i2\pi\nu_n t, \quad (U_n = E_x^{(n)} \exp. - i\alpha_n) \tag{2 a}$$

$$E_y(t) = \sum_{n=1}^{\infty} V_n \exp. i2\pi\nu_n t, \quad (V_n = E_y^{(n)} \exp. - i\beta_n), \quad (0 \leq t < T) \tag{2 b}$$

or,

$$E_x(t) = \sum_{n=1}^{\infty} (A_n + iB_n) \exp. i2\pi\nu_n t \tag{3 a}$$

$$E_y(t) = \sum_{n=1}^{\infty} (C_n + iD_n) \exp. i2\pi\nu_n t. \tag{3 b}$$

(We shall for convenience continue to refer to each monochromatic constituent in this complex representation as a Fourier component.)

In what follows we fix our attention first on the x-component, for example, by considering the 'resolved' beam transmitted by an analyser with its vibration-axis parallel to Ox. Further, by the term 'intensity' of a Fourier component (of the resolved disturbance) we shall mean the average value of the square of the (real) electric intensity for that Fourier

component, which is equal to $|U_n|^2$ (for the resolved disturbance). While the 'intensity' and absolute phase of any particular Fourier component are necessarily constant (for the duration of observation), it yet becomes necessary to assume that the 'intensities' and phases of the successive Fourier components vary in a random manner.

The basic statistical features associated with the light disturbance represented for a duration T ought to be true whatever be the magnitude of T and whatever instant we choose as the origin of time (in the case of a stationary optical field). This leads immediately to certain other conclusions. Representing the disturbance from T to 2T again as a Fourier series in the form (2) the new amplitude ($|U_n'|$) and phase α_n' of the Fourier component of frequency ν_n for this period cannot be the same as $|U_n|$ and α_n for the previous observation period (0 to T). For if this were the case a Fourier analysis of the x-component for the combined duration 0 to 2T would show that the intensities of the alternate Fourier components (now more closely spaced in frequency) would vanish instead of having essentially unpredictable values as described in the last para. If now we consider a succession (or ensemble) of independent observation periods, each of duration T, the amplitude and phase of the Fourier component of the same frequency ν_n would be distributed statistically according to some distribution law which we now state. As was first pointed out by Einstein (Einstein and Hopf, 1910; Einstein, 1915) it must be assumed that *the Fourier coefficients A_n and B_n* in (3 *a*) *(which are the real and imaginary parts of U_n) have independent normal distributions about zero*:—

$$\left.\begin{aligned} &\langle A_n^2\rangle = \langle B_n^2\rangle; \quad \langle A_n\rangle = \langle B_n\rangle = 0 \\ &\langle A_nB_n\rangle = \langle A_nB_m\rangle = 0 \\ &\langle A_nA_m\rangle = \langle B_nB_m\rangle = 0, \quad (n \neq m) \end{aligned}\right\} \tag{4}$$

where the sharp brackets denote the average over an ensemble of independent observation periods, each of duration T. [This would be true for equilibrium radiation, though not for modulated radiation emitted for example by atoms in a double resonance experiment, Dodd *et. al.* (1959).]

Considering the disturbance during some particular observation period, it will be seen that the above distribution ensures that the constant value of the phase α_n ($= \tan^{-1} - B_n/A_n$) of a Fourier component may have any value from 0 to 2π with equal probability, the phases of the different Fourier components being distributed at random within this range. Further if independent Gaussian distributions for A_n and B_n hold for the disturbance at one

point in the beam it will also hold for any other point in the beam [even if the field variation is of the general form (1)] and will continue to hold when two or more such beams (fields) are superposed and allowed to interfere.

The intensity of a monochromatic Fourier component is not directly observable in usual experiments (hence our use of quotation marks whenever referring to the same). The properties of radiation appropriate to a frequency ν that are usually determined in any experiment really represent the properties averaged over a physically infinitesimal range $\delta\nu$ about ν. Thus for example considering the resolved beam transmitted by a linear analyser, *the basic macroscopically measurable quantity is the spectral density* $I_{xx}(\nu)$ associated with the frequency ν: if one imagines an ideal spectral filter transmitting only Fourier components with frequencies within the range $\nu \pm \frac{1}{2}\delta\nu$, then the mean value of the square of the (real) electric intensity for the disturbance transmitted by the filter gives $I_{xx}(\nu)\,\delta\nu$. Within the physically infinitesimal interval $\delta\nu$ there can yet be a very large number of Fourier components.[a] The transmitted intensity $I_{xx}(\nu)\,\delta\nu$ will then be equal to the sum of the 'intensities' of the monochromatic components lying within that frequency interval—since the interference or 'beating' of monochromatic constituents of different frequencies will not contribute to the average intensity but only to fluctuations about the average (Brown and Twiss, 1957).

The macroscopic or average parameters pertaining to a range $\delta\nu$ could normally be determined by a single observation of sufficiently large duration; but in theoretically evaluating the relevant averages it is convenient (as in all statistical problems) to use the fact that the average characteristics do not change for successive observations, so that we may in addition average over the ensemble of observations. (Such a procedure should obviously yield the correct average value even if the individual observation period is so short as to yield fluctuations from the average—a situation which we shall not consider.) As an example consider the total intensity of the x-component transmitted through a spectral filter allowing the frequency range $\nu \pm \frac{1}{2}\delta\nu$. This intensity [being defined as the mean value of the square of the (real) electric vector of the transmitted x-component] can also be written as:

[a] This implies that the observation period T is chosen sufficiently large, this being in any case necessary if a single observation is to give directly the average transmitted intensity and not a fluctuation from the average.

$$I_{xx}(\nu)\,\delta\nu$$

$$= \frac{1}{T}\int_0^T \left\langle \sum_n (A_l + iB_l)\, e^{i\omega_l t} \cdot \sum_m (A_m - iB_m)\, e^{-i\omega_m t} \right\rangle dt$$

$$= \sum_m (\langle A_m{}^2\rangle + \langle B_m{}^2\rangle) = \sum_m \langle U_m U_m{}^*\rangle$$

where we have used (3 *a*) and (4), the summations being only over the Fourier components in the range $\nu \pm \frac{1}{2}\delta\nu$. If ν_n be a central frequency in the range $\delta\nu$ then (neglecting higher order quantities) the right-hand side of (5) can be written as $\langle U_n\, U_n{}^*\rangle$ times the number of terms in the summation, which is $\delta\nu / \triangle\nu$; hence,

$$I_{xx}(\nu_n) = \frac{\langle U_n U_n{}^*\rangle}{\triangle\nu} \tag{6}$$

i.e., *the ensemble average of the 'intensity' of the Fourier component of frequency* ν *divided by the spectral separation* $\triangle\nu$ *between successive Fourier components gives the spectral density associated with that frequency*—a result which we shall frequently use. While the above result has been shown for the x-component of the vibration it obviously holds also for the beam as a whole, consisting of both x and y-components [*see* Eq. (7 *a*) below].

It should perhaps be emphasised that the expression (1) describes the light disturbance only for a certain specific duration 0 to T and not for all time. If we let $T \to \infty$ in (1) or (2) the light disturbance for all time appears to be represented as a Fourier integral or analytic signal and indeed such a representation is also commonly used in the literature (Born and Wolf, 1959; Hopkins, 1957). Nevertheless it should be carefully borne in mind that, for example, the phase of a Fourier component never becomes a *continuous* function $\alpha(\nu)$ of frequency; nor does a characteristic such as the 'intensity' of a monochromatic constituent ever become the same (or proportional to) the observable characteristic $I(\nu)$ associated with the same frequency, however much we may increase T. By way of contrast, we shall have occasion, both in Parts I and II, to introduce certain continuous functions of frequency, which form macroscopically measurable functions.[b]

[b] To avoid frequent repetition later we add here a few remarks on these spectral functions: firstly, the continuity and measurability of these functions will follow once it is shown that they can be expressed in terms of the spectral densities of certain disturbances; and secondly, it is physically clear that such *continuous* functions can be defined with all the needful accuracy by defining their values at the frequencies ν_n of the Fourier components, since the latter form practically a continuum (*see* previous footnote).

§ 3. PARTIAL POLARISATION OF A POLYCHROMATIC BEAM

In this section we shall consider what are the macroscopically observable parameters or spectral functions which may be said to characterise the state of a polychromatic beam.

The x and y-components of the disturbance at a point in the light beam may be expressed as the sum of Fourier components as given by Eq. (2), over the duration T. Since the x and y-components of the monochromatic constituent of frequency ν_n have a constant phase difference $(\beta_n - \alpha_n)$ and a constant amplitude ratio $|V_n|/|U_n|$ this monochromatic constituent is necessarily *completely polarised*—in general, elliptically polarised—the azimuth of the major axis being λ_n and the ellipticity β_n (say). The 'intensity' $I^{(n)}$ of this monochromatic constituent (*i.e.*, the mean value of the square of the electric vector associated with this monochromatic constituent) is clearly given by

$$I^{(n)} = |U_n|^2 + |V_n|^2. \tag{7 a}$$

The state of polarisation of this elliptic vibration can be characterised symbolically by a representative point P_n on the surface of the so-called Poincare sphere of centre 0 (λ_n and β_n being the latitude and longitude of the representative point). Alternatively the completely polarised elliptic vibration of frequency ν_n could be completely characterised by four hypothetical 'Stokes parameters', $S_0^{(n)}$, $S_X^{(n)}$, $S_Y^{(n)}$, $S_Z^{(n)}$, defined by:

$$\left.\begin{aligned} S_0^{(n)} &= I^{(n)}, \\ S_X^{(n)} &= I^{(n)} \cos 2\beta_n \cos 2\lambda_n \\ S_Y^{(n)} &= I^{(n)} \cos 2\beta_n \sin 2\lambda_n \\ S_Z^{(n)} &= I^{(n)} \sin 2\beta_n, \end{aligned}\right\} \tag{7 b}$$

As the notation suggests, the last three parameters may be looked upon as the rectangular components of a vector $S^{(n)}$ laid off from the origin O in the three-dimensional Poincare space: the length of this 'Stokes-Poincare vector' is $I^{(n)}$, the 'intensity' of the monochromatic vibration; while the point of intersection of the vector with the Poincare sphere is the point P_n referred to above. [For a fuller description of these representations of polarised light see *e.g.*, Pancharatnam (1956 *a*, *b*) and Ramachandran and Ramaseshan (1961).]

It should be emphasised that each monochromatic constituent is necessarily perfectly polarised: the possibility of partial polarisation arises only from the circumstance that the totally polarised states [$I^{(n)}$, $S^{(n)}$ and $I^{(n+1)}$, $S^{(n+1)}$] of successive monochromatic constituents may differ in an unpredict-

able manner. We now assert that the macroscopic state of a single beam or disturbance can be characterised by four continuous functions (*see* footnote *b.*), *viz.*, the *Stokes spectral functions* which we define by

$$S_0(\nu_n) = \frac{\langle I^{(n)} \rangle}{\triangle \nu} \; ; \quad S_i(\nu_n) = \frac{\langle S_i^{(n)} \rangle}{\triangle \nu} \; ; \quad i = X, Y, Z, \tag{8}$$

i.e., we may say that each Stokes parameter per unit spectral width to be associated with any frequency is defined as the ensemble average (over a succession of independent durations T) of the corresponding ' Stokes parameter ' for a monochromatic constituent of that frequency, divided by the spectral strip $\triangle \nu$ to be ' associated ' with that monochromatic constituent. As is seen from (6), (7) and (8), the first Stokes function is also the spectral density $I(\nu)$, which may be defined thus: $I(\nu)\,\delta\nu$ is the mean value of the square of the (real) electric intensity of the beam transmitted by a spectral filter transmitting the spectral range $\nu \pm \frac{1}{2}\delta\nu$. The existence of the latter three averages $S_i(\nu)$ [which may be regarded as components of a vector $\mathbf{S}(\nu)$ in the Poincare space] may be shown from physical arguments. Suppose a beam in the state $I(\nu)$, $\mathbf{S}(\nu)$ is incident on an elliptic analyser which transmits completely disturbances in the elliptic state of polarisation given by a unit vector $\mathbf{A}$ in the Poincare space. If $(I^{(n)}, \mathbf{S}^{(n)})$ be the elliptic state of the monochromatic constituent of frequency ν_n in the incident beam, the ' intensity ' of the transmitted vibration of the same frequency (resolved to the state $\mathbf{A}$) is given by a formula essentially due to Stokes [see *e.g.*, Ramachandran and Ramaseshan, 1961, Eq. (8·4)]:

$$I_A^{(n)} = \tfrac{1}{2}(I^{(n)} + \mathbf{A} \cdot \mathbf{S}^{(n)}). \tag{9}$$

Taking an average over an ensemble of independent durations T and dividing by $\triangle \nu$ (the frequency interval between Fourier components) we get using (6) and (8)

$$I_A(\nu) = \tfrac{1}{2}[I(\nu) + \mathbf{A} \cdot \mathbf{S}(\nu)] \tag{10}$$

where $I_A(\nu)$ is the spectral density of the resolved disturbance transmitted by the analyser $\mathbf{A}$.

Thus the macroscopic state of my any polychromatic beam may be said to be completely characterised by four Stokes spectral functions defined in (8)—since these enable us to predict [from (10)] the spectral density of the beam transmitted by an arbitrary elliptic analyser introduced in its path. These spectral parameters may also be operationally defined in terms of the measured *spectral* densities transmitted in turn by various sets of ideal analysers introduced in the path of the beam. (The proce-

dure would be entirely analogus to that used for measuring the customary Stokes parameters [see *e.g.*, Ramachandran and Ramaseshan 1961, Eq. (9·11)]).

§ 4. INADEQUACY OF FOUR STOKES PARAMETERS FOR A QUASI-MONOCHROMATIC BEAM

It is usually thought that the polarisation state of a sensibly monochromatic beam can be characterised as a whole by four parameters (which are *not* spectral functions), *viz.*, the customary Stokes parameters, which we shall refer to as the *integrated Stokes parameters* for reasons given in § 5. As has been indicated elsewhere (Pancharatnam, 1962) the inadequacy of these four integrated Stokes parameters can be revealed in a certain class of experiments which do not even involve a direct spectral analysis of the beam. Consider for example a beam transmitted normally through a birefringent plate whose principal planes are equally inclined to the plane of polarisation of the beam incident normally on it. Suppose the path retardation introduced by the birefringent plate between the fast and slow components is large compared with the coherence length of the light disturbance. Though the integrated Stokes parameters of the emergent light will, when measured, be found to have the same values as for unpolarised light, the emerging disturbance is *not* optically equivalent to unpolarised light. For by passage through a similar birefringent plate (with fast and slow axes interchanged) the light can be rendered plane-polarised—a property which natural unpolarised light does not have.

The example quoted above illustrates that a knowledge of the four customary Stokes parameters of a quasi-monochromatic beam is insufficient for answering the following question: what would be the total intensity transmitted if the beam is passed through a birefringent plate (introducing a relative retardation of arbitrary magnitude) and then through an analyser? On the other hand general problems of this sort can be handled if we know the Stokes *spectral functions* $\mathbf{I}(\nu)$, $\mathbf{S}(\nu)$ for the beam—quasi-monochromatic *or polychromatic*—incident on any arbitrary non-depolarising optical system. Let $\underset{\sim}{S}^{(n)}$ be the hypothetical 'Stokes column vector' characterising the state of the monochromatic constituent of frequency ν_n in the incident disturbance [the components of the column vector being the *four* hypothetical 'Stokes parameters' introduced in Eq. (7 *b*)]. The corresponding 'Stokes vector' $\underset{\sim}{S}'^{(n)}$ in the emerging disturbance will be given by a linear relation of the form:

$$\underset{\sim}{S}'^{(n)} = \hat{\mathbf{M}}(\nu_n)\,\underset{\sim}{S}^{(n)}$$

where $\hat{M}(\nu_n)$ is a 4×4 matrix operator—the so-called Mueller matrix representing the effect of the instrument on a monochromatic disturbance of frequency ν_n [Perrin, 1942; Shurcliff (1962)]. Taking the ensemble average of this equation and dividing by $\triangle\nu$ we get [since the operator $\hat{M}(\nu_n)$ representing the action of the optical system does not change for a succession of independent durations]

$$\underline{S}'(\nu) = \hat{M}(\nu)\,\underline{S}(\nu) \tag{11}$$

where we have used Eq. (8). Thus a knowledge of the Mueller matrix $\hat{M}(\nu)$ *as a function of frequency* enables us to determine the Stokes spectral functions $S_i'(\nu)$ of the emerging radiation from those of the incident radiation [*viz.*, $S_i(\nu)$]—even when the latter is a polychromatic partially polarised beam.

The example quoted in the first para of this sub-section represents a case where the elements of the Mueller spectral matrix $M_{ij}(\nu)$ change very rapidly with frequency even within the spectral range covered by the quasi-monochromatic disturbance; *i.e.*, it can be easily shown for the problem in question (Pancharatnam, 1962) that the relative phase retardation between the fast and slow components introduced by the birefringent plate varies over many multiples of 2π within the spectral range covered. Hence if a spectral analysis is made of the so-called unpolarised light produced, it will be found that each part of the spectrum is completely polarised, but that the state of polarisation alters periodically many times within the spectral width.

§ 5. The Integrated Stokes Parameters for a Quasi-Monochromatic Beam

For many purposes it is sufficient to characterise the 'state' of a quasi-monochromatic beam as a whole by four integrated Stokes parameters $(\bar{I}, \bar{S}_X, \bar{S}_Y, \bar{S}_Z)$ (as opposed to spectral functions) which we define thus:

$$\bar{I} = \int_0^\infty I(\nu)\,d\nu;\quad \bar{S}_i = \int_0^\infty S_i(\nu)\,d\nu, \tag{12}$$

i.e., each integrated Stokes parameter is defined as the integral of the corresponding Stokes spectral function over the spectral line-width of the disturbance—the first parameter being obviously the integrated or total intensity of the beam. These integrated Stokes parameters must be identical with the customary Stokes parameters introduced for characterising the state of a quasi-monochromatic beam; for they lead to the same expression for the

total intensity $\bar{I}_A$ transmitted by an elliptic analyser **A** introduced in the path of a beam which is in the partially polarised state $(\bar{I}, \bar{S})$

$$\bar{I}_A = \tfrac{1}{2}(\bar{I} + \bar{S} \cdot \underset{\sim}{A}), \tag{13}$$

this being obtained by integrating Eq. (10) over frequency.

Suppose a quasi-monochromatic beam is passed through a birefringent retardation plate for which the relative phase retardation between the components varies negligibly within the spectral line-width of the light used. In such problems it is sufficient to know the customary or integrated Stokes parameters of the incident beam in order to predict the integrated parameters for the emerging beam. For by integrating (11) we get:

$$\bar{\underset{\sim}{S}}' = \hat{M}(\bar{\nu})\,\bar{\underset{\sim}{S}} \tag{14}$$

where $\hat{M}(\bar{\nu})$ stands for the Mueller matrix representing the action of the optical instrument for the mean frequency $\bar{\nu}$ of the light.

6. Spectral Coherency Matrix for Characterising the State of Polarisation

As an alternative to the procedure of using four integrated Stokes parameters for characterising the state of a quasi-monochromatic beam, Wolf has advocated the use of four elements forming a 'coherency matrix' for the same purpose (Born and Wolf, 1959). Each of the four elements of the coherency matrix (or density matrix) may be shown to be a linear combination of two of the integrated Stokes parameters; hence the usual coherency matrix cannot really be regarded as characterising the 'state' of a quasi-monochromatic beam except in the same restricted sense as the integrated Stokes parameters (§§ 4, 5). However it may be readily shown that the limitation may be removed and that Wolf's coherency matrix representation may be easily extended so as to characterise the state of a polychromatic beam as a function of frequency. [See also wolf (1956).]

Let the x and y components of the disturbance be regarded as the sum of a number of monochromatic constituents as in (2)—U_n and V_n being the complex amplitudes of the x and y components for the Fourier constituent of frequency ν_n. We may now define the *spectral coherency matrix*, whose elements $I_{ij}(\nu)$ are functions of the frequency ν (*see* footnote, *b*) by

$$I_{xx}(\nu_n) = \frac{\langle U_n U_n^* \rangle}{\Delta\nu} \; ; \; I_{xy}(\nu_n) = \frac{\langle U_n V_n^* \rangle}{\Delta\nu}$$

$$I_{yx}(\nu_n) = \frac{\langle V_n U_n^* \rangle}{\triangle \nu} \; ; \quad I_{yy}(\nu_n) = \frac{\langle V_n V_n^* \rangle}{\triangle \nu}. \tag{15}$$

To establish the connection with the Stokes spectral functions (and incidentally to prove that the above averages exist) we make use of the following relations concerning the completely polarised monochromatic constituent of frequency ν_n, which may be readily proved [Ramachandran and Ramaseshan, 1961, Eq. (16.6)]:

$$\begin{aligned} I^{(n)} &= U_n U_n^* + V_n V_n^*; & S_X^{(n)} &= U_n U_n^* - V_n V_n^* \\ S_Y^{(n)} &= (U_n V_n^* + V_n U_n^*); & S_Z^{(n)} &= i(V_n U_n^* - U_n V_n^*). \end{aligned} \tag{16}$$

By taking an ensemble average of these relations, dividing by $\triangle \nu$ and making use of the definitions of the Stokes spectral functions and the spectral coherency matrix, [Eqs. (8) and (15)], we get the relation between them, which can be solved to give finally,

$$\begin{aligned} I_{xx}(\nu) &= \tfrac{1}{2}[I(\nu) + S_X(\nu)]; & I_{xy}(\nu) &= \tfrac{1}{2}[S_Y(\nu) + iS_Z(\nu)] \\ I_{yx}(\nu) &= \tfrac{1}{2}[S_Y(\nu) - iS_Z(\nu)]; & I_{YY}(\nu) &= \tfrac{1}{2}[I(\nu) - S_X(\nu)]. \end{aligned} \tag{17}$$

Since the elements of the spectral coherency matrix are seen to be independent linear combinations of the Stokes spectral functions, they may equally well be used to characterise the polarisation state of a polychromatic beam, as a function of frequency. The elements of the spectral coherency matrix are also equal to twice the Fourier transform of the corresponding real auto- and cross-correlation functions for the x and y disturbances, as can be shown from Part II eq. (9). Finally, we may mention that the action of a non-depolarising instrument, such as a crystal plate, on a polychromatic beam passing through it can be completely characterised by a spectral matrix—the Jones matrix (see *e.g.*, Shurcliff, 1962)—whose elements may be written as functions of frequency. Following Parrent and Roman (1960) it can be easily shown that this is also the transformation matrix which connects the spectral coherency matrices of the polychromatic radiation leaving and entering the optical device.

§ 7. Summary

The statistical properties of the light disturbance (regarded as a superposition of monochromatic constituents covering a spectrum of frequencies) is described. It is shown that the state of partial polarisation of a polychromatic beam can be completely characterised by four *Stokes spectral functions* which are generalisations of the customary Stokes parameters and

which express the state of partial polarisation as a function of frequency; an equivalent method of description is by a spectral coherency matrix. It is pointed out that the *customary* set of Stokes parameters (or the equivalent coherency matrix) used for characterising the optical state of a quasi-monochromatic beam is really inadequate for the purpose, since beams having the same set of Stokes parameters may *not* be optically equivalent for a certain class of experiments.

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