GENERALIZED THEORY OF INTERFERENCE, AND ITS APPLICATIONS

Part I. Coherent Pencils

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§ 1. INTRODUCTION

THE investigations of which the results are presented in this paper arose during the study of certain specific problems in crystal optics. As investigators in this field are well aware, the simplest procedures for studying the optical properties of anisotropic media (*e.g.*, examination under the polarising microscope) generally involve the use and study of polarised light. The complexity of the peculiar interference phenomena exhibited and also of their customary theoretical analysis (by algebraic methods) become quite considerable even in the case of transparent optically active crystals like quartz—as may be seen by a reference to the treatises of Mascart (1891) and Walker (1904); this is because the two waves propagated along any direction in such a medium are no longer linearly polarised at right angles to one another, but are elliptically polarised. Nevertheless, the types of 'oppositely' polarised waves propagated in such media must be termed simple compared to the elliptically polarised waves propagated in *absorbing biaxial crystals*.

The remarkable interference phenomena exhibited by absorbing biaxial crystals may be easily studied by looking at an extended source through a plate (cut normal to an optic axis); a suitable material is the mineral iolite—which the author had the opportunity of investigating experimentally (Pancharatnam, 1955). (1) With the incident light polarised, and *even without the use of an analyser*, interference rings are seen, which are feeble but are nevertheless easily visible. (2) When, in addition, an analyser is also introduced, the biaxial interference figures seen are notably different from those seen in transparent crystals under the same conditions. (3) With the analyser in position and with no polariser—*i.e., with the incident light completely unpolarised*—feeble interference rings are again easily visible. (4) Finally, even when both analyser and polariser are absent, incipient traces of an interference pattern may be discerned.

Viewing these particular phenomena from a slightly broader perspective, we see that their analysis is connected with certain general questions concerning the properties of two polarised beams travelling along the same direction. We shall now formulate these problems since they form the main content of the paper. The study of the effects with a polariser alone leads us to investigate the following questions: the interference of two coherent beams in different states of elliptic polarisation (§ 3); the resolution of any polarised beam into two beams in given states of polarisation—which occurs at the first face of the crystal plate (§ 4); and the composition of two coherent beams of different polarisation—at the second face of the plate (§§ 5, 6). The problem involved when an analyser is also introduced (keeping the incident light polarised) reduces to the following: the interference of two coherent polarised beams which are ' brought to the same state of elliptic vibration ' by the use of a suitable analyser (§ 8). In § 9 we shall consider the addition of n coherent beams in different states of polarisation.

An attempt to formulate in general terms the problems associated with (3) and (4) leads rather unexpectedly into the subject of the partial coherence of polarised beams. We shall leave the discussion of this subject for Part II.

§ 2. THE POINCARÉ SPHERE AND THE STOKES PARAMETERS

For problems—such as the one we are dealing with—where we require to handle elliptic vibrations with the same facility as linear vibrations, the indirect specification of the polarisation of an elliptic vibration by giving the equation to its rectangular components is obviously unsatisfactory: the procedure not only leads to cumbersome calculations lacking in elegance, as has been pointed out by other authors, but often ceases to give physical insight into the cause of the phenomena actually observed. Two other powerful methods for specifying the state of polarisation of a beam have been used extensively—an analytical method due to Stokes (1901), and a geometrical one due to Poincaré. The conventional theoretical presentation of the subject of the ' Stokes parameters ' may be found in Chandrasekhar (1949) and in Rayleigh (1902); that of the Poincaré sphere and some of its more well-known properties may be found in Pockels (1906), Walker (1904), Ramachandran and Ramaseshan (1952), and Jerrard (1954). In Part I of the present paper, only the Poincaré representation is used; and we may mention that this part constitutes in itself a self-contained derivation of the properties of the Poincaré sphere by a new procedure. The Stokes representation will be required only in Part II where the subject of partial polarisation naturally enters; but the entire subject of the Stokes representation is introduced there, in a new manner, through the Poincaré representation itself—by developing the ideas of Fano (1949) and Ramachandran (1952).

The state of polarisation of a completely polarised beam is directly specified by the form of the ellipse traced by the tip of the displacement vector—this being invariant for a completely polarised beam, unlike the intensity and absolute phase which may be subject to statistical fluctuations. Poincaré introduced a mapping whereby any particular form of elliptic vibration is represented by a specific point on the surface of a sphere—the points on the 'Poincaré sphere' exhausting all the conceivable forms of elliptic vibrations. The definition of the mapping allows the ellipse represented by a point P to be visualised directly in terms of the longitude 2λ and latitude 2ω of the point: for λ is the azimuth of the major axis of the elliptic vibration and $\tan \omega$ the ellipticity. Alternatively the point P may be specified in cartesian co-ordinates instead of in polar co-ordinates—with the XY plane coinciding with the equatorial plane. As Perrin (1942) had observed, the *three parameters introduced by Stokes for characterising a completely polarised beam are proportional to the cartesian co-ordinates of the point P*—the constant of proportionality being the intensity of the beam which is taken as the fourth Stokes parameter.

Two elliptical vibrations whose states of polarisation are represented by diametrically opposite points on the Poincaré sphere will be said to be oppositely polarised. The conventional procedure of decomposing any elliptic vibration into two rectangular linear vibrations (represented by two diametrically opposite points on the equator) is a particular case of decomposing it into two elliptical vibrations of opposite polarisation. The fundamental property of the Poincaré sphere relates to such a decomposition (*see* Fig. 1) and is the following:—

I. *When a vibration of intensity I in the state of polarisation C is decomposed into two vibrations in the opposite states of polarisation A and A′, the intensities of the 'A-component' and the 'A′-component' are* $I \cos^2 \frac{1}{2} CA$ *and* $I \sin^2 \frac{1}{2} CA$ *respectively.*

All the results proved in this paper are deduced from the above theorem. We shall not give the proof of the theorem since a proposition equivalent to the above has been proved by other authors (*see* § 8 below).

Since according to Theorem I, the sum of the intensities of the oppositely polarised component beams are together equal to the intensity I of the resultant beam we deduce the following well-known result.

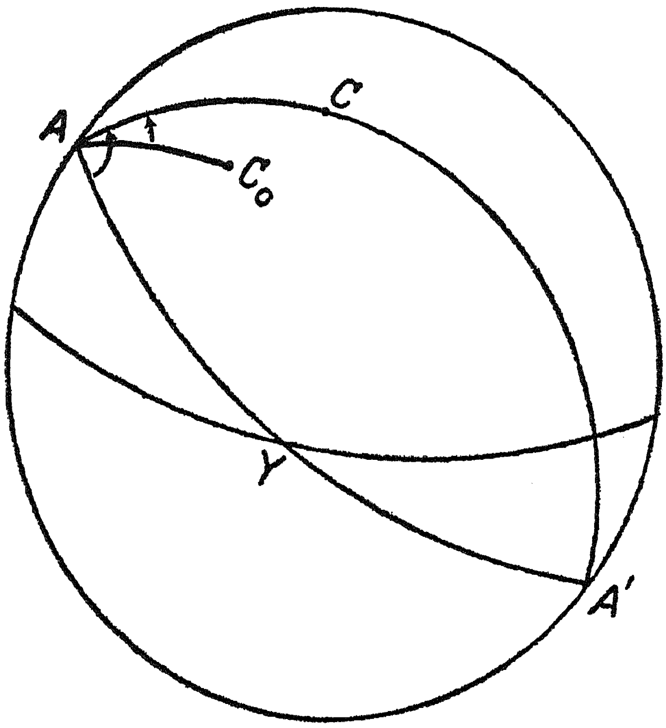

FIG. 1

II. *Two oppositely polarised beams cannot constructively or destructively interfere.*

§ 3. INTERFERENCE OF TWO NON-ORTHOGONALLY POLARISED VIBRATIONS

The interference of two *linear* vibrations not orthogonal to one another can be analysed by resolving the first vibration into two vibrations which are respectively parallel and orthogonal to the second vibration. The interference of any two elliptically polarised vibrations will now be handled in a similar manner. We shall prove the following proposition.

III. The intensity I of the beam obtained on combining two mutually coherent beams 1 and 2, of intensities I_1 and I_2 in the states of polarisation A and B respectively, will be given by the general interference formula:—

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \tfrac{1}{2}c \cos \delta \tag{1}$$

Here c is the angular separation of the states A and B on the Poincaré sphere (*see* Fig. 2). Thus $\cos^2 \frac{1}{2}c$ is a ' similarity factor ' between the states of polarisation, which determines the extent of interference and which varies from unity (for identically polarised beams) to zero (for oppositely polarised beams). The significance of the ' phase difference ' δ between the beams will be elucidated below.

The above relation may be obtained as follows. The vibration 2 may be replaced by two oppositely polarised vibrations—one in the state of polarisation A of the first beam, and the second in the state of polarisation A′ orthogonal to A. These vibrations (which we shall denote by 2_A and $2_{A'}$ res-

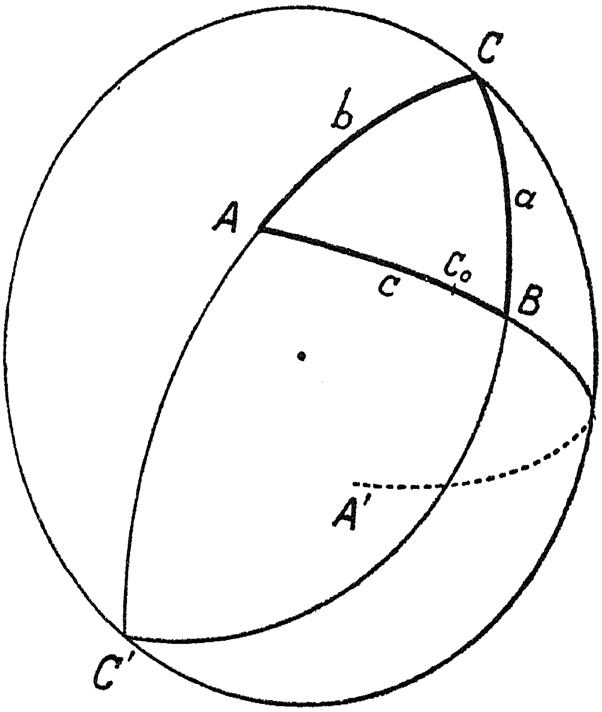

FIG. 2

pectively) will be coherent with vibration 1 and will have intensities $I_2 \cos^2 \frac{1}{2}c$ and $I_2 \sin^2 \frac{1}{2}c$ respectively (§ 2, I). The vibration 1 being in the same state of polarisation as vibration 2A (over which it has a phase advance δ, say), can be combined with it to yield a vibration (1 + 2A) of intensity

$$I_1 + I_2 \cos^2 \tfrac{1}{2}c + 2\sqrt{I_1 I_2} \cos \tfrac{1}{2}c \cos \delta$$

We are thus left with the vibration (1 + 2A) and the vibration 2A′ which are in the opposite states of polarisation A and A′. The resultant intensity I, obtained merely by adding their intensities (§ 2, II), is that given in formula (1) which we wished to deduce.

An explicit expression for the similarity factor between any two states of polarisation (in terms of the azimuths λ_1, λ_2 of the major axes and the ellipticities $\tan \omega_1$, $\tan \omega_2$ of the two vibrations) may, if necessary, be obtained from the following relation (obtained by spherical trigonometry):—

$$\cos c = \sin 2\omega_1 \sin 2\omega_2 + \cos 2\omega_1 \cos 2\omega_2 \cos 2 (\lambda_2 - \lambda_1) \qquad (2)$$

It may be noted that $\cos \frac{1}{2}c$ is equal to the visibility of fringes (as defined by Michelson) obtained under the conditions $I_1 = I_2$.

In the above discussion the quantity δ has been introduced as the phase advance of the first beam over the A-component of the second beam. There are two properties of δ however which enable us to speak of it as the absolute difference of phase between the two beams themselves—though they are in different states of polarisation. In the first place we note that if the first beam is subjected to a particular path retardation relative to the second, then δ as defined above decreases by the corresponding phase angle; in the second place we note that as long as no path retardation is introduced between the two beams, any alteration of the intensities of the two beams will not change the value of δ as defined above. Hence we will be guilty of no internal inconsistency if we make the following statement by way of a definition: *the phase advance of one polarised beam over another* (not necessarily in the same state of polarisation) *is the amount by which its phase must be retarded relative to the second, in order that the intensity resulting from their mutual interference may be a maximum.*

This phase advance is identically equal to δ, and the above definition holds only for non-orthogonal vibrations.

By picturing an elliptic vibration as being made up of its rectangular components, it becomes apparent that the coherent addition contemplated in this section, of two beams in different states of polarisation will yield a resultant beam which is also elliptically polarised. The intensity of this having been determined, the next step logically would be to determine its state of polarisation. This problem we shall relegate to § 5, and take up in the next section the converse problem which is simpler to handle.

§ 4. DECOMPOSITION OF A POLARISED BEAM INTO TWO BEAMS IN GIVEN STATES OF POLARISATION

The method of approaching the general problem described in the heading is immediately made clear by regarding the following particular case, illustrated in Fig. 3. Suppose that a *linear* vibration C has to be split into two *linear* vibrations A and B (with which it makes angles $\frac{1}{2}b$ and $\frac{1}{2}a$ respectively). The intensities of the A- and B-vibrations may be obtained by the parallelogram law; more specifically, by equating the projections of the B- and C-vibrations in the direction orthogonal to the A-vibration, we obtain the intensity of the B-vibration as proportional to $(\sin \frac{1}{2}b / \sin \frac{1}{2}c)^2$, where $\frac{1}{2}c$ is the angle between the A and B vibrations.

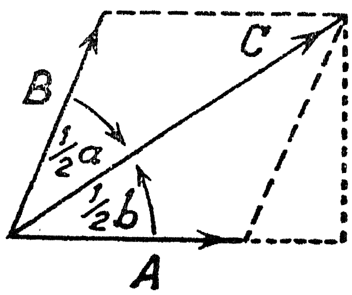

FIG. 3

By similar reasoning we shall prove the following proposition.

IV. If a beam of intensity I in the state of polarisation C is decomposed into two coherent beams 1 and 2, in the states of polarisation A and B respectively, the intensities of these beams will be given by

$$I_1 = I \cdot \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}c}; \qquad I_2 = I \cdot \frac{\sin^2 \frac{1}{2}b}{\sin^2 \frac{1}{2}c} \tag{3}$$

where a, b and c are the angular separations BC, CA and AB respectively (*see* Fig. 2).

To prove the proposition, we decompose each vibration into two vibrations of polarisation A and A′. The sum of the A′-components of the vibrations 1 and 2 must together be the same as the A′-component of the given vibration 3. The vibration 1, however, has no A′-component since we have chosen A′ opposite to A. Hence the A′-components of vibrations 2 and 3 are identical. Equating their intensities (as given by § 2, I) we have $I_2 \sin^2 \frac{1}{2}c = I \sin^2 \frac{1}{2}b$. Similarly, we can show that $I_1 \sin^2 \frac{1}{2}c = I \sin^2 \frac{1}{2}a$, thus establishing relations (3) above. [The expressions (3) for the intensities I_1 and I_2 could be written explicitly in terms of the constants of the elliptic vibrations by substituting the expression (2) for cos c and similar expressions, obtained by cyclic permutation, for cos a and cos b. It can thus be shown that the expressions (3) are equal to the more lengthy expressions deduced directly by Stokes (*loc. cit.*).]

The beams 1 and 2 will have a definite phase relationship with one another which, as we shall see, depends in a remarkable fashion on the mutual configuration of the points A, B and C.

From (1) we have

$$\cos \delta = \frac{I - (I_1 + I_2)}{2\sqrt{I_1 I_2} \cos \frac{1}{2}c}$$

Substituting for I_1 and I_2 from (3),

$$-\cos \delta = \frac{\sin^2 \frac{1}{2}a + \sin^2 \frac{1}{2}b - \sin^2 \frac{1}{2}c}{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \cos \frac{1}{2}c} \tag{4}$$

If C′ be the point diametrically opposite to C (Fig. 2), then denoting by a', b', c the sides of the triangle ABC′, we will have

$$-\cos \delta = \frac{\cos^2 \frac{1}{2}a' + \cos^2 \frac{1}{2}b' + \cos^2 \frac{1}{2}c - 1}{2 \cos \frac{1}{2}a' \cos \frac{1}{2}b' \cos \frac{1}{2}c}$$

The expression on the right-hand side is the cosine of half the solid angle subtended by the triangle C′BA at the centre of the sphere (*see* M'Clelland and

Preston, 1897, Part II, Ch. 7, p. 50, Ex. 1). Since the Poincaré sphere has unit radius, we arrive at the following unexpected geometrical result.

V. When a beam of polarisation C is decomposed into two beams in the states of polarisation A and B respectively, the phase difference δ between these beams is given by

$$|\,\delta\,| = \pi - \tfrac{1}{2}\,|\,E'\,| \tag{5 a}$$

where the angle $|\,E'\,|$ is numerically equal to the area of the triangle C′BA colunar to ABC. (E′ is also the spherical excess of the triangle C′BA, *i.e.*, the excess of the sum of its three angles over π.)

Hitherto we have fixed the state of zero phase difference between two non-orthogonally polarised beams by the criterion that the resultant intensity produced by mutual interference should be a maximum. We can equally well use the fact (shown by relation 5 *a*) that *the beams in the state of polarisation A and B will have zero phase difference when the resultant state of polarisation C* (produced by their mutual interference) *lies on the arc directly joining A and B* (for then the colunar triangle will have the area of a hemisphere). On the other hand, the beams will be opposed in phase ($\delta = \pm\pi$), when the resultant state of polarisation C lies on the *greater* segment of the great circle through A and B (colunar triangle has zero area).

The relation (5 *a*) does not give the sign of δ — the phase advance of vibration 1 over vibration 2. We shall now resolve this ambiguity. It is clear (from considerations of continuity) that the sign of δ remains the same for all positions of the resultant polarisation C lying on any one side of the great circle AB, the magnitude of δ being between 0 and π; and that δ changes sign (without discontinuity) as the resultant polarisation C crosses from one side of the great circle AB to the other. (This is forced by the physical requirement that the addition of the beams 1 and 2 with specific intensities and a specific phase relationship should lead unambiguously to a unique state of polarisation C of the resultant beam.) We can now show that the phase advance of vibration 1 over 2 will be positive if (as is drawn in Fig. 2) the point C appears to the left of AB as we proceed from A to B on the surface of the sphere. To prove this proposition it is sufficient to show that this rule of signs is true for a *particular* pair of non-orthogonal states A and B. (For it can then be shown to hold for an adjacent pair of points A and B, from continuity arguments, and hence for *any* pair of states A and B.) As a particular case which proves the general rule, we may see in Fig. 3 that if the *linear* vibration A has a positive advance of phase over the *linear* vibration B, the resultant vibration will be left-elliptic. The results of the discussion of this paragraph can also be summarised by using the sign convention that the area E′ of the triangle C′BA be counted positive only when the sequence of points C′BA are described in a counter-clockwise sense on the surface of the sphere. We can then write

$$\delta = \pi - \tfrac{1}{2}E' \tag{5 b}$$

§ 5. The Composition of Non-Orthogonally Polarised Pencils

When two coherent beams of intensities I_1 and I_2 (in the states of polarisation A and B respectively) are combined, the resultant state of polarisation C may be specified by the angular distances b and a of the point C from the points A and B (*see* Fig. 2). According to (3) these are given by

$$\sin^2 \tfrac{1}{2}a = \frac{I_1}{I} \cdot \sin^2 \tfrac{1}{2}c; \qquad \sin^2 \tfrac{1}{2}b = \frac{I_2}{I} \cdot \sin^2 \tfrac{1}{2}c \tag{6}$$

where I is the resultant intensity given by (1). On proceeding from A to B the point C appears to the left or right according as δ is positive or negative. An alternative method of finding the state C will be given in Part II.

The following geometrical facts are useful for qualitatively locating the state C. When the phase difference between the two beams is altered without altering the ratio of their intensities, the state C moves along the locus $\sin^2 \tfrac{1}{2}a/\sin^2 \tfrac{1}{2}b$ = constant; this is a small circle (with its centre on the great circle through AB) which cuts the arc AB internally and externally according to this ratio (*see* M'Clelland and Preston, *loc. cit.*, Part I, Ch. III, p. 66, Ex. 1). On the other hand, when the ratio of the intensities of the beams is altered without altering their phase difference, the state C moves along the locus, E′ = constant; this is a small circle passing through A and B with its centre on the great circle which is the perpendicular bisector of the arc AB (M'Clelland and Preston, *loc. cit.*, Part II, § 101). The point C is thus determined by the intersection of these two families of small circles.

§ 6. The Composition of Two Oppositely Polarised Beams

When a polarised beam is split into two orthogonally polarised components, A and A′, the fundamental property of the Poincaré sphere enunciated in § 2, I, gives us information regarding the intensities of these components but tells us nothing about their relative phase relationship. When we wish to enquire into the latter, we run into the apparent difficulty that the state of relative phase between the two beams which could be taken as the standard or 'zero' with respect to which any additional phase differences could be measured, cannot be defined as in § 3 by their interference properties—because there is no such interference to talk of. We can however avoid the difficulty by restricting ourselves to the following query which alone is of practical importance (*see* Fig. 1).

If, after decomposing a beam of polarisation C_0 into two oppositely polarised beams of polarisation A and A′ respectively, we retard the phase of the A′-component by an amount Δ (relative to the other), and also alter their intensities in any specified manner, what will be the resultant state of polarisation C?

To find the distance of C from A (*see* Fig. 1) constitutes no problem; for, from the first theorem (§ 2, I) itself we see that *the ratio of the* (final) *intensities of the A′ and A beams must be equal to* $\tan^2 \frac{1}{2}CA$. Our question therefore really concerns the magnitude of the angle CAC_0—regarding which we shall prove the following proposition.

VI. The angle CAC_0 is equal to the phase retardation Δ introduced.

It is remarkable that this second important property of oppositely polarised vibrations follows in the ultimate analysis as a consequence of the first fundamental property itself—for we shall prove it is a limiting case of the properties of two non-orthogonally polarised vibrations as they tend towards states of opposite polarisation.

For convenience let us regard the initial and final states of polarisation C_0 and C as given (*see* Fig. 2). We first decompose the beam of polarisation C_0 into two beams of polarisation A and B respectively, where B is chosen such that the arc AB contains C_0. By introducing a phase retardation δ between these beams of polarisation A and B, altering their relative intensities suitably and then compounding them, we can produce a beam of polarisation C. According to § 4, V, we have $\delta = \pi - \frac{1}{2}E'$, where E' is the area of the triangle C′BA. We wish to find the limit towards which δ tends, as the state of polarisation B tends towards the state A′ opposite to A. As the point B moves towards A′ and ultimately coalesces with it, the area of the triangle C′BA obviously becomes equal to the area of the *lune* enclosed between the great circular arc AC_0A' and $AC'A'$, this area being $2\angle C_0AC'$. Hence we have

$$\Delta = \pi - \angle C_0AC' = C\hat{A}C_0$$

thus proving the required proposition. It is clear from (5 *b*) that the angle CAC_0 must be measured positive in the counter-clockwise sense as indicated in Fig. 1, in order that it may have the same sign as Δ (which is the amount by which the A-component is *advanced* in phase).

The particular property of the Poincaré sphere which has led to its extensive application in tracing the passage of polarised light through transparent birefringent media follows as a corollary of Theorem VI above. On passage through any plate of such a medium, the emergent state of polarisation C can be obtained from the incident state C_0 by a *rotation* of the sphere by an anticlockwise angle Δ about the axis AA′, where A represents the state of elliptic polarisation of the faster of the two orthogonally polarised waves. (The author has not come across a general proof of this much-used property of the Poincaré sphere in any of the references quoted.)

A second corollary of the proposition VI is the following; when the ratio of the intensities of the orthogonally polarised beams is altered without altering their phase relationship, the locus of the resultant state of polarisation C is the great circular arc ACA′.

§ 7. Definition of the 'Phase Difference' Between Oppositely Polarised Vibrations

It will be a great convenience in connection with a later discussion to set up an arbitrary standard with respect to which the relative phase relationship between two orthogonal vibrations may be measured.

When two orthogonal linear vibrations of equal intensity combine to yield a linear vibration bisecting the right angle included between the directions OX and OX′ of their vibrations, we customarily say that the linear vibrations are in phase. (There will be no ambiguity regarding whether the vibrations are to be regarded in phase or opposed in phase, if we choose both the radii vectors OX and OX′ within the interval, $\pi/2 \geqslant \theta > -\pi/2$, with respect to a fixed radius vector Or on the wave-front.)

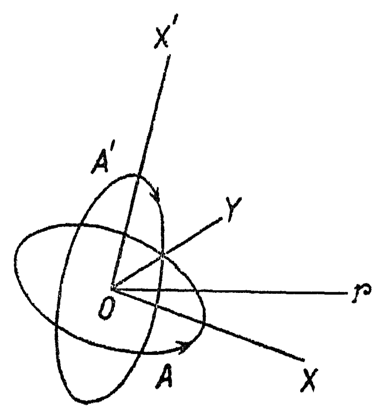

Fig. 4

In conformity with this, we may define two orthogonal elliptic vibrations of equal intensity as being in the same phase when they combine to yield a linear vibration OY bisecting the right angle included between the directions OX and OX′ of their major axes (*see* Fig. 4). The radii vectors OX and OX′ are taken parallel to the major axes of the left- and right-elliptic vibrations respectively and are chosen such that OXX′Z forms a right-handed system—OZ being the direction of propagation. In terms of the Poincaré representation (*see* Fig. 1), the above definition has the following significance. The states of polarisation of two oppositely polarised vibrations are represented by the points A and A′ on the upper and lower hemisphere respectively, while the linear vibration OY is represented by a point Y. The point Y is one of the two opposite points on the equator at an angular distance of $\pi/2$ from both A and A′; on proceeding from A to Y the upper pole appears to the left, as indicated in the figure. The A- and A′-components of the vibration Y (and hence of any vibration represented by a point on the great circular arc AYA′) are defined to be in the same phase.

The case of opposite circular vibration is not covered in the above definition. We may define two such vibrations of equal intensity to be in the same phase when they yield a linear vibration OY parallel to a fixed radius vector Or on the wave-front.

The phase advance of one polarised vibration A over an orthogonally polarised vibration A′ is then equal to the angle CAY (measured positive in the counter-clockwise sense), where C represents the resultant state of vibration obtained on compounding the two. Henceforward we may speak of any one polarised beam as having a definite phase advance δ over another coherent polarised beam without implying that the two beams are non-orthogonally polarised.

§ 8. INTERFERENCE OF THE COMPONENTS OF TWO POLARISED BEAMS TRANSMITTED BY AN ANALYSER

It is well known that light of any arbitrary elliptic polarisation C′ may be extinguished by means of an appliance consisting of a suitably oriented quarter-wave plate followed by a linear analyser at the proper setting; when light of the opposite polarisation C is incident on the same appliance the *intensity* of the transmitted light will be equal to that of the incident as may be directly shown (Stokes, 1901), without using any property of the Poincaré sphere. Any appliance having both the above properties will be referred to as an *analyser C*.

Since light of any other polarisation P may be decomposed into two coherent beams of polarisation C and C′, the intensity transmitted by an analyser C will be equal to the C-component of the incident beam. If we assume the fundamental Theorem I, § 2, it follows that an analyser C transmits a fraction $\cos^2 \frac{1}{2}PC$ of the intensity of light of polarisation P. Conversely, one of the simplest proofs of the fundamental Theorem I, lies in the direct analytical proofs of this property of an analyser given by Fano (1949) and also by Ramachandran and Ramaseshan (1952).

As was pointed out in the Introduction, the problem to be now discussed arises when we wish to consider the phenomena exhibited by a plate of an absorbing biaxial crystal when kept between a polariser and an analyser. Emerging from the crystal plate along any particular direction will be a coherent mixture of two beams 1 and 2. Let their states of polarisation be A and B, their intensities I_1 and I_2, and let the phase advance of the first beam over the second be δ. (The state of polarisation of the resultant beam will not be required here and will *not* be denoted by C.)

In order to compute the intensity transmitted by an analyser C, we first resolve each of the two vibrations into the opposite states of polarisation C and C′ (*see* Fig. 2). The C-components of the beams of polarisation A and B will have intensities $I_1 \cos^2 \frac{1}{2}b$ and $I_2 \cos^2 \frac{1}{2}a$ respectively (§ 2, I). Since these components will have some definite phase difference δ', say, they can interfere. Hence the resultant beam obtained by combining the beams 1 and 2 will have a C-component whose intensity I_C is given by

$$I_C = I_1 \cos^2 \tfrac{1}{2}b + I_2 \cos^2 \tfrac{1}{2}a + 2\sqrt{I_1 I_2} \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \delta' \qquad (7)$$

Similarly the C′-components of the beams 1 and 2 will have intensities $I_1 \sin^2 \frac{1}{2}b$ and $I_2 \sin^2 \frac{1}{2}a$ respectively, and a definite phase difference δ'', say. Hence the C′-component of the resultant beam will have an intensity $I_{C'}$, given by

$$I_{C'} = I_1 \sin^2 \tfrac{1}{2}b + I_2 \sin^2 \tfrac{1}{2}a + 2\sqrt{I_1 I_2} \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \delta'' \qquad (8)$$

Since an analyser transmits only the intensity I_C it remains to determine δ' in terms of the phase difference between the beams 1 and 2, and the analyser position C.

Now δ' represents the phase advance of the C-component of the beam of polarisation A over the C-component of the beam of polarisation B; while δ'' is the phase advance of the C′-component of the beam of polarisation A over the C′-component of the beam of polarisation B. Hence *it follows from a consideration of VI*, § 6, *that* $\delta'' - \delta' = \pm \hat{C}$ (where the positive sign has to be taken if C lies on the side of AB shown in Fig. 2). Furthermore, the intensity I of the resultant beam obtained on compounding 1 and 2 is equal to the sum of the intensities of the C- and C′-components (§ 2, II).
Hence adding (7) and (8)

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\{\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \delta' + \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos (\delta' \pm C)\}$$

Expanding cos $(\delta' \pm C)$, it can be shown by applying the standard expressions for the spherical excess of a triangle (M'Clelland and Preston, *loc. cit.*, Part II, p. 37, Art. 104) that the above relation reduces to

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \tfrac{1}{2}c \cos (\delta' + \tfrac{1}{2}E)$$

where E represents the area of the triangle ABC itself (counted positive or negative according as the sequence of points A, B, C describe the periphery of the triangle in the counter-clockwise or clockwise sense). The intensity I is also given directly in terms of the phase difference δ between the beams 1 and 2 by the expression (1) of § 3. Comparing the two we get the value of δ' to be substituted in (7):

$$\delta' = \delta - \tfrac{1}{2}E \tag{9}$$

VII. Hence when a mixture of two coherent beams of intensities I_1 and I_2 in the states of polarisation A and B respectively, is incident on an analyser C, the transmitted intensity I_C is given by

$$I_C = I_1 \cos^2 \tfrac{1}{2}b + I_2 \cos^2 \tfrac{1}{2}a + 2\sqrt{I_1 I_2} \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos (\delta - \tfrac{1}{2}E) \tag{10}$$

where δ is the phase advance of the beam of polarisation A over the other, *a*, *b* and *c* being the angular sparations BC, CA and AB respectively and E the area of the triangle ABC.

It is to be noted that the above result must also hold in the limiting case when the two beams incident on the analyser are in the orthogonal states of polarisation A and A'. In this case, if AYA' be the great circular arc (Fig. 1) with respect to which the phase difference δ is measured (ref. §7), then E (which now becomes the area of the lune AYA'CA) is equal to twice the angle CAY. The expression (9) may in this case be further simplified by the substitution $\cos \tfrac{1}{2}a = \sin \tfrac{1}{2}b$.

§ 9. THE ADDITION OF *n* COHERENT BEAMS

Suppose a mixture of 3 coherent beams of polarisation A_1, A_2, A_3 is incident on an analyser C. Let $2\theta_i$ denote the length of the arc CA_i, δ_{ij} denote the phase *lag* of the beam *i* over the beam *j*, and E_{ij} be the area of the triangle A_iCA_j. We have obviously

$$\delta_{ij} = -\delta_{ji}; \qquad E_{ij} = -E_{ji} \tag{11}$$

The C-component of the first vibration may be written as $\sqrt{I_1} \cos \theta_1 \, \mathbf{c}\, e^{i\omega t}$ where $\mathbf{c}\, e^{i\omega t}$ is a vibration of unit intensity in the state of polarisation C†.

† The components of the vector **c** are the complex amplitudes of the components of the elliptic vibration.

If $A\,c\,e^{i\omega t}$ denote the C-component of the resultant vibration obtained on compounding the three vibrations, then according to (9) we will have,

$$A = \sqrt{I_1}\cos\theta_1 + \sqrt{I_2}\cos\theta_2 \exp i(\delta_{12} - \tfrac{1}{2}E_{12})$$
$$+ \sqrt{I_3}\cos\theta_3 \exp i(\delta_{13} - \tfrac{1}{2}E_{13})$$

Since the phase lag of the C-component of the second beam over the C-component of the third is according to (8), given by $(\delta_{23} - \frac{1}{2}E_{23})$ we have

$$-(\delta_{23} - \tfrac{1}{2}E_{23}) = (\delta_{12} - \tfrac{1}{2}E_{12}) - (\delta_{13} - \tfrac{1}{2}E_{13}) \qquad (12)$$

with similar expressions connecting all the δ_{ij}. Hence the intensity I_C transmitted by an analyser C being equal to AA^*, will be given by

$$I_C = \Sigma I_i \cos^2\theta_i + \sum_{i \neq j} \sqrt{I_i I_j}\cos\theta_i \cos\theta_j \cos(\delta_{ij} - \tfrac{1}{2}E_{ij}) \qquad (13)$$

The resultant intensity I of the beam obtained on compounding the three beams will be equal to the sum of the intensities I_C and $I_{C'}$, where $I_{C'}$ is the intensity transmitted by the orthogonal analyser C′. The resultant intensity can in this manner be shown to be given by

$$I = \Sigma I_i + \sum_{i \neq j} \sqrt{I_i I_j}\cos\tfrac{1}{2}c_{ij}\cos\delta_{ij} \qquad (14)$$

It is obvious that the same argument holds when there are a mixture of n coherent beams; the formula (13) gives the intensity transmitted by an analyser C, while (14) gives the resultant intensity. The state of polarisation of the resultant beam will be deduced incidentally in Part II. We may here merely note that by deducing the intensity transmitted by any analyser C we have a method of deducing the resultant polarisation of the beam: for example, we could find the particular analyser for which the transmitted intensity is zero.

In conclusion the author wishes to express his deep sense of indebtedness to Professor Sir C. V. Raman, F.R.S., N.L., without whose encouragement this work could not have been written.

§ 10. Summary

The superposition of two coherent beams in different states of elliptic polarisation is discussed in a general manner. If A and B represent the states of polarisation of the given beams on the Poincaré sphere, and C that of the resultant beam, the result is simply expressed in terms of the sides, a, b, c of the spherical triangle ABC. The intensity I of the resultant beam is given by:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\tfrac{1}{2}c\cos\delta;$$

the extent of mutual interference thus varies from a maximum for identically polarised beams ($c = 0$), to zero for oppositely polarised beams ($c = \pi$). The state of polarisation C of the resultant beam is located by $\sin^2 \frac{1}{2}a = (I_1/I) \sin^2 \frac{1}{2}c$ and $\sin^2 \frac{1}{2}b = (I_2/I) \sin^2 \frac{1}{2}c$. The 'phase difference' δ is equal to the supplement of half the area of the triangle C′BA (where C′ is the point diametrically opposite to C). These results also apply to the converse problem of the decomposition of a polarised beam into two others.

The interference of two coherent beams after resolution into the same state of elliptic polarisation by an elliptic analyser or compensator is discussed; as also the interference (direct, *and* after resolution by an analyser) of n coherent pencils in different states of polarisation.

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