ACHROMATIC COMBINATIONS OF BIREFRINGENT PLATES

Part I. An Achromatic Circular Polarizer

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1. INTRODUCTION

WHEN polarised light of any form is incident on a birefringent plate, the phase retardation δ introduced between the two waves during their passage through the plate, is far from being the same for all wave-lengths; in fact, being given by $\delta = (2\pi/\lambda)(\mu_1 - \mu_2) t$, this retardation of phase varies almost inversely as the wave-length λ, that is, if the birefringence $(\mu_1 - \mu_2)$ does not disperse notably with wave-length. So much so, that what acts as a quarter-wave retardation plate for the deep red end of the spectrum will, for the wave-lengths in the deep violet, behave practically as a half-wave plate.

The question of the achromatisation of devices in this field has engaged the attention of several workers. References to their investigations may be found cited in the paper entitled, " *Réalisation d'un quart d'onde quasi-achromatique par juxtaposition de deux lames crystallines de même nature* " by Destriau and Prouteau[1]; the two plates referred to, do not, of course, have their principal planes parallel, and the 'compound plate', comprising the two superposed plates, can transform incident circularly polarised light to plane polarised light vibrating at a certain azimuth, or *vice-versa*. But the device described by these authors would more properly be called an achromatic circular polarizer (or analyser); to call it an achromatic quarter-wave plate would be incorrect since the combination does not have the usual at ributes of a quarter-wave plate—it cannot, for example, be used for the analysis of elliptically polarised light in the usual manner of an ordinary $\lambda/4$ retardation plate.

Nevertheless it is as a circular polarizer or analyser that a quarter-wave plate is often used—as when it is inserted with its principal planes at an angle of 45° to those of a nicol, in a petrographic microscope. And we shall reserve for the second part, the problem of superposing birefringent plates in such a manner that the combination as a whole behaves as an achromatic quarter-wave plate. In this paper we shall describe a circular polarizer which is expected to have a much higher degree of achromatism than the one discussed in the paper quoted above. The description of the device may be found at the end of the next section; its applicability is not limited to the visible. The birefringent plates in our discussion are all of the same material; the dispersion of the birefringence of this material is not assumed to be negligible—though such an assumption would indeed be justified for plates of muscovite mica, at least for the visible and ultra-violet wave-lengths.[2]

2. The Achromatic Circular Polarizer

The Poincaré sphere* lends itself very conveniently for the theoretical discussion of the two problems at hand—especially the one to be treated later. (As is well known, any point on the surface of this sphere, of latitude 2ω and longitude $2l$, represents an elliptic vibration of ellipticity $|\tan \omega|$ the major axis of which makes an angle l with a fixed reference direction. Passage through a birefringent plate of retardation δ, the orientation of whose principal planes are given by O and O′, is equivalent to the operation of rotating the sphere by an angle δ about the equatorial diameter OO′.)

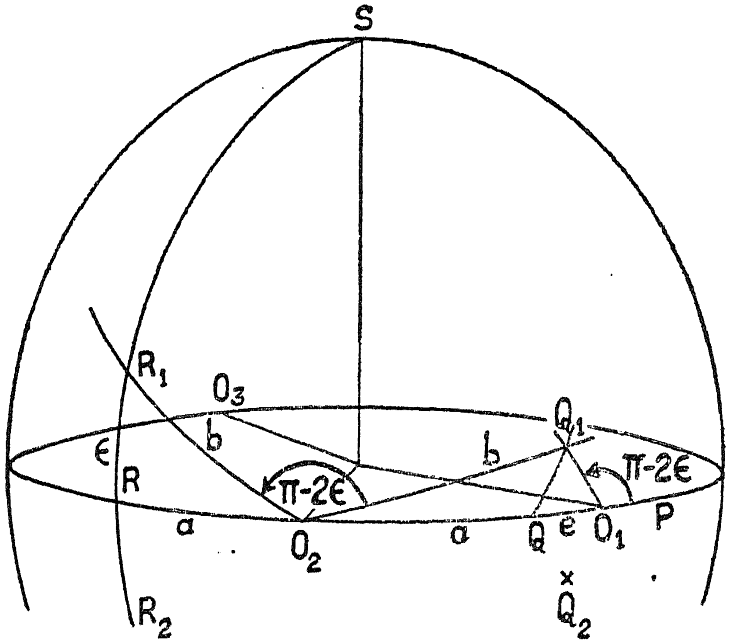

Fig. 1

A quarter-wave plate with its fast vibration direction in the orientation O_3 (Fig. 1) can transform the incident linear vibration R—inclined at 45° to the principal planes—to circularly polarised light represented by the pole S—but only for the wave-length λ for which its retardation is exactly

* Expositions in English, of the properties of the Poincaré sphere are apparently not so common in text-books, but may be found in numerous recent articles: see *e.g.*, Ramachandran and Ramaseshan, *J. Opt. Soc. Am.*, 1952, **42**, 49.

$\pi/2$. For the wave-lengths λ_1 and λ_2 on either side of λ, for which the respective retardations are $\pi/2 - \epsilon$ and $\pi/2 + \epsilon$, the emergent light will be circularly polarised only if the incident vibrations are given respectively by R_1 and R_2; these two points lie on the meridian SR and are equidistant from R (arc $R_1R = \epsilon$). Thus if the ellipticity of the incident vibrations were dispersed roughly along R_1R_2 in a proper fashion, the quarter-wave plate would transform the vibrations for all the wave-lengths to approximately circular polarised light.

The required dispersion of the ellipticity of the vibrations incident on the quarter-wave plate, it may be seen, can be effected by allowing plane polarised light P to first pass successively through two half-wave plates, the orientation of whose fast axes are given by O_1 and O_2 respectively and are to be determined. In order that the light emerging from the third retardation plate (*i.e.*, the quarter-wave plate) should be exactly circularly polarised for all the three wave-lengths λ_1, λ and λ_2, the state of polarization of the light incident on the second half-wave plate for these three wave-lengths should be given by Q_1, Q and Q_2 respectively—got by the following construction. The point Q is marked off on the equator such that $O_2Q = O_2R$ ($= a$ say); the arc O_2Q_1 is drawn such that $R_1O_2Q_1 = \pi - 2\epsilon$, and $O_2Q_1 = O_2R_1$ ($= b$ say); Q_2 is a point symmetrically placed with respect to Q_1 on the lower hemisphere. The first half-wave plate has to be oriented in a position depending on the quantity a which determines the orientation of the second; for we must have $O_1Q_1 = O_1Q_2 = O_1Q$ ($= e$ say); and further we must choose a such that $P\hat{O}_1Q_1 = \pi - 2\epsilon$. If the plane of the incident vibration P be now set such that $O_1P = e$, then the light emerging from the combination will be circularly polarized for the three wave-lengths λ_1, λ and λ_2 for which the retardations of a quarter-wave plate would be $\pi/2 - \epsilon$, $\pi/2$, and $\pi/2 + \epsilon$ respectively; ϵ is an arbitrarily chosen parameter on which will depend the two opposing characteristics of the combination: the range and the degree of its achromatism.

By the use of spherical trigonometry we shall get an equation for a involving the parameter ϵ. Thus, from the equilateral triangle Q_1O_1Q, denoting arc Q_1Q by c

$$\cos 2\epsilon = -\cos^2 Q_1QO_1 + \sin^2 Q_1QO_1 \cos c$$

or

$$1 + \cos 2\epsilon = 2\sin^2 Q_1QO_1 \cos^2 c/2$$

From the triangle Q_1O_2Q we have, writing $Q_1O_2Q = C$,

$$\frac{\sin c}{\sin C} = \frac{\sin b}{\sin Q_1QO_1}$$

i.e.,

$$\sin Q_1QO_1 \cos \frac{c}{2} = \frac{\sin b \sin C}{2 \sin \frac{c}{2}}$$

We thus obtain

$$\sin^2 b \sin^2 C = 2 \sin^2 \frac{c}{2} (1 + \cos 2\epsilon) \quad (1)$$

We shall express the quantities on the left-hand side in terms of a. Denoting R_1O_2R by C′

$$\sin C = \sin (2\epsilon - C') = \sin 2\epsilon \cos C' - \cos 2\epsilon \sin C' \quad (2)$$

From the right-angled triangle R_1O_2R

$$\sin C' = \frac{\sin \epsilon}{\sin b}$$

and

$$\cos C' = \frac{\cos \epsilon - \cos a \cos b}{\sin a \sin b} = \cos \epsilon \cdot \frac{\sin a}{\sin b}$$

since

$$\cos b = \cos \epsilon \cos a \quad (3)$$

Substituting the expressions obtained for cos C′ and sin C′ in (2)

$$\sin C = \frac{\sin \epsilon}{\sin b} (2 \cos^2 \epsilon \sin a - \cos 2\epsilon)$$

Introducing this in (1)

$$(2 \cos^2 \epsilon \sin a - \cos 2\epsilon)^2 = 2 \cot^2 \epsilon (1 - \cos c) \quad (4)$$

It remains to express cos c in terms of a. From triangle Q_1QO_2

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$\cos C = \cos (2\epsilon - C') = \cos 2\epsilon \cos C' + \sin 2\epsilon \sin C'$$

$$= \frac{\cos \epsilon}{\sin b} (\cos 2\epsilon \sin a + 2 \sin^2 \epsilon)$$

Usin_ this relation, together with (3), the expression for cos c becomes,

$$\cos c = \cos \epsilon (1 + 2 \sin^2 \epsilon \sin a - 2 \sin^2 \epsilon \sin^2 a)$$

Substituting for cos c in (4), and rearranging, we finally get an equation in sin a of the form:

$$A \sin^2 a + B \sin a + C = 0 \quad (I)$$

where

$$A = 4\cos^3\epsilon\,(1 - \cos\epsilon)$$

$$B = 4\cos^2\epsilon\,(\cos 2\epsilon - \cos\epsilon)$$

$$C = 2\cot^2\epsilon\,(1 - \cos\epsilon) - \cos^2 2\epsilon$$

An expression for e may be obtained from triangle Q_1O_1Q

$$\cos c = \cos^2 e + \sin^2 e \cos 2\epsilon$$

or

$$\sin^2 e = \frac{1 - \cos c}{1 - \cos 2\epsilon}$$

Using (4),

$$\sin e = \frac{2\cos^2\epsilon \sin a - \cos 2\epsilon}{2\cos\epsilon} \qquad \text{(II)}$$

The combination can produce exactly circularly polarized light for three wave-lengths. If we neglect the dispersion of the birefringence, the two extreme wave-lengths will obviously be $(1 + f)\,\lambda$ and $(1 - f)\,\lambda$—where we have denoted the ratio $\epsilon : \pi/2$ by f; the achromatism of the combination may however be considered to extend up to two wave-lengths *outside* this interval, for which the deviation from circular polarisation is roughly the same as the maximum deviation attained within this interval—which latter may be expected to occur in the neighbourhood of the wave-lengths $(1 \pm \cdot 5f)\,\lambda$. Thus we may with some arbitrariness, consider the combination to be achromatic within the range $(1 \pm 1\cdot 25f)\,\lambda$.

Choosing $2\epsilon = 47°$ (which means $1\cdot 25f = \cdot 325$), equation (I) when solved gives $\sin a = \cdot 6642$ or $a = 41°\,37'$.

The corresponding value of e obtained from (II) is $e = 13°\,43'$.

On the other hand, if we choose $2\epsilon = 36°$ (which means $1\cdot 25f = \cdot 25$) we obtain

$$a = 43°\,8'; \; e = 13°.$$

Thus to summarize our results, an achromatic arrangement for producing circularly polarised light is obtained by allowing parallel plane polarised light to pass normally through a combination of three superposed plates of the same material; the first two should be half-wave plates, and the last, a quarter-wave plate for the wave-length λ in the centre of the spectral range to be covered. Let θ be the angle by which the fast vibration direction of the first plate is turned with respect to the azimuth of the incident vibration; θ_1, the angle by which the fast axis of the second plate is turned with respect to the first; and θ_2 the corresponding angle between the fast axes of the third and second plates—all the angles being measured in the same sense. The set of values

$$\theta_2 = \tfrac{1}{2}(\pi/2 + a) = 65^\circ\, 49'; \;\; \theta_1 = \tfrac{1}{2}(a + e) = 27^\circ\, 40'; \;\; \theta = \tfrac{1}{2}\, e = 6^\circ 52'$$

can be used for covering the range from $1 \cdot 325\ \lambda$ to $\cdot 675\ \lambda$ and is therefore suited for covering the entire visible spectrum if we choose $\lambda = 6000$ Å. A greater degree of achromatism can be attained at the expense of restricting the range to the major portion of the visible. Thus the set of values

$$\theta_2 = 66^\circ\, 34'; \;\; \theta_1 = 28^\circ\, 4'; \;\; \theta = 6^\circ\, 30'$$

can be used for covering the range from $1 \cdot 25\ \lambda$ to $\cdot 75\ \lambda$. If the dispersion of the birefringence of the material used is not negligible, the values given above for the range over which the combination may be considered achromatic would have to be altered.

On turning the polarizing nicol to a perpendicular position, circularly polarized light of the opposite sense is produced. In fact in passage through any succession of 'elliptically birefringent' plates, two orthogonal states of the incident polarization will correspond to two orthogonal states of the emergent polarization. This is a consequence of a more general fact; the angular separation on the Poincaré sphere of two possible states of the incident polarization will be equal to the angular separation of the two corresponding states of the emergent polarization, since angular relationships remain invariant under any number of rotations. Thus if the nicol had been turned by 45°, the emergent wave-lengths would all have been practically plane polarized, but with the azimuths of the vibrations dispersed—an interesting illustration of the fact that the combination does not have the properties of a quarter-wave plate.

3. Experimental Verification

Two half-wave plates and a quarter-wave plate, all prepared of mica, were cemented together (using copal varnish) with their principal planes at inclinations slightly different from the first set of values given above—in order to cover an even wider range. The combination was laid on a mirror. With the aid of another mirror, white light from a point source was made to pass normally through a polaroid and the achromatic combination, and then back again—after which it reached the eye. As the polaroid is rotated a position is reached where the image of the source is completely extinguished; this is the position where a single passage through the compound plate would give circularly polarized light, while the double passage gives plane polarized light which is crossed out by the polarizer itself. When the experiment is repeated with a single quarter-wave plate the image is never completely extinguished, but is highly coloured.

The two $\lambda/2$ retardation plates may first be cemented together at the proper angle θ_1, using the fact that the two together will rotate the plane of polarization of *any* incident linear vibration (of wave-length λ) by an angle $\pi - 2\theta_1$; (this can be proved by a construction given in Part II); the quarter-wave plate may then be cemented and adjusted, while the cement is still wet, to the position of best achromatism—as determined by the test given above.

SUMMARY

Circularly polarised light is obtained by superposing two half-wave plates and one quarter-wave plate, all of the same material, such that the fast vibration-directions of the successive plates make specific angles θ_1, θ_2, θ_3 with the azimuth of the linear vibration incident on the first plate. The required range of achromatism determines the optimum values of the angles. Thus, using mica retardation plates prepared for Na 5890, the range from 4000 Å to 7800 Å is covered with $\theta_1 = 6° 52'$, $\theta_2 = 34° 32'$, $\theta_3 = 100° 20'$; while the range 4400 Å to 7400 Å can be covered with superior achromatism by taking $\theta_1 = 6° 30'$, $\theta_2 = 34° 34'$, $\theta_3 = 101° 8'$.

REFERENCES

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2. Mathieu .. *Bull. Soc. Franc. Min.*, 1934, **57**, 233.