Nonequilibrium growth problems

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We discuss the features of nonequilibrium growth problems, their scaling description and their differences from equilibrium problems. The emphasis is on the Kardar–Parisi–Zhang equation and the renormalization group point of view. Some of the recent developments along these lines are mentioned.

HOW to characterize the degree of roughness of a surface as it grows and how the roughness varies in time have evolved into an important topic due to diverse interests in physics, biology, chemistry and in technological applications. One crucial aspect of these nonequilibrium growth processes is the scale invariance of surface fluctuations similar to the scale invariance observed in equilibrium critical point phenomena. Although different kinds of growths may be governed by distinct natural processes, they share a common feature that the surface, crudely speaking, looks similar under any magnification and at various times. This nonequilibrium generalization of scaling involving space and time (called 'dynamic scaling') makes this subject of growth problems important in statistical mechanics.

Growth problems are both of near-equilibrium and nonequilibrium varieties. Therefore, they provide us with a fertile ground to study the differences and the extra features that might emerge in a nonequilibrium situation[1,2]. Take for example the case of crystal growth. In equilibrium, entropic contributions generally lead to a rough or fluctuating surface, an effect called thermal roughening, but, for crystals, because of the lattice periodicity a roughening transition from a smooth to rough surface occurs at some temperature. The nature of the growth of a crystal close to equilibrium expectedly depends on whether the surface is smooth or rough. One can also think of a crystal growth process which is far away from equilibrium by subjecting it to an external drive, for instance by random deposition of particles on the surface. The roughening that occurs in the nonequilibrium case is called kinetic roughening. Is the nature of the surface any different in kinetic roughening? Crystals are definitely not the only example of growth processes; some other examples of such nonequilibrium growths would be the growth of bacterial colonies in a petri dish, sedimentation of colloids in a drop, the formation of clouds in the upper atmosphere, and so on. Note the large variation of length scales of these problems. In many such examples, it is difficult if not impossible to think of an equilibrium counterpart.

Scale invariance in interface fluctuations implies that fluctuations look statistically the same when viewed at different length scales. A quantitative measure of the height fluctuation (height measured from an arbitrary base) is provided by the correlation function

$$C(\mathbf{x}, t) = \langle [h(\mathbf{x} + \mathbf{x}_0, t + t_0) - h(\mathbf{x}_0, t_0)]^2 \rangle, \quad (1)$$

where $\mathbf{x}$ and t denote the d-dimensional coordinate on the substrate and time, respectively. The averaging in eq. (1) is over all $\mathbf{x}_0$, and, by definition, $C(\mathbf{x}, t)$ is independent of the choice of the arbitrary base. In simple language, scale invariance then means that when the system is, say, amplified by a scaling $x \to bx$ and $t \to b^z t$, the height fluctuations reveal the same features as the original, up to an overall scale factor. Quantitatively, there exists a generalized scaling

$$C(x, t) = b^{-2\chi} C(bx, b^z t), \quad (2)$$

where b is a scale factor, and χ and z are known as the roughening and dynamic exponents which are also universal. As a direct consequence of eq. (2), a scaling form for $C(x, t)$ can be obtained by choosing $b = 1/x$

$$C(x, t) = x^{2\chi} \hat{C}\ (t/x^z), \quad (3)$$

a form that also explains the origin of the name 'dynamic exponent' for z. The power law behaviour (as opposed to say exponential decay) of the correlation function implies absence of any scale, neither in space nor in time. All the underlying length scales required to define the problem dropped out of the leading behaviour in eq. (3). Such a scale invariance is one of the most important features of equilibrium phase transitions and is observed when a parameter, say the temperature, approaches its critical value. However, here there is no special tuning parameter; the scale invariance appears from the interplay of competing processes which in the simplest case can be the surface tension and noise present due to inherent randomness in the growth. There can be, of course, more complex events like a phase transition between surfaces with different roughness but scale invariance (not only of the correlation function but of any physical quantity) is generically preserved in all these surfaces.

It is worth emphasizing the enormous simplification that

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occurs in the scaling description. It is only a very few quantities that define the asymptotic behaviour of the system. Consequently, the idea of studying the universal aspects of growth processes is to classify and characterize the various universality classes as determined by the exponents, e.g. χ and z, the scaling function and if necessary certain other important universal quantities.

At this point, it might be helpful to compare the equilibrium and nonequilibrium cases again. In equilibrium, thanks to thermal energy (or 'random kicks' from a heat reservoir), all configurations of a system are accessible and do occur, but no net flow of probability between any two states is expected (called 'detailed balance'). Consequently, the knowledge of the states (and the energies) of a system allows one to obtain the thermodynamic free energy by summing over the Boltzmann factors $\exp(-E/k_B T)$, where E is the energy of the state, T the temperature and k_B is the Boltzmann constant. In a nonequilibrium situation, either or both of the above two conditions may be violated, and the framework of predicting the properties of a system from free energy is not necessarily available. A dynamic formulation is needed. By assigning a time-dependent probability for the system to be in a configuration at a particular time, one may study the time evolution of the probability. The equilibrium problem can be viewed from a dynamical point also. This description must give back the Boltzmann distribution in the infinite time steady state limit. This is the Fokker–Planck approach. The probabilistic description comes from the ensemble picture where identical copies of the same system exchange energy with the bath independently. An alternative approach which finds easy generalization to the non-equilibrium cases is the Langevin approach where one describes the time evolution of the degrees of freedom, in our example $h(\mathbf{x}, t)$, taking care of the random exchange of energy by a noise. The dynamics we would consider is dissipative so that the system in absence of any noise would tend to a steady state. However, for it to reach the equilibrium Boltzmann distribution in the presence of noise, it is clear that the noise must satisfy certain conditions (Einstein relation) connecting it to the system parameters. The nonequilibrium case does not have any thermodynamic free energy as a guiding light and therefore, there is no requirement to reach the Boltzmann distribution. In the Langevin approach, the noise term can be completely independent. In the equilibrium case, the Langevin equation will be determined by the Hamiltonian or the free energy of the system, but for nonequilibrium cases there might be terms which cannot be obtained from Hamiltonians. Since for $t \to \infty$, the probability distribution for equilibrium cases attains the Boltzmann distribution, the roughness exponent χ is determined even in dynamics by the stationary state while the details of the dynamics is encoded in the dynamic exponent z. In other words, the two exponents χ and z are independent quantities. In the nonequilibrium case, there is no compulsion to reach any predetermined stationary state and therefore the surface roughness is related to the growth, i.e. χ and z need not be independent. We see below that there is in fact a specific relation connecting these two exponents.

The existence of scale invariance and universal exponents implies that as far as the exponents are concerned, the theory should be insensitive to the microscopic details, or, in other words, one may integrate out all the small length scale features. The universal exponents come out as an output of this process of coarse graining of say the Langevin equation, followed by a length rescaling that brings the system back to its original form. The new system will however have different values of the parameters and one can study the flow of these parameters in the long length and time scale limit. This is the basic idea behind the renormalization group (RG). In this approach, the importance of an interaction or a term is judged not by its numerical value but by its relevance. One may start with any physically possible process in dynamics and see how it appears as the length scale or resolution changes. For large length scales, one is left only with the relevant terms that grow with length, and marginal terms that do not change; the irrelevant terms or interactions that decay with length scale automatically drop out from the theory. The exponents are determined at the fixed points of the flows in the parameter space. These fixed points, which remain invariant under renormalization, characterize the macroscopic or asymptotic behaviour of the system. Clearly from this viewpoint of RG, one can explain why the microscopic details can be ignored and how the idea of 'universality' emerges. All systems whose dynamical behaviour would flow to the same fixed point under RG transformation will have identical scaling behaviour. The various universality classes can then be associated with the various fixed points of RG transformations, and phase transitions or criticality with unstable fixed points or special flows in the parameter space. An RG approach therefore seems rather natural and well suited for studying any scale invariant phenomena in general, and growth problems in particular. Quite expectedly, the modern approach to growth problems is based on these views of RG.

For a quantitative discussion, we consider two simple equations that, from the historical point of view, played a crucial role in the development of the subject in the last two decades.

A simple Langevin equation describing the dynamics of a surface is the Edwards–Wilkinson (EW) equation[3]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(\mathbf{x}, t), \tag{4}$$

where $\mathbf{x}$ represents in general the coordinate on the d-dimensional substrate. ν is the coefficient of the diffusion term trying to smoothen the surface and η is the Langevin noise which tries to roughen the surface[3]. One may add a constant current c to the right hand side, but by going over to a moving frame of reference ($h \to h + ct$) one recovers eq.

(4). The noise here is chosen to have zero mean and short-range correlation as $\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t')\rangle = 2D\delta(\mathbf{x}-\mathbf{x}')\ \delta(t-t')$. One of the important assumptions in this equation is that the surface is single-valued and there are no overhangs. One can solve eq. (4) exactly just by taking the Fourier transform and obtain the exponents $\chi = (2-d)/2$ and $z = 2$. That the dynamic exponent z is 2 follows from the simple fact that the equation involves a first derivative in time but a second derivative in space. The surface is logarithmically rough at $d = 2$. For $d > 2$, fluctuations in the height are bounded and such a surface is more or less flat, better called 'asymptotically flat'. From the growth equation one can also derive the stationary probability distribution for the height $h(\mathbf{x})$ which takes the form of a Boltzmann factor $P(h(\mathbf{x})) \propto \exp[-(\nu/D)\int(\nabla h)^2 \mathrm{d}^d x]$ resembling an equilibrium system at a temperature given by $D = k_B T$. This is the Einstein relation that noise should satisfy to recover equilibrium probability distribution. Conversely, given a hamiltonian of the form $\int(\nabla h)^2 \mathrm{d}^d x$, the equilibrium dynamics will be given by eq. (4) with D determined by the temperature. Nevertheless, if we do not ascribe any thermal meaning to D, eq. (4) is good enough to describe a nonequilibrium dynamic process as well. Such a nonequilibrium growth will have many similarities with equilibrium processes, differing only in the origin of the noise, e.g. the expected symmetry $h \to -h$, with $\langle h\rangle = 0$ in equilibrium will be preserved in the nonequilibrium case also. The growing surface with a correlation $C(\mathbf{x}, t) = |\mathbf{x}|^{(2-d)/2}\ (t/|\mathbf{x}|^2)$ will be similar in both cases for $d > 2$.

A genuine nonequilibrium process will involve breaking the up-down symmetry which in equilibrium follows from detailed balance. It should therefore be represented by a term involving even powers of h. We already saw that a constant current (zeroth power) does not add anything new. Since the origin in space or time or the position of the basal plane should not matter, the first possible term is $(\nabla h)^2$. By looking at the geometry of a rough surface, it is easy to see that such a term implies a lateral growth that would happen if a deposited particle sticks to the first particle it touches on the surface. One gets the Kardar–Parisi–Zhang (KPZ) equation[4]

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x}, t). \qquad (5)$$

As a consequence of its mapping to the noisy Burger's equation, to the statistical mechanics of directed polymer in a random medium and other equilibrium and nonequilibrium systems, the KPZ equation has become a model of quite widespread interest in statistical mechanics. Though we focus on growth problems in this paper, the KPZ equation is also applicable in erosion processes.

Taking cue from the development in understanding the growth phenomenon through the KPZ equation, a vast class of simulational and analytical models have evolved to explain different experimentally observed growth processes. Diverse technical tools ranging from simulations with various dynamical rules to different versions of RG techniques, mode coupling theory, transfer matrix techniques, and scaling arguments have been employed to understand kinetic roughening. In this review we attempt to provide an overview of this phenomenon of roughening of a growing surface. It is almost beyond the scope of this review to describe in detail various models and their experimental relevance. Rather, we focus our attention on a few examples which may broadly represent a few different routes along which research has continued.

The plan of this article is as follows. In the next section we focus on the KPZ equation and its RG description. We also point out the connection of the KPZ equation to some other problems of physics. Next, a more generalized growth mechanism involving nonlocal interactions is presented. Finally, the progress in understanding the roughening and super roughening transitions which appear in a very distinct class of models involving lattice pinning potential is presented.

KPZ equation and more

Let us first look at the origin of the various terms in eqs (4) and (5). In both the equations, the noise term represents random deposition, the fluctuation around the steady value. As already mentioned, a steady current can be removed from the equation by going over to a moving frame. The term involving second derivative of h can represent either of the two processes. It could be a surface tension controlled diffusion process, in which a particle comes to the surface and then does a random walk on the surface to settle at the minimum height position, thereby smoothening the surface. An alternative interpretation would be that there is desorption from the surface and the process is proportional to the chemical potential gradient. The chemical potential of the particles on the surface cannot depend on h or gradient of h because it is independent of the arbitrary base or its tilt. The chemical potential then is related to the second derivative of h. This also has a geometric meaning that $\nabla^2 h$ is related to the local curvature. The larger the curvature, the higher is the chance to desorb because of a lesser number of neighbours. In the KPZ equation, the nonlinear term represents lateral growth. The diffusion-like term can then be thought of either (a) as an alternative that a particle coming to the surface instead of sticking to the first particle it touches, deposits on the surface and then diffuses, or (b) as a random deposition process with desorption. In either case, the noise term tends to roughen the surface, the diffusion term, of whatever origin, smoothens it while the nonlinear term leads to a laterally growing surface. Even if the smoothening linear term is not present, RG or the scaling argument indicates that such a term is generated on a large length scale.

The KPZ equation has a special symmetry not present in the EW case. This is the tilt symmetry (often called Galilean invariance – a misnomer, though, in this context). If we tilt the surface by a small angle, then with a reparametrization $h' = h + \varepsilon \cdot \mathbf{x}$ and $\mathbf{x} = \mathbf{x} + \lambda \varepsilon t'$ and $t = t'$, the equation remains invariant for small ε. This transformation depends only on λ, the coefficient of the nonlinear term and fails for $\lambda = 0$. Since this tilt symmetry is to be maintained no matter at what lengthscale we look at, λ must be an RG invariant.

Let us now perform a length rescaling analysis. Under a change of scale as $x \to bx$, $t \to b^z t$ and $h \to b^\chi h$, KPZ equation transforms as

$$b^{\chi - z} \frac{\partial h}{\partial t} = \nu b^{\chi - 2} \nabla^2 h + \frac{\lambda}{2} b^{2\chi - 2} (\nabla h)^2 + b^{-d/2 - z/2} \eta, \qquad (6)$$

where the noise correlation has been used to obtain the scaling of the noise term. Therefore under this scale transformation different parameters scale as $\nu \to b^{z-2} \nu$, $D \to b^{z-d-2\chi} D$ and $\lambda \to b^{\chi + z - 2} \lambda$. For $\lambda = 0$, the equation remains invariant provided $z = 2$ and $\chi = (2 - d)/2$. These are just the exponents one expects from the EW model. (Such surfaces with anisotropic scaling in different directions like x and h are called self-affine.) Though we cannot predict the exponents from eq. (6) when $\lambda \neq 0$, it does tell us that a small nonlinearity added to the EW equation scales with a scaling dimension $\chi + z - 2$. This term is always relevant in one-dimension, because it scales like $b^{1/2}$. This type of scaling argument also shows that no other integral powers of derivatives of h need be considered in eq. (5) as they are all irrelevant, except $(\partial h/\partial x)^3$ at $d = 1$, which, however, detailed analysis shows to be marginally irrelevant. Based on this analysis, we reach an important conclusion that the nonequilibrium behaviour in one-dimension, and in fact for any dimension below two, would be distinctly different from the equilibrium behaviour. For dimensions greater than two, EW or equilibrium surfaces, as already mentioned, are asymptotically flat with $\chi = 0$, $z = 2$, and so, a small nonlinearity is irrelevant because it will decay with b. In other words, the growth in higher dimensions for small λ would be very similar to equilibrium problems because the EW model is stable with respect to a small perturbation with nonlinearity. The simple scaling argument does not tell us if the nature of the surface changes for large λ for $d > 2$, but an RG analysis shows that it does change. That the nonequilibrium growth is always different in lower dimensions and in higher dimensions (greater than two), and that there will be a dynamic phase transition from an equilibrium-like to a genuine nonequilibrium behaviour, explains the source of excitement in this minimal KPZ equation, in the last two decades.

If the nonlinear parameter λ is to remain an invariant, i.e. independent of b in eq. (6), then $\chi + z = 2$, a relation which need not be satisfied by the equilibrium growth. It is this relation connecting the two exponents of the scaling function of eq. (2) that distinguishes nonequilibrium growth from equilibrium, the former requiring one less exponent than the latter. We wonder if such an exponent relation is generally true for all nonequilibrium systems.

Though we are far away from a complete understanding of all the nuances and details of the KPZ equation, the RG analysis has been very successful in identifying different phases, nature of phase transitions, and, in certain cases, relevant exponents. In brief, the various results obtained from RG analysis are as follows. In one-dimension and for $d < 2$, even a small nonlinearity, as already mentioned, being relevant in the RG sense, leads to new values of roughening and dynamic exponents, and is characterized by a RG fixed point. Beyond $d = 2$, there is a phase transition demarcating two different types of surfaces. A small nonlinearity is irrelevant around EW model and the surface is almost flat with $\chi = 0$ and $z = 2$. A strong nonlinear growth, however, drives the system to a different phase with rougher surface where $\chi \neq 0$. Several aspects of this phase transition can be studied from RG but the strong λ regime is still out of reach, because of the absence of any RG fixed point.

The KPZ equation in $d = 1$ has distinct nonequilibrium behaviour, and the scaling behaviour is the same no matter how small or large λ is. More peculiar is the existence of a stationary probability distribution of the height in one-dimension which is the same as for the linear EW model. This is not just an accident but a consequence of certain subtle relations valid only in one-dimension. We do not go into those issues here. The same stationary distribution implies that the nonlinearity does not affect the stationary state solution, and $\chi = 1/2$. The two models, however, differ in the dynamic exponent which, in the case of KPZ growth, has to satisfy $\chi + z = 2$. This leads to an exact answer $z = 3/2$. Its significance can be grasped if we compare various known cases. For ballistic motion, distance goes linearly with time so that the dynamic exponent is $z = 1$ while for diffusive motion or in quantum mechanics (e.g. a nonrelativistic free quantum particle), $z = 2$ as is also the case for EW. Here is an example where the nonequilibrium nature of the problem leads to a completely new exponent connecting the scaling of space and that of time.

Dynamic renormalization group analysis

A dynamic RG analysis is a more general approach applicable for dynamics which e.g. may be governed by the Langevin equation for the appropriate dynamical variable. For our problem, it is easier to work in Fourier coordinates $\mathbf{q}$, and ω conjugate to space and time. Long distance, long time implies $\mathbf{q} \to 0$ and $\omega \to 0$, and q can be taken as the inverse wavelength at which the height variable is probed. The magnitude of wave vector $\mathbf{q}$ varies from 0 to Λ, where the upper cutoff is determined by the underlying microscopic length scale like lattice spacing or size of particles, etc. In the Fourier space, different Fourier modes in the linear EW model get decoupled so that each $h(\mathbf{q}, \omega)$ for each $(\mathbf{q}, \omega)$ behaves independently. It is this decoupling that allows the

simple rescaling analysis of eq. (6) or dimensional analysis to give the correct exponents. For the KPZ equation, the nonlinear term couples heights of various wavelengths and therefore any attempt to integrate out the large ($\mathbf{q}$, ω) modes will affect h with low values of ($\mathbf{q}$, ω). This mixing is taken into account in the RG analysis which is implemented in a perturbative way. One thinks of the noise and the nonlinear term as disturbances affecting the EW-like surface. If we know the response of such a surface to a localized disturbance we may recover the full response by summing over the disturbances at all the points and times. However, this disturbance from the nonlinear term itself depends on the height, requiring an iterative approach that generates successively a series of terms. By averaging over the noise, one then can compute any physical quantity. At this stage only degrees of freedom with $\mathbf{q}$ in a small shell $e^{-l}\Lambda < q < \Lambda$ is integrated out. In real space this corresponds to integrating out the small scale fluctuation. The contribution from this integration over the shell is absorbed by redefining the various parameters ν, λ and D. These are the coupling constants for a similar equation as eq. (5) but with a smaller cutoff Λe^{-l}. A subsequent rescaling then restores the original cutoff to Λ. Following this procedure, the flow equations for different parameters ν, D, and λ can be obtained[5]. Using the exponent relation predicted from the Galilean invariance and the RG invariance of ν, the flow equations for all the parameters can be combined into a single flow equation for $\bar{\lambda}^2 = \lambda^2 D/\nu^3$ (with $\Lambda = 1$). This is the only dimensionless parameter that can be constructed from λ, ν, D, and Λ, and it is always easier to work with dimensionless quantities. Its recursion relation is

$$\frac{d\lambda}{dl} = \frac{2-d}{2}\bar{\lambda} + K_d \frac{2d-3}{4d}\bar{\lambda}^3, \tag{7}$$

where K_d is the surface area of a d-dimensional sphere divided by $(2\pi)^d$. The invariance of ν under the RG transformation implies $z = 2 - K_d \bar{\lambda}^2 \frac{2-d}{4d}$, and the Galilean invariance provides the value of $\chi = 2 - z$ once the value of z is known. To be noted here is that the dynamic exponent is different from 2 by a term that depends on λ coming from the renormalization effects.

A few very important features are apparent from eq. (7). From the fixed point requirement $d\bar{\lambda}/dl = 0$, we find that at $d = 1$, there is a stable fixed point $\bar{\lambda}^2 = 2/K_1$. At this fixed point $z = 3/2$ and $\chi = 1/2$ supporting the results predicted from the symmetry analysis. At $d = 2$, the coupling is marginally relevant, indicating a strong coupling phase not accessible in a perturbation scheme. At $d > 2$, the flow equation indicates two different regimes, namely a weak coupling regime where $\bar{\lambda}$ asymptotically vanishes leading to a flat EW phase with $\chi = 0$, $z = 2$, and a strong coupling rough phase, the fixed point of which cannot be reached by perturbation analysis.

Owing to this limitation of the RG analysis based on the perturbation expansion, the scaling exponents in this strong coupling phase cannot be determined by this RG scheme. Different numerical methods yield $z = 1.6$ at $d = 2$. The phase transition governed by the unstable fixed point of $\bar{\lambda}$ is well under control with $z = 2$ for all $d > 2$. To explore the strong coupling phase, techniques like self-consistent mode coupling approach, functional RG, etc. have been employed, but even a basic question whether there is an upper critical dimension at which z will again become 2 remains controversial.

Relation with other systems

The relation of the KPZ equation with other quite unrelated topics in equilibrium and nonequilibrium statistical mechanics is impressive. Here, we provide a very brief account of these systems.

Noisy Burgers equation: By defining a new variable $\mathbf{v} = \nabla h$, we obtain an equation

$$\frac{\partial \mathbf{v}}{\partial t} = D\nabla^2\mathbf{v} + \lambda\mathbf{v}\cdot\nabla\nu + \mathbf{f}(\mathbf{x}, t), \tag{8}$$

where the noise term $\mathbf{f} = \nabla\eta$. The above equation represents the noisy Burgers equation for vortex free ($\nabla \times \mathbf{v} = 0$) fluid flow with a random force. This equation is very important in studies of turbulence. The tilt invariance of the KPZ equation turns out to be the conventional Galilean invariance for the Burgers equation (for $\lambda = 1$), and that is how the name stayed on.

Directed polymer in a random medium: A directed polymer, very frequently encountered in different problems in statistical mechanics, is a string-like object which has a preferred longitudinal direction along which it is oriented, with fluctuations in the transverse direction. The flux lines in type II or high T_c superconductors are examples of such directed polymers in 3-dimensions, while the steps on a vicinal or miscut crystal surface or the domain walls in a uniaxial two-dimensional system are examples in two-dimensions. The formal mathematical mapping to such objects follows from a simple (Cole–Hopf) transformation of the KPZ equation using $W(\mathbf{x}, t) = \exp$ The Cole–Hopf transformation linearizes the nonlinear KPZ equation and the resulting linear diffusion equation (or imaginary time Schrödinger equation) is identical to that satisfied by the partition function of a directed polymer in a random potential. For such random problems, one is generally interested in the averages of thermodynamic quantities like the free energy and we see that the noise averaged height $\langle h(\mathbf{x}, t)\rangle$ gives the average free energy of a directed polymer of length t with one end at the origin and the other end at $\mathbf{x}$. This is a unique example of a system where the effect of such quenched averaging of free energy can be studied without invoking any tricks (like

the replica method). This has led to many important results and enriched our understanding of equilibrium statistical mechanics. Recently, this formulation has been extended to study details of the properties of the random system near the phase transition point and overlaps in lower dimensions[6,7]. It turns out that one needs an infinite number of exponents to describe the statistical behaviour of the configurations of the polymer in the random medium[8]. We do not go into this issue as this is beyond the scope of this article.

An interesting connection between the $1+1$-dimensional KPZ equation and the equilibrium statistical mechanics of a two-dimensional smectic-A liquid crystal has been recently established by Golubovich and Wang[9]. This relationship further provides exact approach to study the anomalous elasticity of smectic-A liquid crystals.

Apart from these, there are a number of other relations between KPZ equation and kinetics of annihilation processes with driven diffusion, the sine–Gordon chain, the driven diffusion equation and so on.

Beyond KPZ

Conservation condition: The situation encountered in molecular-beam epitaxy (MBE) for growth of thin films is quite different from the mechanism prescribed by the KPZ equation[2]. In MBE, surface diffusion takes place according to the chemical potential gradient on the surface, respecting the conservation of particles. If the particle concentration does not vary during growth, then a mass conservation leads to a volume conservation and the film thickness is governed by an underlying continuity equation

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{j} = \eta, \tag{9}$$

where $\mathbf{j}$ is the surface diffusion current which states that the change of height at one point is due to flow into or out from that point. The current is then determined by the gradient of the chemical potential, and since the chemical potential has already been argued to be proportional to the curvature $\nabla^2 h$, the growth equation thus becomes a simple linear equation involving $\nabla^4 h$ which, like the EW model, is exactly solvable. Taking into account the effect of nonlinearity the full equation can be written as

$$\frac{\partial h}{\partial t} = -\nabla^2 \left[\nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \right] + \eta(\mathbf{x}, t), \tag{10}$$

where the noise correlation is $\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle = 2D\nabla^2 \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$, if the noise also maintains conservation (if it originates from the stochasticity of diffusion) or would be the same white noise as in the KPZ equation, if the noise is from random deposition. It goes without saying that the exponents are different from the EW model even for the linear theory. The invariance of λ in this case leads to a different relation between χ and z. At the dimension of physical interest $d = 2$, this growth equation leads to an enhanced roughness than the KPZ case and may explain the results of experiments of high temperature MBE.

Quenched noise: A different type of generalization of the KPZ equation was to explore the motion of domain walls or interfaces in a random medium. In this case, the noise is not explicitly dependent on time but on the spatial position and the height variable. Such a noise has been called quenched noise because the noise is predetermined and the interface or the surface moves in this random system. The simple features of the KPZ equation and the EW model are lost. Functional RG analysis and numerical studies are attempted to clarify the question of universality classes and details of dynamics in such cases. The important concept that emerged in this context is the depinning transition so that the surface remains pinned by the randomness until the drive exceeds a certain critical value. Interface depinning is an example of a nonequilibrium phase transition. The velocity of the surface near this depinning transition also has critical-like behaviour with long-range correlations. Below the threshold, the dynamics is sluggish, while just above the threshold, the velocity is in general not proportional to the drive but obeys a power law with a universal exponent. For a very strong drive (or large velocity of the interface) the moving surface encounters each site only once, and therefore the noise is effectively like a space–time dependent noise rather than the quenched one. The nature of the surface would then be like KPZ.

Coloured noise: In the previous section we discussed the KPZ equation with white noise. If the noise is coloured in the sense that there is correlation in space or time or both, the universal behaviour, the phase transitions and the properties are different but still can be studied by the same RG technique. Several aspects of the problem, especially the role of noise correlation, have been explored[10].

All of the above seem to suggest that if there is no conservation law, then the KPZ equation is the equation to describe any nonlinear or nonequilibrium growth process and all phenomena can be put in one of the known universality classes. However, experimentally KPZ exponents seem to elude us so far[1,2,11]. Since results are known exactly in one-dimension, special one-dimensional experiments were conducted like paper burning, interface motion in paper, colloid suspension, etc., but KPZ exponents have not been seen. In the colloid experiment[11], the surface formed by the depositing colloids on the contact line ($d = 1$) between the colloid latex film and a glass slide was measured from video images. This method yields $\chi = 0.71$ but cannot determine the dynamic exponent. A recent analysis[12] of tropical cumulus clouds in the upper atmosphere, from satellite and space shuttle data from 0.1 to 1000 km, seems to agree with the KPZ results in $d = 2$.

Kinetic roughening with nonlocality

In spite of a tremendous conceptual and quantitative success of the KPZ equation in describing the nonequilibrium growth mechanism, the agreement with experimentally observed exponents is rather unsatisfactory. One wonders whether there is any relevant perturbation that drives the systems away from the KPZ strong coupling perturbation. One goal of this section is to point out that indeed there can be long-range interactions that may give rise to non-KPZ fixed points.

In many recently studied systems involving proteins, colloids, or latex particles the medium-induced interactions are found to play an important role[13]. This nonlocal interaction can be introduced by making a modification of the nonlinear term in the KPZ equation. Taking the gradient term as the measure of the local density of deposited particles, the long-range effect is incorporated by coupling these gradients at two different points. The resulting growth equation is a KPZ-like equation with the nonlinear term modified as[14] $\frac{1}{2}\int d\mathbf{r}'\ V(\mathrm{r}')\nabla h(\mathbf{r}+\mathbf{r}',t)\cdot\nabla h(\mathbf{r}-\mathbf{r}',t)$. For generality, we take $V(r')$ to have both short- and long-range parts with a specific form in Fourier space as $V(k) = \lambda_0 + \lambda_\rho k^{-\rho}$ such that in the limit $\lambda_\rho \to 0$, KPZ results are retrieved. The aim is to observe whether the macroscopic properties are governed by only λ_0 and hence KPZ-like or the behaviour is completely different from KPZ due to the relevance of λ_ρ around the KPZ fixed points.

A scaling analysis as done in eq. (6) clearly indicates different scaling regimes and the relevance of λ_0 and λ_ρ for $d > 2$ at the EW fixed point. For any $\lambda_\rho(\neq 0)$ with $\rho > 0$, the local KPZ theory (i.e. $\lambda_\rho = 0$ and $\chi + z = 2$) is unstable under renormalization and a non-KPZ behaviour is expected. For $2 < d < 2 + 2\rho$, only λ_ρ is relevant at the EW fixed point. The exponents of the non-KPZ phases can be obtained by performing a dynamic RG calculation[14]. By identifying the phases with the stable fixed points, we then see the emergence of a new fixed point where the long-range features dominate ($\chi + z = 2 + \rho$). Most importantly, at $d = 2$, the marginal relevance of $\bar{\lambda}$ is lost and there is a stable fixed point (LR) for $\rho > 0.0194$.

On the experimental side, there are experiments on colloids with $\chi = 0.71$ which is the value also obtained from paper burning exponents. For colloids, hydrodynamic interactions are important. Similar long-range interactions could also play a role in paper burning experiment due to the microstructure of paper. With this χ our exponents suggest $\rho = -0.12$ at $d = 1$ at the long-range fixed point. Further experiments on deposition of latex particles or proteins yielding the roughness of growing surface have not been performed. Probably such experiments may reveal more insights on this growth mechanism.

More recently, the effect of coloured noise in presence of nonlocality has been studied[15] and the nature of the phases and the various phase transitions clarified. A conserved version of the nonlocal equation has also been considered and it shows rich behaviour[16].

Roughening transition in nonequilibrium

It is interesting to study the impact of equilibrium phase transitions on the nonequilibrium growth of a surface. This is the situation observed experimentally in growth of solid ^{4}He in contact with the superfluid phase[17]. There is an equilibrium roughening transition at $T_R = 1.28$ K. For $T > T_R$ the growth velocity is linear in the driving force F (chemical potential difference), but for $T < T_R$ the velocity is exponentially small in the inverse of the driving force. For infinitesimal drive, the mobility which is the ratio of the growth velocity and F vanishes with a jump from a finite value at the transition. With a finite force the transition is blurred and the flat phase below T_R in equilibrium becomes rougher over large length scale.

The equilibrium roughening transition is an effect of discrete translational symmetry of the lattice. The equilibrium dynamics in this case is essentially governed by the Langevin equation

$$\frac{\partial h}{\partial t} = K\nabla^2 h(\mathbf{r},t) - V\sin\left[\frac{2\pi}{a}h(\mathbf{r},t)\right] + \zeta(\mathbf{r},t), \qquad (11)$$

where the sine term favours a periodic structure of spacing a. Extensive investigations have been done on this equilibrium model. At low temperature, this periodic potential is relevant and it ensures that minimum energy configuration is achieved when ϕ is an integer multiple of lattice periodicity. In this phase, the surface is smooth and the roughness is independent of length. In the high temperature phase the equilibrium surface is thermally rough and the roughness is logarithmic

$$C(L, \vartheta) \sim \ln[Lf(\vartheta L^z)]. \qquad (12)$$

The critical point is rather complicated and goes by the name of Kosterlitz–Thouless transition, first discussed in the context of defect mediated transitions in two-dimensional XY magnets[18].

For a nonequilibrium crystal growth problem, one needs to introduce the KPZ nonlinear term in eq. (11). There is no longer any roughening transition. The fact that away from equilibrium the roughening transition is blurred is manifested by the domination of the nonlinear term and the suppression of the pinning potential in the asymptotic regime[19].

A very nontrivial situation arises when the surface contains quenched disorder which shifts the position of the minima of the pinning potential in an arbitrary random fashion[20,21]. In this case, there is a new phase transition which is drastically different from the equilibrium roughening transition. This transition is called super roughening. Above the transition temperature, i.e. for $T > T_{sr}$, the surface is logarithmically rough as it is in the high temperature phase of the pure problem. However in the low temperature phase, i.e. for $T < T_{sr}$, the surface is no longer flat and is even rougher than the high temperature

phase. Recent numerical treatments suggest that the surface roughness behaves as $(\ln L)^2$. In the nonequilibrium situation, the linear response mobility vanishes continuously at the transition temperature unlike the jump discontinuity in the pure case. A general treatment with a correlated disorder elucidates the connection between the roughening and super roughening transition and one observes that the roughening turns into a super roughening transition if the disorder correlation decays sufficiently fast. Away from equilibrium, the super roughening transition is essentially dominated by the KPZ nonlinearity and instead of the logarithmic roughness, an asymptotic power law behaviour of the roughness is found over all temperature ranges.

In a similar situation in the nonequilibrium case, one needs to study the role of the KPZ nonlinearity with long-range disorder correlation[22]. A functional renormalization scheme with an arbitrary form of the disorder correlation turns out to be useful, though a detailed solution is not available. It is found that the flow of the KPZ nonlinearity under renormalization, with power law form of the disorder correlation, is such that it decays with length. This implies that nonequilibrium feature does not set in over a certain length scale. Over this scale one would then expect usual roughening transition. However, there is generation of a driving force due to the nonlinearity, and the growth of this force with length scale would invalidate use of perturbative analysis. For large length scales, one expects a KPZ-type power law roughness of the surface. Nevertheless, the initial decay of the nonlinearity with the length scale due to the long-range correlation of the disorder is an interesting conclusion that seems to be experimentally detectable.

Remarks

In this brief overview, we attempted to focus on the difference between equilibrium and nonequilibrium growth problems with an emphasis on the scaling behaviour and RG approach. Many details with references to pre-1995 papers can be found in Halpin-Healy and Zhang[1], and Barbási and Stanley[2], which should be consulted for more detailed analysis. Though the success story of the KPZ equation is rather impressive, there are still many unresolved, controversial issues. In fact for higher dimensions, the behaviour is not known with as much confidence as for lower dimensions. Developments in this direction are awaited.

Note added in proof:

1. The growth mechanism of metal-organic films deposited by the Langmuir–Blodgett technique has been studied in ref. 23 by X-ray scattering and atomic force microscopy. The results have been interpreted by a combination of 1-dimensional EW equation (eq. (4)) and 2-dimensional linear conserved equation (eq. (10)) with conserved noise.
2. For effects of nonlocality in equilibrium critical dynamics, see ref. 24.

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