ON GENERAL DIFFERENTIABLE STRUCTURE, NIJENHUIS TENSOR

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In this paper the authors have discussed various forms of Nijenhuis tensor and its properties.

1. Introduction

We consider a differentiable manifold V_n of class $\overset{\infty}{C}$. Let there be a vector-valued linear function F of class $\overset{\infty}{C}$, such that

$$\overline{\overline{X}} = a^2 X \tag{1.1 a}$$

for arbitrary vector field X, where

$$\bar{X} \underset{\Longrightarrow}{\text{def}} F(X) \tag{1.1 b}$$

and a is any complex number.

Let us agree to say that F gives to V_n, a differentiable structure, briefly GF-structure, defined by the algebraic equation (1.1 a). It is well-known that V_n is endowed with a π-structure (Legrand 1956) or an almost product structure or an almost complex structure (Mishra 1967) or an almost tangent structure (Eliopoulas 1965) according as $a \neq 0$ or $a = 1$ or $a = i$ or $a = 0$.

The rank of F in the first three cases is n and in the last case is $n/2$. In the last two cases n has to be even.

Agreement 1.1—All the equations which follow, hold for arbitrary vector fields X, Y, Z......, etc,

If the given GF-structure is endowed with a Hermite metric g, such that

$$g(\overline{X}, \overline{Y}) + a^2 g(X, Y) = 0,$$

then we say that (F, g) gives to V_n, a Hermite structure briefly H-structure, subordinate to GF-structure.

Let us consider on V_n, equipped with H-structure, a tensor f of the type (0, 2), such that

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$$f(X,Y) \underset{=}{\text{def}} g(\overline{X}, Y) = -g(X, \overline{Y}). \tag{1.2a}$$

Then the following results hold :

$$f(\overline{X}, Y) = -f(X, \overline{Y}) = a^2 g(X, Y) \tag{1.2 b}$$

$$f(\overline{X}, \overline{Y}) = -a^2 g(\overline{X}, Y) = a^2 g(X, \overline{Y}) = -a^2 f(X, Y). \tag{1.2 c}$$

Since g is symmetric, eqns (1.2a) and (1.2c) imply that f is skew-symmetric.

If for an H-structure

$$(D_X F)(Y) = 0, \ (D_{\overline{X}} F)(\overline{Y}) = 0 \tag{1.3 a}$$

is satisfied, then we say that V_n is a Kähler manifold.

If for an H-structure

$$(D_X F)(Y) + (D_Y F)(X) = 0 \tag{1.3 b}$$

is satisfied, then we say that V_n is an almost Tachibana manifold in the broad sense.

A bilinear function ψ is said to be pure in the two slots, if

$$\psi(\overline{X}, \overline{Y}) - a^2 \psi(X, Y,) = 0. \tag{1.4a}$$

It is said to be hybrid in the two slots, if

$$\psi(\overline{X}, \overline{Y}) + a^2 \psi(X, Y) = 0. \tag{1.4 b}$$

From the above we note that f is hybrid in X and Y.

2. Nijenhuis Tensor

Nijenhuis tensor with respect to F is a vector valued bilinear function N given by (Yano 1965)

$$N(X, X) = [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}} - \overline{[X, \bar{Y}]} - \overline{[\bar{X}, Y]}$$

$$=[\bar{X}, \bar{Y}] + a^2 [X, Y] - \overline{[X, \bar{Y}]} - \overline{[\bar{X}, Y]} \tag{2.1}$$

where $[X, Y] = D_X Y - D_Y X$, and D is Riemannian connexion.

Then the following equation hold (Duggal 1971)

$$N(X. Y) = -N(Y, X) \tag{2.2 a}$$

$$N(X, \overline{Y}) = N(\bar{X}, Y) = -\overline{N(X, Y)} \tag{2.2b}$$

$$N(\overline{X}, \overline{Y}) = a^2 N(X, Y) = -\overline{N(X, \overline{Y})} = -\overline{N(\overline{X}, Y)}. \tag{2.2 c}$$

From (2.2 *c*), it is clear that $N(X, Y)$ is pure in X and Y. Also, if V_n is equipped with an almost tangent structure, then

$$N(\bar{X}, \bar{Y}) = \overline{N(X, \bar{Y})} = \overline{N(X, Y)} = 0.$$

Theorem 2.1—Let us put

$$P(X, Y) \underline{\underline{\text{def}}} [\bar{X}, \bar{Y}] - \overline{[\bar{X}, Y]}. \tag{2.3}$$

Then

$$\overline{P(X, \bar{Y})} = -a^2 P(X, Y) = a^2 (\overline{[\bar{X}, Y]} - [\bar{X}, \bar{Y}]) \tag{2.4 a}$$

$$\overline{P(\bar{X}, \bar{Y})} = -a^2 P(\bar{X}, \bar{Y}) = a^4 (\overline{[X, Y]} - [X, \bar{Y}]) \tag{2.4 b}$$

$$\overline{P(X, Y)} = -P(X, \bar{Y}) = \overline{[\bar{X}, \bar{Y}]} - a^2 [\bar{X}, Y]) \tag{2.4 c}$$

$$P(\bar{X}, \bar{Y}) = -\overline{P(\bar{X}, Y)} = a^2 (a^2 [X, Y] - \overline{[X, \bar{Y}]}). \tag{2.4 d}$$

Consquently

$$\overline{P(X, \bar{Y})} + \overline{P(\bar{X}, Y)} = -a^2 N(X, Y) = -N(\bar{X}, \bar{Y}) \tag{2.5 a}$$

$$P(\bar{X}, \bar{Y}) + a^2 P(X, Y) = -\overline{N(\bar{X}, Y)} = a^2 N(X, Y) \tag{2.5 b}$$

$$a^2 \overline{P(X, Y)} + \overline{P(\bar{X}, \bar{Y})} = -a^2 N(X, \bar{Y}) \tag{2.5 c}$$

$$a^2 (P(\bar{X}, Y) + P(X, \bar{Y})) = -\overline{N(\bar{X}, \bar{Y})} \tag{2.5 d}$$

Proof: Barring (2. 3) throughout or different vectors in it and using (1.1 *a*) we get (2.4 *a*) – (2.4 *d*).

Again, using (1.1*a*), (2.2) and (2.4) in the following equations :

$$N(X, Y) = [\bar{X}, \bar{Y}] + a^2 [X, Y] - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]}$$

$$\overline{N(X, Y)} = \overline{[\bar{X}, \bar{Y}]} + a^2 \overline{[X, Y]} - a^2 [\bar{X}, Y] - a^2 [X, \bar{Y}]$$

$$N(\bar{X}, Y) = a^2 [X, \bar{Y}] + a^2 [\bar{X}, Y] - \overline{[\bar{X}, \bar{Y}]} - a^2 \overline{[X, Y]}$$

$$\overline{N(\bar{X}, Y)} = a^2 \overline{[X, \bar{Y}]} + a^2 \overline{[\bar{X}, Y]} - a^2 [\bar{X}. \bar{Y}] - a^4 [X, Y]$$

$$N(X, \bar{Y}) = a^2 [\bar{X}, Y] + a^2 [X, \bar{Y}] - a^2 \overline{[X, Y]} - \overline{[\bar{X}, \bar{Y}]}$$

$$\overline{N(X, \bar{Y})} = a^2 \overline{[\bar{X}, Y]} + a^2 \overline{[X, \bar{Y}]} - a^4 [X, Y] - a^2 [\bar{X}, \bar{Y}]$$

$$N(\bar{X}, \bar{Y}) = a^2 (a^2 [X, Y] + [\bar{X}, \bar{Y}] - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]})$$

$$\overline{N(\bar{X}, \bar{Y})} = a^2 (a^2 [X, Y] + \overline{[X, \bar{Y}]} - a^2 [\bar{X}, Y] - a^2 X [X, \bar{Y}])$$

we get (2.5 *a*) – (2.5 *d*).

Note 2.1 : Some more relations of the type (2.4) and (2.5) can be obtained but they reduce to (2.4 *a*, *b*, *c*, *d*) and (2.5 *a*, *b*, *c*, *d*).

Remark 2.1 : If V_n is equipped with an almost tangent structure then from (1.4 *b*) and (2.5 *b*), it follows that $P(X, Y)$ is hybrid in X and Y.

Theorem 2.2—Let us put

$$Q(X, Y) \underline{\underline{\text{def}}}\, a^2 [X, Y] - \overline{[X, \bar{Y}]}. \tag{2.6}$$

Then

$$Q(X, \bar{Y}) = -\overline{Q(X, Y)} = a^2 ([X, \bar{Y}] - \overline{[X, Y]}) \tag{2.7 a}$$

$$\overline{Q(X, \bar{Y})} = -a^2 Q(X, Y) = a^2 (\overline{[X, \bar{Y}]} - a^2 [X, Y]) \tag{2.7 b}$$

$$\overline{Q(\bar{X}, Y)} = -Q(\bar{X}, \bar{Y}) = a^2 ([\bar{X}, Y] - \overline{[\bar{X}, \bar{Y}]}) \tag{2.7 c}$$

$$\overline{Q(\bar{X}, \bar{Y})} = -a^2 Q(\bar{X}, Y) = a^2 (\overline{[\bar{X}, \bar{Y}]} - a^2 [\bar{X}, Y]). \tag{2.7 d}$$

Consequently

$$Q(\bar{X}, \bar{Y}) = +a^2 Q(X, Y) = N(\bar{X}, \bar{Y}) = a^2 N(X, Y) \tag{2.8 a}$$

$$Q(\bar{X}, Y) - \overline{Q(X, Y)} = N(\bar{X}, Y) \tag{2.8 b}$$

$$\overline{Q(\bar{X}, \bar{Y})} - a^2 Q(X, \bar{Y}) = \overline{N(\bar{X}, \bar{Y})} \tag{2.8 c}$$

$$Q(X, \bar{Y}) + Q(\bar{X}, Y) = N(X, \bar{Y}). \tag{2.8 d}$$

Proof : The proof of these equations follows the pattern of the proof of the Theorem 2.1.

Corollary 2.1—We have in *GF*-structure

$$P(X, \bar{Y}) = Q(\bar{X}, Y) = -\overline{P(X, Y)} \tag{2.9 a}$$

$$P(\bar{X}, Y) = -\overline{Q(X, Y)} = -a^2 Q(X, \bar{Y}) \tag{2.9 b}$$

$$\overline{P(X, \bar{Y})} = -\overline{Q(\bar{X}, Y)} = -a^2 P(X, Y) \tag{2.9 c}$$

$$P(\bar{X}, \bar{Y}) = a^2 Q(X, Y). \tag{2.9 d}$$

Proof : The statement follows from (2.4), (2.7) and (1.1 *a*).

Corollary 2.2—We have in *GF*-structure

$$N(X, Y) = P(X, Y) + Q(X, Y) \tag{2.10 a}$$

$$N(\bar{X}, Y) = P(X, \bar{Y}) - \overline{Q(X, Y)} \tag{2.10 b}$$

$$N(\bar{X}, \bar{Y}) = -\overline{P(X, \bar{Y})} + a^2 Q(X, Y) \tag{2.10 c}$$

$$\overline{N(\bar{X}, \bar{Y})} = -a^2 P(X, \bar{Y}) - a^2 Q(X, \bar{Y}). \tag{2.10 d}$$

Proof: Equation (2.10 a) is the consequence of the equations (2.1 a) (2.3) and (2.6). The relation (2.10 b) is obtained by (2.8 b) and (2.9 a). We get (2.10 c) by using (2.7 c) and (2.9 c) in (2.8 a). Barring (2.10 c) throughout, using (1.1 a) and (2.7 a), we get (2.10 d).

Note 2.2: Some other relations of the type (2.9 a, b, c, d) and (2.10 a, b, c, d,) can be obtained but they reduce to them.

Remark 2.2: If V_n is equipped with an almost tangent structure then due to (1.4 b) eqn. (2.8 a) shows that $Q(X, Y)$ is hybrid in X and Y.

Theorem 2.3—If we put

$$U(X, Y) \underset{=}{\text{def}}\, a^2 [X, Y] - \overline{[\bar{X}, Y]}. \tag{2.11}$$

Then

$$\overline{U(\bar{X}, Y)} = -a^2 U(X, \bar{Y}) = a^2 ([\bar{X}, \bar{Y}] - a^2 [X, \bar{Y}]). \tag{2.12}$$

Consequently

$$U(X, \bar{Y}) - \overline{U(X, Y)} = N(\bar{X}, Y). \tag{2.13}$$

Proof: Barring X, Y in (2.11) and then throughout tneresulting equation obtained and using (1.1 a), we get (2.12). Barring X in (2.1 a) and using (1.1 a) we get (2.12). Barring X in (2.1 a) and using (1.1 a), we have

$$\begin{aligned} N(\bar{X}, Y) &= a^2 [X, \bar{Y}] + a^2 [\bar{X}, Y] - \overline{[\bar{X}, \bar{Y}]} - a^2 \overline{[X, Y]} \\ &= (a^2 [X, \bar{Y}] - \overline{[\bar{X}, \bar{Y}]}) - (a^2 \overline{[X, Y]} - a^2 [\bar{X}, Y]) \\ &= U(X, \bar{Y}) - \overline{U(X, Y)} \end{aligned} \tag{2.14}$$

which is (2.13).

Theorem 2.4—Let us put

$$V(X, Y) \underset{=}{\text{def}}\, \overline{[\bar{X}, Y]} + \overline{[X, \bar{Y}]}. \tag{2.15}$$

Then

$$\overline{V(\bar{X}, Y)} = \overline{V(X, \bar{Y})} = a^2 (a^2 [X, Y] + [\bar{X}, \bar{Y}]). \tag{2.16}$$

Consequently

$$\overline{V(\bar{X}, Y)} - V(\bar{X}, \bar{Y}) = a^2 N(X, Y) \tag{2.17}$$

Proof: The statement follows the pattern of the Theorem 2.3.

Note 2.3 : Other relations for $U(X, Y)$ and $V(X, Y)$ can also be established as for $P(X, Y)$ and $Q(X, Y)$.

Remark 2.3 : If V_n is equipped with an almost tangent structure then $U(X, Y)$ is also hybrid in X and Y.

For the following discussions, we suppose V_n to be equipped with an H-structure subordinate to GF-structure unless stated otherwise.

We know that the Nijenhuis tensor for F, with a suitable connexion D with respect to g is given by

$$N(X, Y) = [\bar{X}, \bar{Y}] + a^2 [X, Y] - \overline{[X, \bar{Y}]} - \overline{[\bar{X}, Y]}.$$

If we put

$$'N(X, Y, Z) \underset{=}{\text{def}} - a^2 g(N(X, Y), Z) = - g(N(\bar{X}, \bar{Y}), Z). \tag{2.18}$$

Then

$'N(X, Y, Z)$ is skew-symmetric in X and Y, i. e.

$$'N(X, Y, Z) = - 'N(Y, X, Z) \tag{2.19}$$

$$'N(\bar{X}, \bar{Y}, Z) = 'N(\bar{X}, Y, \bar{Z}) = 'N(X, \bar{Y}, \bar{Z}) = a^2\, 'N(X, Y, Z).$$

Corollary 2.3 : Let us define

$$'P(X, Y, Z) \underset{=}{\text{def}} g(P(X, Y), Z) \tag{2. 20 a}$$

$$'Q(X, Y, Z) \underset{=}{\text{def}} g(Q(X, Y), Z) \tag{2.20 b}$$

$$'U(X, Y, Z) \underset{=}{\text{def}} g(U(X, Y), Z) \tag{2,20 c}$$

$$'V(X, Y Z) \underset{=}{\text{def}} g(V(X, Y), Z). \tag{2.20 d}$$

Then $'N(X, Y, Z)$ can be put in the form

$$a^2\, 'N(X, Y, Z) = a^2\, 'P(X, Y, Z) + 'P(\bar{X}, \bar{Y}, Z) \tag{2.21 a}$$

$$a^2\, 'N(X, Y, Z) = a^2\, 'Q(X, Y, Z) + 'Q(\bar{X}, \bar{Y}, Z) \tag{2.21 b}$$

$$a^2\, 'N(X, Y, Z) = a^2\, 'U(X, Y, Z) + 'U(\bar{X}, \bar{Y}, Z) \tag{2.21 c}$$

$$a^2\, 'N(X, Y, Z) = a^2\, 'V(X, Y Z) + 'V(\overline{X, Y}, Z). \tag{2.21 d}$$

Proof: The equation (2.21 *a*) follows from (2.2 *c*), (2.5 *b*) and 2.20 *a*). By using (2.2 *c*), (2.8 *a*) and (2.20 *a*) we get (2.21 *b*).

The remaining two can be proved similarly.

Corollary 2.4—We have

$$V(\bar{X}, \bar{Y}) = a^2\, V(X, Y) = a^2(\overline{[X, \bar{Y}]} + \overline{[\bar{X}, Y]}). \tag{2.22}$$

Then

$$'V(\bar{X}, \bar{Y}, Z) = a^2\, 'V(X, Y, Z). \tag{2.23}$$

Consequently $'V(X, Y, Z)$ is pure in X and Y.

Proof: Using (2.20d) in (2.22), we have (2.23).

The equation (2.23) together with (1.4a) implies that $'V(X, Y, Z)$ is pure in X, Y.

Note 2.5 : If the given H-structure is subordinate to an almost tangent structure then equations (2.21a)–(2.21d) imply that

$$'P(\bar{X}, \bar{Y}, Z) = 'Q(\bar{X}, \bar{Y}, Z) = U(\bar{X}, \bar{Y}, Z) = 'V(\overline{X, \bar{Y}, Z}) = 0.$$

Theorem 2.5—The necessary and sufficient condition for a manifold with an H-structure subordinate to the GF-structure to be a Kähler manifold is

$$a^2\, D_X Y = \overline{D_X \bar{Y}} \tag{2.24a}$$

equivalently

$$a^2\, D_{\bar{X}} Y = D_{\bar{X}} \bar{Y} \tag{2.24b}$$

$$\overline{D_{\bar{X}} Y} = D_{\bar{X}} \bar{Y} \tag{2.24c}$$

$$\overline{D_X Y} = D_X \bar{Y}. \tag{2.24d}$$

Proof: We know that

$$(D_X F)(Y) + F(D_X Y) = D_X \bar{Y}$$

or

$$(D_X F)(Y) + \overline{D_X Y} = D_X \bar{Y}.$$

Substituting from (1.3a), we have

$$D_X \overline{\bar{Y}} = D_X \bar{Y}.$$

Barring and using (1.1a) in this equation, we get (2.24a).

Barring X in (2.24a), we obtain (2.24b). The equation (2.24c) follows from barring (2.24b) and using (1.1a) and (2.24d) can be had by barring X in (2.24c) after using (1.1a).

Corollary 2.5—For a Kähler manifold, we have

$$a^2[X, Y] = \overline{[X, \bar{Y}]} \tag{2.25a}$$

$$a^2[\bar{X}, Y] = [\bar{X}, \bar{Y}] \tag{2.25b}$$

$$[\bar{X}, \bar{Y}] = \overline{[\bar{X}, Y]} \tag{2.25c}$$

$$[X, \bar{Y}] = \overline{[X, Y]}. \tag{2.25d}$$

Proof: Interchanging X and Y in (2.24a) and subtracting the resulting equation obtained from (2.24a), we get (2.25a). Barring X in (2.25a), we get (2.25b). By barring (2.25b) throughout and using (1.1a), we obtain (2.25c) and (2.25d) follows from barring X in (2.25c) and using (1.1a).

Remark 2.4—Since for a Kähler manifold, Nijenhuis tensor vanishes, we have

$$a^2 P(X, Y) = -P(\bar{X}, \bar{Y}) \tag{2.26a}$$

$$a^2 Q(X, Y) = -Q(\bar{X}, \bar{Y}) \tag{2.26b}$$

$$a^2 U(X, Y) = -U(X, Y) \tag{2.26c}$$

$$a_2 V(X, Y) = \overline{V(X, \bar{Y})}. \tag{2.26d}$$

Theorem 2.6—The necessary and sufficient condition for a manifold with an H-structure subordinate to the GF-structure to be an almost Tachibana manifold is

$$a^2(D_X Y + D_Y X) = \overline{D_X \bar{Y}} + \overline{D_Y \bar{X}} \tag{2.27a}$$

equivalent to

$$a^2(D_{\bar{X}} Y + D_Y \bar{X}) = \overline{D_{\bar{X}} \bar{Y}} + a^2 \overline{D_Y X} \tag{2.27b}$$

$$\overline{D_{\bar{X}} \bar{Y}} + \overline{D_{\bar{Y}} \bar{X}} = a^2(D_{\bar{X}} Y + D_{\bar{Y}} X) \tag{2.27c}$$

$$D_{\bar{X}} \bar{Y} + D_{\bar{Y}} \bar{X} = \overline{D_{\bar{X}} Y} + \overline{D_{\bar{Y}} X}. \tag{2.27d}$$

Proof: We have

$$(D_X F)(Y) + F(D_X Y) = D_X \bar{Y}. \tag{2.28}$$

Interchaning X and Y in (2.28) and adding the resulting equation obtained in (2.28), we have

$$(D_X F)(Y) + (D_Y F)(X) + \overline{D_X Y} + \overline{D_Y X} = D_X \bar{Y} + D_Y \bar{X}.$$

Substituting from (1.3b), we get

$$\overline{D_X Y} + \overline{D_Y X} = D_X \bar{Y} + D_Y \bar{X}.$$

Barring throughout the above equation and using (1.1a), we obtain (2.27a).

Other relations follow from (2.27*a*) by barring different vectors or throughout the equation and using (1.1*a*).

Theorem 2.7—The necessary and sufficient condition that $'N(X, Y, Z)$ is completely skew-symmetric in X, Y, Z in an almost Tachibana manifold is

$$\overline{D_{\bar{X}}\bar{Y}}+\overline{D_{\bar{Y}}\bar{X}} = a^2(D_{\bar{X}}Y+D_{\bar{Y}}X). \tag{2.29}$$

Proof : We know (Duggal 1971) that a necessary and sufficient condition for $'N(X, Y, Z)$ to be completely skew-symmetric in an H-structure is

$$(D_{\bar{X}}F)(Y)+(D_{\bar{Y}}F)(X) = \overline{(D_XF)(Y)+(D_YF)(X)}$$

or

$$\overline{(D_{\bar{X}}F)(Y)}+\overline{(D_{\bar{Y}}F)(X)} = a^2(\,(D_XF)(Y)+(D_YF)(X)\,).$$

But when the manifold is an almost Tachibana, substituting from (1.3*a*), we have

$$\overline{(D_{\bar{X}}F)(Y)} + \overline{(D_{\bar{Y}}F(\,(X)} = 0$$

or

$$D_{\bar{X}}\bar{Y}-\overline{D_{\bar{X}}Y}+D_{\bar{Y}}\bar{X}-\overline{D_{\bar{Y}}X} = 0$$

or

$$\overline{D_{\bar{X}}\bar{Y}}+\overline{D_{\bar{Y}}\bar{X}} = a^2(D_{\bar{X}}Y+D_{\bar{Y}}X)$$

which is the required result.

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