SUBMANIFOLDS OF CODIMENSION $m-n$

By

R. S. Mishra

(Received January 7, 1974)

Summary. In this paper, *I* have studied the submanifolds of codimension $m-n$ in an almost complex and an almost contact manifold and also a differentiable manifold with f-structure of rank r.

Chapter I. Invariant submanifolds

1. Introduction. Let V_m be m-dimensional differentiable manifold. Let there be defined in V_m a vector valued linear function F satisfying

$$F^3+F=0 , \tag{1.1}$$

where

$$\text{a)}\quad F^p=F^{p-1}F , \qquad \text{b)}\quad F^0=I \text{ (identity)}. \tag{1.2}$$

Let the rank $((F))$ be r everywhere. Then r is even and V_m is called a differentiable manifold with f-structure.

We can always introduce a metric G in V_m. Let G satisfy

$$G(F(\lambda), F(\mu))=-G(F^2(\lambda), \mu) . \tag{1.3}$$

We are justified in taking (1.3) as it is because replacing λ by $F^2(\lambda)$ in (1.3) and using (1.1), we get the same equation. Thus there is no inconsistency or contradiction. V_m is, then, called a metric differentiable manifold with f-structure of rank r.

Let us put

$$\text{a)}\quad L\stackrel{\text{def}}{=}-F^2 , \qquad \text{b)}\quad M\stackrel{\text{def}}{=}F^2+I \tag{1.4}$$

Then

$$\text{a)}\ L^2=L,\quad \text{b)}\ M^2=M,\quad \text{c)}\ L(M)=M(L)=0,\quad \text{d)}\ L(F)=F(L)=F,\quad \text{e)}\ M(F)=F(M)=0 . \tag{1.5}$$

We will consider the following three particular cases:

Case 1. rank $(F)=m$

In this case m is even and (1.4) a, b and (1.1) yield

$$(1.6) \qquad \text{a)} \quad L=I \qquad \text{b)} \quad M=F^2+I=0$$

V_m is called an almost complex manifold. If we introduce the metric G in V_m as in (1.3) assumes the form

$$(1.7) \qquad G(F(\lambda), F(\mu))=G(\lambda, \mu)\ ,$$

V_m is then called an almost Hermite manifold and the structure (F, G) is called an almost Hermite structure.

Case 2. rank $(F)=m-1$

In this case m is odd and (1.1) and (1.3) yield

$$(1.8)\ \text{a} \qquad M(\lambda)=F^2(\lambda)+\lambda=A(\lambda)T\ ,$$

where A is a 1-form and T is a vector field. V_m is called an almost contact manifold with the almost contact structure (F, T, A). From (1.8)a, we easily deduce

$$(1.8)\ \text{b} \qquad A(T)=1, \quad \text{c)} \quad A(F)=0, \quad \text{d)} \quad F(T)=0$$

If we introduce the metric G in V_m as in (1.3), then (1.3) assumes the form

$$(1.9) \qquad G(F(\lambda), F(\mu))=G(\lambda, \mu)-A(\lambda)A(\mu)\ .$$

V_m is called an almost contact metric manifold or an almost Grayan manifold and the structure (F, T, A, G) is called an almost contact metric structure.

Case 3. rank $(F)=r<m-1$ (constant everywhere)

In this case r is even and (1.1) and (1.3) yield

$$(1.10)\ \text{a} \qquad M(\lambda)=F^2(\lambda)+\lambda=A^\alpha(\lambda)T_\alpha\ , \quad r+1\leqslant\alpha\leqslant m\ ,$$

where A^α is 1-form and T_α is a vector field. V_m is called a differentiable manifold with f-structure of rank r. From (1.10)a, we may easily deduce

$$(1.10)\ \text{b} \qquad A^\alpha(T_\beta)=\delta^\alpha_\beta, \quad \text{c)} \quad A^\alpha(F)=0, \quad \text{d)} \quad F(T_\alpha)=0\ .$$

If we introduce the metric G in V_m as in (1.3), then (1.3) assumes the form

$$(1.11)\ \text{a} \qquad G(F(\lambda), F(\mu))=G(\lambda, \mu)-{}'M(\lambda, \mu)\ ,$$

where

$$(1.11)\ \text{b} \qquad {}'M(\lambda, \mu)\overset{\text{def}}{=}G(M(\lambda), \mu)=G(\lambda, M(\mu))\ .$$

V_m is called differentiable Riemannian manifold with f-structure of rank r.

In the following D will be taken as the Riemannian connexion:

$$\text{(1.12) a)} \quad \bar{D}_\lambda\mu=\bar{D}_\mu\lambda+[\lambda,\mu], \quad \text{b)} \quad \lambda(G(\mu,\nu))=G(\bar{D}_\lambda\mu,\nu)+G(\mu,\bar{D}_\lambda\nu)$$

If in an almost Hermite manifold V_m

$$\text{(1.13)} \qquad (\bar{D}_\lambda F)(\mu)=0\ .$$

V_m is called a Kahler manifold.

If in an almost Hermite manifold V_m

$$\text{(1.14)} \qquad (\bar{D}_\lambda F)(\mu)+(\bar{D}_\mu F)(\lambda)=0\ ,$$

V_m is called an almost Tachibana manifold.

If in an almost Hermite manifold V_m,

$$\text{(1.15) a} \qquad (\bar{D}'_\lambda F)(\mu,\nu)+(\bar{D}'_\mu F)(\nu,\lambda)+(\bar{D}'_\nu F)(\lambda,\mu)=0$$

where

$$\text{(1.15) b} \qquad {}'F(\lambda,\mu)\overset{\text{def}}{=}G(F(\lambda),\mu)=-{}'F(\mu,\lambda)\ ,$$

V_m is called an almost Kahler manifold.

If in an almost Hermite manifold V_m

$$\text{(1.15) c} \qquad (\bar{D}_\lambda F)(\mu)+(\bar{D}_{F(\lambda)}F)(F(\mu))=0\ ,$$

V_m is called an almost 0-manifold.

Let N be Nijenhuis tensor in V_m:

$$\text{(1.16)} \qquad N(\lambda,\mu)\overset{\text{def}}{=}[F(\lambda),F(\mu)]+F^2([\lambda,\mu])-F([F(\lambda),\mu])-F([\lambda,F(\mu)])\ .$$

If in an almost contact manifold V_m

$$\text{(1.17) a} \qquad N(\lambda,\mu)+dA(\lambda,\mu)T=0\ ,$$

V_m is called an almost normal contact manifold.

If in an almost Grayan manifold V_m

$$\text{(1.17) b} \qquad 2'F=dA\ ,$$

V_m is called an almost Sasakian manifold.

If in an almost Sasakian manifold T is a Killing vector

$$\text{(1.17) c} \qquad {}'F(\lambda,\mu)=(\bar{D}_\lambda A)(\mu)=-(\bar{D}_\mu A)(\lambda)$$

V_m is called a Sasakian manifold.

Let V_n be a sub-manifold of V_m with the immersion

$$b:V_n\to V_m\ ,$$

such that $X \in V_n \Longrightarrow BX \in V_m$.

Agreement (1.1). *In the above and in what follows $\lambda, \mu, \nu, \cdots$ will be taken as arbitrary vector fields in V_m and $X, Y, Z, \cdots$ will be taken as arbitrary vector fields in V_n*

Let g be the induced metric in V_n. Then

(1.18) $$g(X, Y)=G(BX, BY)\circ b$$

Let C be a set of unit normal vectors to V_n. Then

(1.19) a $$G(C_x, C_y)\circ b=\delta_{xy}$$

(1.19) b $$G(C_x, BX)\circ b=0\ .$$

2. Submanifolds

Theorem (2.1). *Let F be a vector valued linear function on V_m. Let us put*

(2.1) a $$F(BX)=B\bar{X}+p^x(X)C_x\ ,$$

where

(2.1) b $$\bar{X}\overset{\text{def}}{=}f(X)\ .$$

f being a vector valued linear function on V_n

(2.2) a $$F(C_x)=-BP_x+r_x^y C_y\ .$$

Then

(2.2) b $$F^2(BX)=B\{\bar{\bar{X}}-p_x(X)P_x\}+\{p^y(\bar{X})+p^z(X)r_z^y\}C_y$$

(2.2) c $$F^3(BX)=B\{\bar{\bar{\bar{X}}}-p^x(\bar{X})P_x-p^x(X)\bar{P}_x-r_x^y p^x(X)P_y\}$$
$$+\{p^y(\bar{\bar{X}})+p_z(\bar{X})r_z^y+p^x(X)r_x^z r_z^y-p^x(X)p^y(P_x)\}C_y$$

(2.2) d $$F^2(C_x)=-B\{\bar{P}_x+r_x^y P_y\}+\{r_x^z r_z^y-p^y(P_x)\}C_y\ ,$$

(2.2) e $$F^3(C_x)=-B\{\bar{\bar{P}}_x+r_x^y\bar{P}_y-p^y(P_x)P_y+r_x^y r_y^z P_z\}$$
$$-\{p^y(\bar{P}_x)+r_x^z p^y(P_z)+p^z(P_x)r_z^y-r_x^z r_z^t r_t^y\}C_y$$

(2.2) f $${}'M(BX, BY)\circ b={}'m(X, Y)-p^x(X)g(P_x, Y)$$

(2.2) a $$g(\bar{X}, \bar{Y})=g(X, Y)-{}'m(X, Y)+p^x(X)g(P_x, Y)-p^x(X)p^x(Y)$$

(2.3) b $$G(F(BX), BY)\circ b=g(\bar{X}, Y)\ ,$$

(2.3) c $$G(F(BX), C_x)\circ b=p_x(X)\ ,$$

(2.3) d $$G(F(BX),F(C_x))\circ b=-g(\bar{X},P_x)+p^y(X)r^y_x\ ,$$

(2.3) e $$G(F(C_x),BY)\circ b=-g(P_x,Y)\ ,$$

(2.3) f $$G(F(C_x),C_y))\circ b=r^y_x\ ,$$

(2.3) g $$G(F(C_x),F(C_y))\circ b=g(P_x,P_y)+r^z_x r^z_y\ .$$

Proof. Premultiplying (2.1)a by F and using (2.1) and (2.2) we obtain (2.2)b. We similarly obtain (2.2)c, d, e.

From (1.11)b and (1.10)a, we have

(2.3) h $$'M(BX,BY)=G(M(BX),BY)=G(F^2(BX),BY)+G(BX,BY)\ .$$

Using (2.2)a, (1.18) and (1.19) in this equation, we get (2.2)f. Substituting (2.1)a, (1.18) and (2.2)c in

$$G(F(BX),F(BY))=G(BX,BY)-{}'M(BX,BY)\ ,$$

we have (2.3)a. (2.3)b-g are the consequences of (2.1) and (2.2).

Theorem (2.2). *Let D be the Riemannian connexion in V_m and D be the induced connexion*

(2.4) a $$\bar{D}_{BX}BY=BD_XY+{}'h_x(X,Y)C_x\ ,$$

(2.4) b $$\bar{D}_{BX}C_x=-Bh_x(X)+\alpha^y_x(X)C_y\ ,$$

such that

(2.4) c $$g(h_x(X),Y)={}'h_x(X,Y)\ ,$$

(2.4) d $$\alpha^y_x+\alpha^x_y=0\ .$$

Then

(2.5) $$\begin{aligned}(\bar{D}_{BX}F)(BY)=&B(D_Xf)(Y)-p^x(Y)Bh_x(X)+{}'h_x(X,Y)BP_x\\&+\{{}'h_x(X,\bar{Y})+(D_Xp^x)(Y)+p^y(Y)\alpha^x_y(X)-{}'h(X,Y)\gamma^x_y\}C_x\end{aligned}$$

(2.6) $$\begin{aligned}(\bar{D}_{BX}F)(C_x)=&B\overline{h_x(X)}+\alpha^y_x(X)BP_y-BD_XP_x-\gamma^y_xBh_y(X)\\&+\{p^z(h_x(X))-\alpha^y_x(X)\gamma^z_y+X(\gamma^z_x)+\gamma^y_x\alpha^z_y(X)-{}'h_z(X,P_x)C_z\end{aligned}$$

We have, using (2.1)a, (2.4)a, b and (2.2)

$$\begin{aligned}(\bar{D}_{BX}F)(BY)&=\bar{D}_{BX}F(BY)-F(\bar{D}_{BX}BY)\\&=\bar{D}_{BX}\{B\bar{Y}+p^x(Y)C_x\}-F\{BD_XY+{}'h_x(X,Y)C_x\}\\&=BD_X\bar{Y}+{}'h_x(X,\bar{Y})C_x+(D_Xp^x)(Y)C_x+p^x(Y)\{-Bh_x(X)+\alpha^y_x(X)C_y\}\\&\quad-B\overline{D_XY}-{}'h_x(X,Y)\{-BP_x+r^y_xC_y\}.\end{aligned}$$

This is the equation (2.5). The equation (2.6) follows similarly.

Let us assume now that the tangent space of the submanifold V_n of codimension $m-n$ in V_m is invariant under the action of F at every point. We call such a manifold an *invariant sub-manifold.*

Theorem (2.3). *Let the submanifold V_n of an m-dimensional manifold V_m be invariant. Then*

(2.7) a $$F(BX)=B\bar{X} ,$$

(2.7) b $$F(C_x)=-BP_x+r_x^y C_y ,$$

(2.8) a $$F^2(BX)=B\bar{\bar{X}}$$

(2.8) b $$F^3(BX)=B\bar{\bar{\bar{X}}}$$

(2.8) c $$F^2(C_x)=-B\bar{P}_x-r_x^y BP_y+r_x^y r_y^z C_z ,$$

(2.8) d $$F^3(C_x)=-B\bar{\bar{P}}_x-r_x^y B\bar{P}_y-r_x^y r_y^z BP_z+r_x^z r_z^t r_t^y C_y ,$$

(2.8) e $$'M(BX, BY)\circ b='m(X, Y) ,$$

(2.9) a $$g(\bar{X}, \bar{Y})=g(X, Y)-'m(X, Y) ,$$

(2.9) b $$G(F(BX), BY)\circ b=g(\bar{X}, Y) ,$$

(2.9) c $$G(F(BX), C_x)\circ b=0 ,$$

(2.9) d $$G(F(BX), F(C_x))\circ b=-g(\bar{X}, P_x) ,$$

(2.9) e $$G(F(C_x), BY)\circ b=-g(P_x, Y) ,$$

(2.9) f $$G(F(C_x), C_y)\circ b=r_x^y ,$$

(2.9) g $$G(F(C_x), F(C_y))\circ b=g(P_x, P_y)+r_x^z r_y^z ,$$

(2.10) a $$(\bar{D}_{BX}F)(BY)=B(D_X f)(Y)+'h_x(X, Y)BP_x +\{'h_x(X, \bar{Y})+(D_X p^x)(Y)+p^y(Y)\alpha_y^x(X)-'h_y(X, Y)r_y^x\}C_x$$

(2.10) b $$(\bar{D}_{BX}F)(C_x)=B\overline{h_x(X)}+\alpha_x^y(X)BP_y-BD_X P_x-\gamma_x^y Bh_y(X) +\{X(r_x^z)-\alpha_x^y(X)r_y^z+r_x^y\alpha_y^z(X)-'h_z(X, P_x)\}C_z$$

Proof. Putting $p^x=0$ in (2.1)a, (2.2), (2.3), (2.5) and (2.6), we obtain (2.7)—(2.10).

3. Almost Hermite enveloping manifold

Theorem (3.1). *Let V_m be almost Hermite. Then the invariant submanifold V_n is also almost Hermite.*

Proof. Since V_m is almost Hermite

(3.1) $$F^2+I=0$$

(3.2) a) $M(BX)=0$, b) $'M(BX, BY)=0$.

Substituting from these equations in (2.8)a, we obtain

(3.3) a $$\bar{\bar{X}}+X=0 .$$

In consequence of (3.2) and (2.8)e, the equation (2.9)a assumes the form

(3.3) b $$g(\bar{X}, \bar{Y})=g(X, Y) .$$

The equations (3.3) prove the statement.

Theorem (3.2). *Let V_m be almost Hermite. Then for the almost Hermite invariant submanifold V_n*

(3.4) $$P_x=0 .$$

(3.5) $$F(C_x)=r^y_x C_y ,$$

(3.6) a $$(\bar{D}_{BX}F)(BY)=B(D_Xf)(Y)+\{'h_x(X, \bar{Y})-'h_y(X, Y)r^x_y\}C_x$$

(3.6) b $$(\bar{D}_{BX}F)(C_x)=B\overline{h_x(X)}-r^y_x Bh_y(X)+\{X(r^z_x)-\alpha^y_x(X)r^z_y+r^y_x\alpha^z_y(X)\}C_z$$

Proof. For the almost Hermite enveloping manifold

$$G(F(BX), C_x)+G(BX, F(C_x))=0 .$$

Substituting from (2.9)c and (2.2)a in this equation, we get

$$g(P_x, X)=0 .$$

Since this equation holds for arbitrary X, we have (3.4). Putting $P_x=0$ in (2.7)b, (2.5) and (2.6), we obtain (3.5) and (3.6).

Corollary (3.1). *The invariant almost Hermite submanifold V_n of a Kahler manifold V_m is Kahler. For such a manifold*

(3.7) $$'h_x(X, \bar{Y})='h(X, Y)r^x_y$$

(3.3) a $$\overline{h_x(X)}=r^y_x h_y(X) .$$

(3.8) b $$X(r^z_x)-\alpha^y_x(X)r^z_y+r^y_x\alpha^z_y(X)=0 .$$

Proof. (3.7) and (3.8) follow from (3.5) and (3.6).

Corollary (3.2). *The invariant almost Hermite submanifold V_n of an almost Tachibana manifold is almost Tachibana. For such a manifold*

(3.9) $$'h_x(X, \bar{Y})+'h_x(\bar{X}, Y)=2'h_x(X, Y)r^x_y .$$

Proof. The proof is obvious.

Corollary (3.3). *The invariant almost Hermite submanifold V_n of an almost Kahler manifold is almost Kahler.*

Proof. The proof is obvious.

4. Almost Grayan enveloping manifold

Theorem (4.1). *Let the enveloping manifold V_m be almost Grayan with the structure (F, T, G, A). Then m is odd and*

(4.1) $$T = Bt' + \beta^x C_x$$

(4.2) a $$A(BX)\circ b = g(t', X)\ ,$$

(4.2) b $$\bar{\bar{X}} + X = a(X)t$$

(4.2) c $$g(t', X)t' + p^x(X)P_x \overset{\text{def}}{=} a(X)t$$

(4.2) d $$g(t, X) = a(X)$$

(4.2) e $$g(t', X)\beta^y = p^y(X) + p^x(X)r^y_x\ ,$$

(4.2) f $$p^x(X) = g(P_x, X)\ ,$$

(4.2) g $$r^y_x + r^x_y = 0\ ,$$

(4.2) h $$g(t', X)g(t', Y) + p^x(X)p^x(Y) = {}'m(X, Y)\ ,$$

(4.2) i $$g(\bar{X}, \bar{Y}) = g(X, Y) - {}'m(X, Y)\ ,$$

(4.2) j $$g(X, \bar{Y}) + g(\bar{X}, Y) = 0$$

(4.2) k $$\bar{t}' = \beta^x P_x\ ,$$

(4.2) l $$p^y(t') + \beta^x r^y_x = 0$$

(4.2) m $$\beta^x = A(C_x)\circ b$$

(4.2) n $$a(P_x) = r^y_x \beta^y$$

(4.2) o $$\delta_{xy} - \beta^x\beta^y = g(P_x, P_y) + r^z_x r^z_y$$

(4.2) p $$1 = a(t) + \beta^x\beta^y$$

(4.2) q $$g(t, \bar{X}) + p^x(X)\beta^x = 0\ ,$$

(4.2) r $$g(t', X) = g(\bar{X}, P_x) - p^y(X)r^y_x\ ,$$

Proof. From (4.1) we have

$$A(BX)=G(T, BX)=G(Bt', BX)$$

This equation yields (4.2)a. We have

$$F^2(BX)=-BX+A(BX)T .$$

Substituting from this equation, (4.1) and (4.2)a in (2.2)b, we obtain (4.2)b, c, d. The equation

$$G(F(BX), C_x)+G(BX, F(C_x))=0 ,$$

yields (4.2)f in consequence of (2.1)a and (2.2)a. Similarly the equation

$$G(F(C_x), C_y)+G(C_x, F(C_y))=0$$

yields (4.2)g in consequence (2.2)a. Plugging in from

$$'M(BX, BY)=A(BX)A(BY)$$

and (4.2)a, f in (2.2)f, we obtain (4.2)h. The equation (4.2)i is a consequence of (2.3)a and (4.2)i. The equations (2.3)b and

$$G(F(BX), BY)+G(BX, F(BY))=0$$

yield (4.2)j. The equations (4.1) and (2.1)a yield

$$0=F(T)=B\bar{t}'+p^x(t')C_x+\beta^x(-BP_x+r^y_xC_y)$$

This equation implies (4.2)k, l. Substituting from (4.1) and (4.2)a in

$$G(T, C_x)=A(C_x)$$

and

$$G(T, F(C_x))=A(F(C_x))=0$$

we obtain (4.2)m, n. In consequence of (2.2)a and (4.2)m the equation

$$G(F(C_x), F(C_y))=G(C_x, C_y)-A(C_x)A(C_y)$$

yields (4.2)o. Since

$$1=A(T)=A(Bt')+\beta^xA(C_x)$$

we have (4.2)p. Substituting from (4.1) and (2.1)a in

$$G(T, F(BX))=0 ,$$

we obtain (4.2)q.

Theorem (4.2). *Let the submanifold V_n of the almost Grayan manifold V_m be invariant. Then*

(4.3) $$p^x=\beta^x=P_x=A(C_x)\circ b=0$$

(4.4) a $$F(BX)=B\bar{X}\,,$$

(4.4) b $$F(C_x)=r_x^y C_y\,,$$

(4.4) c $$T=Bt=Bt'$$

(4.4) d $$A(BX)\circ b=g(t, X)=a(X)$$

(4.4) e $$\bar{\bar{X}}+X=a(X)t\,,$$

(4.4) f $$r_x^y+r_y^x=0$$

(4.4) g $$g(\bar{X}, \bar{Y})-g(X, Y)=a(X)a(Y)$$

(4.4) h $$g(X, \bar{Y})+g(\bar{X}, Y)=0$$

(4.4) i $$\bar{t}=0$$

(4.4) j $$a(t)=1$$

(4.4) k $$a(\bar{X})=0\,.$$

Proof. (4.4) follow from (4.2) by putting $p=0$.

Corollary (4.1). *The invariant submanifold of an almost Grayan manifold is almost Grayan with the structure* (f, a, t, g).

Proof. The equations (4.4)e, g prove the statement.

Theorem (4.3). *Let N and n be Nijenhuis tensors of the almost Grayan manifold V_m and the almost Grayan invariant submanifold V_n respectively. Then*

(4.5) $$N(BX, BY)=Bn(X, Y)\,.$$

Proof. In consequence of (4.4)a

$$[F(BX), F(BY)]=[B\bar{X}, B\bar{Y}]=B[\bar{X}, \bar{Y}]\,,$$

$$F^2([BX, BY])=F^2(B[X, Y])=F(B\overline{[X, Y]})=B\overline{\overline{[X, Y]}}\,,$$

$$F([F(BX), BY])=F([B\bar{X}, Y])=F(B[\bar{X}, Y])=B\overline{[\bar{X}, Y]}$$

Hence

$$\begin{aligned} N(BX, BY)&=[F(BX), F(BY)]+F^2([BX, BY])-F([F(BX), BY])\\ &\quad -F([BX, F(BY)])\\ &=B[\bar{X}, \bar{Y}]+B\overline{\overline{[X, Y]}}-B\overline{[\bar{X}, Y]}-B\overline{[X, \bar{Y}]}\\ &=Bn(X, Y)\,. \end{aligned}$$

Corollary (4.2). *We have*

(4.6) $$(dA)(BX, BY)\circ b=(da)(X, Y)\,.$$

Proof. $(dA)(BX, BY)=BX(A(BY))-BY(A(BX))-A([BX, BY])$.

Therefore

$$(dA)(BX, BY)\circ b=X(a(Y))-Y(a(X))-a([X, Y])$$
$$=(da)(X, Y) .$$

Theorem (4.4). *An invariant submanifold V_n of a normal contact Riemannian manifold V_m is normal contact Riemannian.*

Proof. Since V_m is normal contact Riemannian

(4.7) $$N(\lambda, \lambda)+(dA)(\lambda, \lambda)T=0 ,$$

which implies

$$N(BX, BY)+(dA)(BX, BY)T=0 .$$

In consequence of (4.5), (4.6) and (4.4)c, we have

$$n(X, Y)+(da)(X, Y)t=0 .$$

This equation proves the statement.

Theorem (4.5). *An invariant submanifold V_n of an almost Sasakian manifold is alost Sasakian.*

Proof. Since V_m is almost Sasakian

$$2'F=dA ,$$

which in consequence of (4.5) implies

(4.8) a $$2'F(BX, BY)\circ b=(dA)(BX, BY)\circ b=(da)(X, Y).$$

Also in consequence of (1.15)b, (4.4)a and (1.18),

(4.8) b $$'F(BX, BY)\circ b=G(F(BX),\ BY)\circ b=G(B\bar{X}, BY)\circ b$$
$$=g(\bar{X}, Y)='f(X, Y)$$

From (4.8)a, b, we have

$$2'f=da$$

which proves the statement.

Theorem (4.6). *An invariant submanifold V_n of a Sasakian maifold is Sasakian.*

Proof. For the Sasakian manifold V_m

$$'F(\lambda, \lambda)=(D_\lambda A)(\mu) ,$$

which implies

$$
\begin{aligned}
'F(BX, BY)\circ b &= (\bar{D}_{BX}A)(BY)\circ b \\
&= BX(A(BY))\circ b - A(\bar{D}_{BX}BY)\circ b \\
&= X(a(Y)) - A(BD_XY + 'h_x(X, Y)C_x)\circ b \\
&= X(a(Y)) - a(D_XY) \\
&= (D_Xa)(Y) .
\end{aligned}
$$

But since

$$'F(BX, BY)\circ b = 'f(X, Y) ,$$

we have

$$'f(X, Y) = (D_Xa)(Y) .$$

This equation proves the statement.

5. Enveloping manifold with f-structure

Theorem (5.1). *Let the enveloping manifold V_m be a differentiable manifold with f-structure. Then*

(5.1) a $\qquad \bar{\bar{\bar{X}}} + \bar{X} = p^x(\bar{X})P_x + p^x(X)\bar{P}_x + p^y(X)r_y^x P_x$

(5.1) b $\qquad p^y(\bar{\bar{X}}) + p^x(\bar{X})r_x^y - p^x(X)p^y(P_x) + p^x(X)r_x^z r_z^y = -p^y(X)$

(5.1) c $\qquad 'M(BX, BY)\circ b = 'm(X, Y) - p^x(X)p^x(Y) ,$

(5.1) d $\qquad g(\bar{X}, \bar{Y}) = g(X, Y) - 'm(X, Y) ,$

(5.1) e $\qquad g(\bar{X}, Y) + g(X, \bar{Y}) = 0 ,$

(5.1) f $\qquad p^x(X) = g(P_x, X) ,$

(5.1) g $\qquad r_x^y + r_y^x = 0 .$

Proof. We have

$$F^3(BX) = -F(BX) .$$

Substituting from (2.1)a and (2.2)c in this equation, we obtain (5.1)a, b. We have

$$'F(BX, BY) + 'F(BY, BX) = 0$$

Substituting from (2.1)a in this equation we obtain (5.1)e. Similarly we have

$$'F(BX, C_x) + 'F(C_x, BX) = 0 .$$

Substituting in this equation from (2.1)a and (2.2)a, we get (5.1)f (5.1)c, d are consequences of (5.1)f, (2.2)f and (2.3)a.

Since

$$'F(C_x, C_y)+'F(C_y, C_x)=0\ ,$$

we have (5.1)g.

Theorem (5.2). *Let the submanifold V_n of the differentiable manifold V_m with f-structure be invariant. We then, have*

(5.2) a $$F(BX)=B\bar{X}$$

(5.2) b $$F(C_x)=r_x^y C_y\ ,$$

(5.2) c $$\bar{\bar{\bar{X}}}+\bar{X}=0\ ,$$

(5.2) d $$'M(BX, BY)\circ b='m(X, Y)\ ,$$

(5.2) e $$g(\bar{X}, \bar{Y})=g(X, Y)-'m(X, Y)$$

(5.2) f $$g(X, \bar{Y})+g(\bar{X}, Y)=0$$

(5.2) g $$P_x=0\ ,$$

(5.2) h $$r_x^y r_y^x=0\ .$$

Proof. Putting $p^x=0$ in (5.1), we obtain (5.2).

Corollary (5.1). *The invariant submanifold V_n of the differentiable Riemannian manifold V_m with f-structure is the differentiable Riemannian manifold with f-structure.*

Proof. The statement follows (5.2)c, e.

Corollary (5.2). *For the invariant Riemannian submanifold with f-structure of the Riemannian manifold V_m with f-structure, we have*

(5.3) a $$(\bar{D}_{BX}F)(BY)=B(D_Xf)(Y)+\{'h_x(X, Y)+'h_y(X, Y)r_x^y\}C_x\ ,$$

(5.3) b $$(\bar{D}_{BX}F)(C_x)=B\overline{h_x(X)}-r_x^y Bh_y(X)+\{-\alpha_x^y r_y^z +X(r_x^z)+r_x^y\alpha_y^z-r_x^y r_y^z\}C_z\ .$$

Proof. (5.3)a, b follow from (2.5) and (2.6) by putting $p^x=0$.

6. Almost Hermite manifold with non-invariant submanifold:

Theorem (6.1). *Let V_m be an almost Hermite manifold. Then the necessary and sufficient conditions that the non-invariant submanifold V_n be almost Hermite are*

(6.1) a $$p^x(X)P_x=0$$

(6.1) b $$p^x(X)+p^z(X)r^y_x=0$$

(6.1) c $$p^x(X)=g(P_x, X)\ ,$$

(6.1) d $$r^y_x+r^x_y=0\ ,$$

(6.1) e $$\delta_{xy}=g(P_x, P_y)+r^z_x r^z_y\ .$$

Proof. When V_m and V_n are both almost Hermite,

(6.2) a $$F^2(BX)=-BX$$

(6.2) b $$\bar{\bar{X}}=-X$$

(6.2) c $$'M='m=0$$

(6.2) d $$G(F(BX), C_x)+G(BX, F(C_x))=0$$

(6.2) e $$G(F(C_x), C_y)+G(C_x, F(C_y))=0$$

(6.2) f $$G(F(C_x), F(C_y))=G(C_x, C_y)\ .$$

Substituting these values in (2.2)b, (2.3)c, e, f, g we obtain (6.1).

Corollary (6.1). *Let the non-invariant submanifold of a Kahler manifold be Kahler. Then*

(6.3) a $$'h_x(X, Y)P_x=p^x(Y)h_x(X)\ ,$$

(6.3) b $$'h_x(X, \bar{Y})+(D_X p^x)(Y)+p^y(Y)\alpha^x_y(X)-'h_y(X, Y)r^x_y=0\ ,$$

(6.3) c $$\overline{h_x(X)}+r^y_x(X)P_y-D_X P_x-r^y_x h_y(X)=0$$

(6.3) d $$X(r^z_x)-\alpha^y_x(X)r^z_y+r^y_x\alpha^z_y(X)-'h_z(X, P_x)+p^z(h_x(X))-r^y_x r^z_y=0\ .$$

Proof. (6.3) follow from (2.5) and (2.6).

Corollary (6.2). *Let the non-invariant submanifold of an almost Tachibana manifold be almost Tachibana. Then*

(6.4) a $$2'h_x(X, Y)P_x=p^x(Y)h_x(X)+p^x(X)h_x(Y)$$

(6.4) b $$\begin{aligned}&'h_x(X, \bar{Y})+'h_x(\bar{X}, Y)+(D_X p^x)(Y)+(D_Y p^x)(X)\\&\quad+p^y(Y)\alpha^x_y(X)+p^y(X)\alpha^x_y(Y)\\&=2'h_y(X, Y)r^x_y\ .\end{aligned}$$

Proof. (6.4) follow from (2.5).

Theorem (6.2). *Let the enveloping manifold V_m be almost Grayan with the structure (F, T, G, A). Then the non-invariant submanifold V_n is also almost Grayan with the structure (f, t, g, a) if t, a are given by* (4.2)c, d.

Proof. The statement follows from (4.2)b, c, d, h, i.

Theorem (6.3). *Let the enveloping manifold V_m be almost Grayan with the structure (F, T, G, A). Let the non-invariant submanifold V_n be such that*

(6.5) $$g(t', X)t' + p^x(X)P_x = 0 .$$

Then V_n is almost Hermite.

Proof. In view of (4.2)c, d and (6.5), we have

$$a = 0 = t .$$

Then (4.2)b, h, i assume the forms

(6.6) $$\bar{\bar{X}} + X = 0$$

(6.7) $$'m = 0$$

(6.8) $$g(\bar{X}, \bar{Y}) = g(X, Y) .$$

These equations prove the statement.

Theorem (6.4). *Let the enveloping manifold V_m be a differentiable Riemannian manifold with f-structure. Let the submanifold V_m be non-invariant. Then*

(6.9) a $$\bar{\bar{\bar{X}}} + \bar{X} = p^x(\bar{X})P_x + p^x(X)\bar{P}_x + p^x(X)r_x^y P_y$$

(6.9) b $$p^y(\bar{\bar{X}}) + p^x(\bar{X})r_x^y + p^y(X) + p^x(X)r_x^z r_z^y = p^x(X)p^y(P_x)$$

(6.9) c $$\bar{\bar{p}} + P_x + r_x^y \bar{P}_y + r_x^y r_y^z P_z = p^y(P_x)P_y$$

(6.9) d $$p^y \bar{P}_x + p^z(P_x)r_z^y = r_x^y + r_x^z r_z^t r_t^y$$

(6.9) e $$g(\bar{X}, Y) + g(X, \bar{Y}) = 0$$

(6.9) f $$p^x(X) = g(P_x, X) ,$$

(6.9) g $$'M(BX, C_x)\circ b = p^x(\bar{X}) + p^y(X)r_x^y$$

(6.9) h $$'M(BX, BY)\circ b = {}'m(X, Y) - p^x(X)p^x(Y)$$

(6.9) i $$g(\bar{X}, \bar{Y}) = g(X, Y) - {}'m(X, Y) .$$

(6.9) j $$r_x^y + r_y^x = 0$$

(6.9) k $$'M(C_x, C_y) = \delta_{xy} - p^x(P_y) - r_x^z r_y^z$$

Proof. We have for V_m

$$F^3(BX) = -F(BX) , \quad F^3(C_x) = -F(C_x) .$$

Substituting from these equations in (2.2)c, e and using (2.1)a and (2.2)a, we obtain (6.9)a-d. Using (2.3)b in

$$'F(BX, BY)+'F(BY, BX)=0\ ,$$

we get (6.9)e. Using (2.3)c, e in

$$'F(BX, C_x)+'F(C_x, BX)=0\ ,$$

we get (6.9)f. Similarly using (2.3)f in

$$'F(C_x, C_y)+'F(C_y, C_x)=0$$

we get (6.9)j. (6.9)g follows from (2.2)b. (6.9)h, i are obtained from (2.2)f and (2.3)a by the use of (6.9)f. Since

$$G(F(C_x), F(C_y))=G(C_x, C_y)-'M(C_x, C_y)\ ,$$

we have (6.9)k, by the use of (2.3)g.

Theorem (6.5). *Let the enveloping manifold V_m be a differential Riemannian manifold with f-structure. The necessary and sufficient condition that the non-invariant submanifold V_n be also differentiable Riemannian manifold with f-structure is*

$$p^x(\bar{X})P_x+p^x(X)\bar{P}_x+p^x(X)r^y_xP_y=0\ . \tag{6.10}$$

Proof. The statement follows from (6.9)a, i.

Department of Mathematics,
Banaras Hindu University,
Varanasi 221005.