FIRST ORDER AND SECOND KIND RECURRENT MANIFOLDS

By M. P. S. Rathore and R. S. Mishra

(Department of Mathematics, Banaras Hindu University, Varanasi-5, India)

Received June 24, 1972

1. Introduction

We consider $2n$-dimensional real manifold V_{2n} of differentiability class C^{r+1}, there exists in V_{2n} a vector valued linear function F such that

$$\bar{\bar{X}} + X = 0, \text{ for arbitrary vector field } X, \tag{1.1 a}$$

where

$$\bar{X} \overset{\text{def.}}{=} F(X). \tag{1.1 b}$$

Then F is said to give an almost complex structure to V_{2n} and V_{2n} is called an almost complex manifold.

Agreement (1.1); All the equations which follow hold for arbitrary vector fields X, Y, Z, T, W, ..., etc.

Let the almost complex manifold V_{2n} be also endowed with the Hermitian metric g:

$$g(\bar{X}, \bar{Y}) = g(X, Y), \tag{1.2}$$

then V_{2n} is called an almost Hermite manifold.

Let us put

$$'F(X, Y) \overset{\text{def.}}{=} g(\bar{X}, Y). \tag{1.3}$$

Then from (1.1 a), (1.2) and (1.3), we have

$$'F(\bar{X}, \bar{Y}) = -g(X, \bar{Y}) = g(\bar{X}, Y) = {}'F(X, Y), \tag{1.4 a}$$

$$'F(X, \bar{Y}) = g(X, Y) = -{}'F(\bar{X}, Y),$$

$$'F(X, Y) = -{}'F(Y, X). \tag{1.4 c}$$

Suppose D_X is a Riemannian connexion satisfying:

$$D_X Y - D_Y X = [X, Y], \tag{1.5 a}$$

$$(D_X g)(Y, Z) = 0. \tag{1.5 b}$$

Let K be the curvature tensor of V_{2n} w.r.t. D_X given by

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \tag{1.6}$$

Let Ric be the Ricci tensor of V_{2n} given by

$$\mathrm{Ric}(Y, Z) \overset{\text{def.}}{=} (C_1^1 K)(Y, Z) \tag{1.7}$$

and

$$\mathrm{Ric}(Y, Z) = \mathrm{Ric}(Z, Y). \tag{1.8}$$

Let us put

$$\mathrm{Ric}(Y, Z) \overset{\text{def.}}{=} g\big(r(Y), Z\big) \tag{1.9}$$

and

$$R \overset{\text{def.}}{=} (C_1^1 r). \tag{1.10}$$

where R is the scalar curvature.

In an almost Hermite manifold V_{2n}, we have

$$(D_Y F)(\bar{X}) = -(D_Y F)(X). \tag{1.11}$$

We know that manifold V_{2n} is a birecurrent manifold if

$$(D_Y D_X K)(Z, T, W) = a(X, Y) K(Z, T, W) + (D_{D_Y X} K)(Z, T, W), \tag{1.12}$$

w.r.t. D_X and $a(X, Y)$ is a non-vanishing c^∞ function. The manifold V_{2n} is called Ricci birecurrent manifold if

$$(D_Y D_X \mathrm{Ric})(T, W) = a(X, Y)\, \mathrm{Ric}(T, W) + (D_{D_Y X} \mathrm{Ric})(T, W), \tag{1.13 a}$$

which yields

$$(D_Y D_X r)(T) = a(X, Y)\, r(T) + (D_{D_Y X} r)(T). \tag{1.13 b}$$

The projective curvature tensor *W, the conformal curvature tensor V, the conharmonic curvature tensor L, the concircular curvature tensor C are given by [2]

$$*W(Z,T,W) = K(Z,T,W) - \frac{1}{(2n-1)}[Z\,\mathrm{Ric}(T,W) - T\,\mathrm{Ric}(Z,W)], \tag{1.14}$$

$$\begin{aligned} V(Z,T,W) &= K(Z,T,W) - \frac{1}{2(n-1)}[Z\,\mathrm{Ric}(T,W) \\ &\quad - T\,\mathrm{Ric}(Z,W) + r(Z)\,g(T,W) \\ &\quad - r(T)\,g(Z,W)] + \frac{R}{2(2n-1)(n-1)} \\ &\quad \times [Zg(T,W) - Tg(Z,W)], \end{aligned} \tag{1.15}$$

$$\begin{aligned} L(Z,T,W) &= K(Z,T,W) - \frac{1}{2(n-1)}[Z\,\mathrm{Ric}(T,W) \\ &\quad - T\,\mathrm{Ric}(Z,W) + r(Z)\,g(T,W) \\ &\quad - r(T)\,g(Z,W)], \end{aligned} \tag{1.16}$$

$$C(Z,T,W) = K(Z,T,W) - \frac{R}{2n(2n-1)}[Zg(T,W) - Tg(Z,W)], \tag{1.17}$$

respectively.

2. Basic Formulas

We can make use of the following result (Ricci identities) wherever necessary,

$$(D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z) = K(Y,X,\bar{Z}) - \overline{K(Y,X,\bar{Z})}, \tag{2.1}$$

$$(D_Y D_X{}'F)(Z,T) - (D_X D_Y{}'F)(Z,T) - (D_{[Y,X]}{}'F)(Z,T) = -\,'F(K(Y,X,Z),T) - {}'F(Z,K(Y,X,T)). \tag{2.2}$$

$$\begin{aligned} (D_Y D_X P)(Z,T,W) &- (D_X D_Y P)(Z,T,W) - (D_{[Y,X]} P)(Z,T,W) \\ &= K(Y,X,P(Z,T,W)) - P(K(Y,X,Z),T,W) \\ &\quad - P(Z,K(Y,X,T),W) - P(Z,T,K(Y,X,W)). \end{aligned} \tag{2.3}$$

where P is any one of the curvature tensors K, *W, V, L or C.

$$(D_Y D_X Ric)(Z, T) - (D_X D_Y Ric)(Z, T) - (D_{[Y,X]} Ric)(Z, T)$$
$$= - Ric(K(Y, X, Z), T) - Ric(Z, K(Y, X, T)). \tag{2.4}$$
$$(D_Y D_X r)(Z) - (D_X D_Y r)(Z) - (D_{[Y,X]} r)(Z)$$
$$= K(Y, X, r(Z)) - r(K(Y, X, Z)).$$

3. DEFINITIONS

Let P, a vector valued trilinear function be any one of the curvature tensors K, *W, V, L or C. Then the almost Hermite manifold V_{2n} is said to be

(i) Birecurrent in P (first order and second kind recurrent) if

$$(D_Y D_X P)(\bar{Z}, T, W) + (D_X P)((D_Y F)(Z), T, W) - (D_{D_Y X} P)(\bar{Z}, T, W)$$
$$+ (D_Y P)((D_X F)(Z), T, W) + P((D_Y D_X F)(Z), T, W))$$
$$= a(X, Y) P(\bar{Z}, T, W) + P((D_{D_Y X} F)(Z), T, W). \tag{3.1}$$

(ii) (1, 2) birecurrent in P if:

$$(D_Y D_X P)(\bar{Z}, \bar{T}, W) + (D_X P)((D_Y F)(Z), \bar{T}, W) - (D_{D_Y X} P)(\bar{Z}, \bar{T}, W)$$
$$+ (D_Y P)(\bar{Z}, (D_Y F)(T), W) + (D_Y P)((D_X F)(Z), \bar{T}, W)$$
$$- P((D_{D_Y X} F)(Z), \bar{T}, W) + P((D_Y D_X F)(Z), \bar{T}, W)$$
$$- P((D_X F)(Z), (D_Y F)(T), W) + (D_Y P)(\bar{Z}, (D_X F)(T), W)$$
$$- P((D_Y F)(Z), (D_X F)(T), W) - P(\bar{Z}, (D_{D_Y X} F)(T), W)$$
$$- P(\bar{Z}, (D_Y D_X F)(T), W) = a(X, Y) P(\bar{Z}, \bar{T}, W). \tag{3.2}$$

On interchanging X and Y in (3.2) and subtracting the resulting equation thus obtained from (3.2), we have

$$(D_Y D_X P)(\bar{Z}, \bar{T}, W) - (D_X D_Y P)(\bar{Z}, \bar{T}, W) + P((D_Y D_X F)(Z)$$
$$- (D_{[Y,X]} F)(Z) - (D_X D_Y F)(Z), \bar{T}, W)$$

$$\begin{aligned}&+ P\big(\bar{Z}, (D_Y D_X F)(T) - (D_{[Y,X]} F)(T) - (D_X D_Y F)(T), W\big)\\ &= A(X, Y)\, P(\bar{Z}, \bar{T}, W) + (D_{[Y,X]} P)(\bar{Z}, \bar{T}, W),\end{aligned}$$

where

$$A(X, Y) \overset{\text{def.}}{=} a(X, Y) - a(Y, X).$$

(iii) (1, 3) birecurrent in P if

$$\begin{aligned}&(D_Y D_X P)(\bar{Z}, T, \bar{W}) - (D_X D_Y P)(\bar{Z}, T, \bar{W}) + P\big((D_Y D_X F)(Z)\\ &\quad - (D_{[Y,X]} F)(Z) - (D_X D_Y F)(Z), T, \bar{W}\big)\\ &\quad + P\big(\bar{Z}, T, (D_Y D_X F)(W) - (D_{[Y.X]} F)(W) - (D_X D_Y F)(W)\big)\\ &= A(X, Y)\, P(\bar{Z}, T, \bar{W}) + (D_{[Y,X]} P)(\bar{Z}, T, \bar{W}).\end{aligned} \tag{3.3}$$

(iv) (1, 2, 3) birecurrent in P if

$$\begin{aligned}&(D_Y D_X P)(\bar{Z}, \bar{T}, \bar{W}) - (D_X D_Y P)(\bar{Z}, \bar{T}, \bar{W}) + P\big((D_Y D_X F)(Z)\\ &\quad - (D_{[Y,X]} F)(Z) - (D_X D_Y F)(Z). \bar{T}, \bar{W}\big)\\ &\quad + P\big(\bar{Z}, (D_Y D_X F)(T) - (D_{[Y,X]} F)(T) - (D_X D_Y F)(T), \bar{W}\big)\\ &\quad + P\big(\bar{Z}, \bar{T}, (D_Y D_X F)(W) - (D_{[Y,X]} F)(W) - (D_X D_Y F)(W)\big)\\ &= A(X, Y)\, P(\bar{Z}, \bar{T}, \bar{W}) + (D_{[Y,X]} P)(\bar{Z}, \bar{T}, \bar{W}).\end{aligned} \tag{3.4}$$

(v) (1, 2, 3, 4) birecurrent in P if

$$\begin{aligned}&(D_Y D_X\, {}'P)(\bar{Z}, \bar{T}, \bar{W}, \bar{U}) - (D_X D_Y\, {}'P)(\bar{Z}, \bar{T}, \bar{W}, \bar{U})\\ &\quad - (D_{[Y,X]}\, {}'P)(\bar{Z}, \bar{T}, \bar{W}, \bar{U}) + {}'P\big((D_Y D_X F)(Z)\\ &\quad - (D_X D_Y F)(Z), \bar{T}, \bar{W}, \bar{U}\big) - {}'P\big(D_{[Y,X]} F(Z), \bar{T}, \bar{W}, \bar{U}\big)\\ &\quad + {}'P\big(\bar{Z}, (D_Y D_X F)(T) - (D_X D_Y F)(T), \bar{W}, \bar{U}\big)\\ &\quad - {}'P\big(\bar{Z}, (D_{[Y,X]} F)(T), \bar{W}, \bar{U}\big) + {}'P\big(\bar{Z}, \bar{T}, (D_Y D_X F)(W)\end{aligned}$$

$$- (D_X D_Y F)(W), \bar{U}) - {}'P(\bar{Z}, \bar{T}, (D_{[Y,X]}F)(W), \bar{U})$$

$$- {}'P(\bar{Z}, \bar{T}, \bar{W}, (D_Y D_X F)(U) - (D_X D_Y F)(U))$$

$$- {}'P(\bar{Z}, \bar{T}, \bar{W}, (D_{[Y,X]}F)(U)) = A(X, Y)\,{}'P(\bar{Z}, \bar{T}, \bar{W}, \bar{U}), \tag{3.5 a}$$

where

$${}'P(Z, T, W, U) \overset{\text{def.}}{=} g(P(Z, T, W), U). \tag{3.5 b}$$

We can similarly define (3), (4), (1, 4), (2, 3), (2, 4), (3, 4), (1, 2, 4), (1, 3, 4), (2, 3, 4) birecurrence in P.

Definitions (3.1 *b*).—V_{2n} is said to be generalized (1), (1, 2), (1, 3), (1, 2, 3) and (1, 2, 3, 4) birecurrent symmetric if $A(X, Y) = 0$, *i.e.*, $a(X, Y) = a(Y, X)$ in (3.1), (3.2), (3.3), (3.4) or (3.5 *a*).

Theorem (3.1).—For an almost Hermite manifold V_{2n}, we have

$$(D_Y D_X F)(\bar{Z}) - (D_X D_Y F)(\bar{Z}) = -\overline{(D_Y D_X)F(Z)} - \overline{(D_X D_Y F)(Z)}. \tag{3.1}$$

Proof.—We have

$$(D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]}F)(Z)$$

$$= K(Y, X, \bar{Z}) - \overline{K(Y, X, \bar{Z})}. \tag{3.1 a}$$

Barring Z in (3.1 *a*), we have

$$(D_Y D_X F)(\bar{Z}) \cdot (D_X D_Y F)(\bar{Z}) - (D_{[Y,X]}F)(\bar{Z})$$

$$= -K(Y, X, Z) - \overline{K(Y, X, Z)}. \tag{3.1 b}$$

Barring (3.1 *b*) throughout and comparing this with (3.1 *a*), we have

$$(D_Y D_X F)(\bar{Z}) - \overline{(D_X D_Y F)(\bar{Z})} - \overline{(D_{[Y,X]}F)(\bar{Z})}$$

$$= (D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]}F)(Z). \tag{3.1 c}$$

Barring (3.1 *c*) throughout, we get

$$(D_Y D_X F)(\bar{Z}) - (D_X D_Y F)(\bar{Z}) - (D_{[Y, X]} F)(\bar{Z})$$

$$= -[\overline{(D_Y D_X F)(Z)} - \overline{(D_X D_Y F)(Z)} - \overline{(D_{[Y,X]} F)(Z)}]. \quad (3.1\,d)$$

We know from (1.11) that

$$(D_X F)(\bar{Z}) = -\overline{(D_X F)(Z)}. \quad (3.1\,e)$$

From (3.1 *d*) and (3.1 *e*), we get

$$(D_Y D_X F)(\bar{Z}) - (D_X D_Y F)(\bar{Z}) = -[\overline{(D_Y D_X F)(Z)} - \overline{(D_X D_Y F)(Z)}], \quad (3.1\,f)$$

which proves the statement.

Theorem (3.2).—The necessary and sufficient condition that an almost Hermite manifold V_{2n} be (I) birecurrent in K, (K*b*I) is:

$$(D_Y D_X K)(\bar{Z}, T, W) - (D_X D_Y K)(\bar{Z}, T, W) + K\Big((D_Y D_X F)(Z) - (D_{[Y,X]} F)(Z) - (D_X D_Y F)(\bar{Z}), T, W\Big)$$
$$= A(X, Y) K(\bar{Z}, T, W) + (D_{[Y,X]} K)(\bar{Z}, T, W) \quad (3.2\,a)$$

or

$$(D_Y D_X \mathrm{Ric})(\bar{Z}, W) - (D_X D_Y \mathrm{Ric})(\bar{Z}, W) + \mathrm{Ric}\Big((D_Y D_X F)(Z) - (D_{[Y,X]} F)(Z) - (D_X D_Y F)(Z), W\Big)$$
$$= A(X, Y) \mathrm{Ric}(\bar{Z}, W) + (D_{[Y,X]} \mathrm{Ric})(\bar{Z}, W), \quad (3.2\,b)$$

or

$$(D_Y D_X \mathrm{Ric})(Z, W) - (D_X D_Y \mathrm{Ric})(Z, W) - \mathrm{Ric}\Big((D_Y D_X F)(\bar{Z}) - (D_{[Y,X]} F)(\bar{Z}) - (D_X D_Y F)(\bar{Z}), W\Big)$$
$$= A(X, Y) \mathrm{Ric}(Z, W) + (D_{[Y,X]} \mathrm{Ric}) Z, W), \quad (3.2\,c)$$

or

$$(D_Y D_X r)(\bar{Z}) - (D_X D_Y r)(\bar{Z}) + r\Big((D_Y D_X F)(Z) - (D_X D_Y F)(Z)\Big)$$
$$- r\Big((D_{[Y,X]} F)(Z)\Big) = A(X, Y)\, r(\bar{Z}) + (D_{[Y,X]} r)(\bar{Z}), \quad (3.2\,d)$$

or

$$(D_Y D_X r)(Z) - (D_X D_Y r)(Z) - r\big((D_Y D_X F)(\bar{Z}) - (D_X D_Y F)(\bar{Z})\big)$$
$$+ r\big((D_{[Y,X]} F)(\bar{Z})\big) = A(X, Y)\, r(Z) + (D_{[Y,X]} r)(Z), \quad (3.2\,e)$$

or

$$(D_Y D_X R) - (D_X D_Y R) + C_1^1\big(r\,(\overline{D_Y D_X F} - \overline{D_X D_Y F}) - (\overline{D_{[Y,X]} F})\big)$$
$$= A(X, Y)\, R + (D_{[Y,X]} R). \quad (3.2\,f)$$

Proof.—(3.2 *a*) follows from the definition (3.1). Contracting (3.2 *a*), we get (3.2 *b*) to (3.2 *f*). Using theorem (3.1) in (3.2 *e*) and contracting, we get (3.2 *f*).

Theorem (3.3).—The necessary and sufficient condition that an almost Hermite manifold V_{2n} be (1) projective birecurrent (WK*b*I) is:

$$A(X, Y)\,{}^*W(\bar{Z}, T, W) + (D_{[Y,X]}{}^*W)(\bar{Z}, T, W)$$
$$+ {}^*W\big((D_{[Y,X]} F)(Z), T, W\big)$$
$$= (D_Y D_X K)(\bar{Z}, T, W) - (D_X D_Y K)(\bar{Z}, T, W)$$
$$+ K\big((D_Y D_X F)(Z) - (D_X D_Y F)(Z), T, W\big)$$
$$- \frac{1}{(2n-1)}\Big[\big((D_Y D_X F)(Z) - (D_X D_Y F)(Z)\big)\,\mathrm{Ric}(T, W)$$
$$+ \bar{Z}\big((D_Y D_X \mathrm{Ric})(T, W) - (D_X D_Y \mathrm{Ric})(T, W)\big)$$
$$- T\{(D_Y D_X \mathrm{Ric})(\bar{Z}, W) - (D_X D_Y \mathrm{Ric})(\bar{Z}, W)$$
$$+ \mathrm{Ric}\big((D_Y D_X F)(Z) - (D_X D_Y F)(Z), W\big)\}\Big]. \quad (3.3)$$

Proof.—The Weyl curvature tensor is given by (Mishra, 1965).

$${}^*W(\bar{Z}, T, W) = K(\bar{Z}, T, W) - \frac{1}{(2n-1)}[\bar{Z}\,\mathrm{Ric}(T, W)$$
$$- T\,\mathrm{Ric}(\bar{Z}, W)]. \quad (3.3\,a)$$

From (3.3 *a*), we have

$$(D_X{}^*W)(\bar{Z}, T, W) + {}^*W\big((D_XF)(Z), T, W\big)$$
$$= (D_XK)(\bar{Z}, T, W) + K\big((D_XF)(Z), T, W\big)$$
$$- \frac{1}{(2n-1)}\big[(D_XF)(Z)\,\mathrm{Ric}(T, W) + \bar{Z}\,(D_X\mathrm{Ric})(T, W)$$
$$- T\big((D_X\mathrm{Ric})(\bar{Z}, W)\big) + \mathrm{Ric}\big((D_XF)(Z), W\big)\big],$$

and

$$(D_YD_X{}^*W)(\bar{Z}, T, W) + (D_X{}^*W)\big((D_YF)(Z), T, W\big)$$
$$+ (D_Y{}^*W)\big((D_XF)(Z), T, W\big) + {}^*W\big((D_YD_XF)(Z), T, W\big)$$
$$= (D_YD_XK)(\bar{Z}, T, W) + (D_XK)\big((D_YF)(Z), T, W\big)$$
$$+ (D_YK)\big((D_XF)(Z), T, W\big) + K\big((D_YD_XF)(Z), T, W\big)$$
$$- \frac{1}{(2n-1)}\big[(D_YD_XF)(Z)\,\mathrm{Ric}(T, W)$$
$$+ (D_XF)(Z)\,(D_Y\mathrm{Ric})(T, W) + \big((D_YF)(Z)\big)(D_X\mathrm{Ric})(T, W)$$
$$+ \bar{Z}\,(D_YD_X\mathrm{Ric})(T, W) - T\,\{(D_YD_X\mathrm{Ric})(\bar{Z}, W)$$
$$+ (D_X\mathrm{Ric})\big((D_YF)(Z), W\big) + (D_Y\mathrm{Ric})\big((D_XF)(Z), W\big)$$
$$+ \mathrm{Ric}\big((D_YD_XF)(Z), W\big)\}\big]. \qquad (3.3\,b)$$

Let the manifold be (1) projective birecurrent and then interchanging X and Y and subtracting the resulting equation from (3.3 *b*), we get

$$A(X, Y)^*W(\bar{Z}, T, W) + (D_{[Y,X]}{}^*W)(\bar{Z}, T, W)$$
$$+ {}^*W\big((D_{[Y,X]}F)(Z), T, W\big)$$
$$= (D_YD_XK)(\bar{Z}, T, W) - (D_XD_YK)(\bar{Z}, T, W)$$
$$+ K\big(((D_YD_XF)(Z) - (D_XD_YF)(Z)), T, W\big)$$
$$- \frac{1}{(2n-1)}\big[\big((D_YD_XF)(Z) - (D_XD_YF)(Z)\big)(\mathrm{Ric}\,T, W)$$

$$+ \bar{Z}\left((D_Y D_X \mathrm{Ric})(T, W) - (D_X D_Y \mathrm{Ric})(T, W)\right)$$

$$- T\{(D_Y D_X \mathrm{Ric})(\bar{Z}, W) - (D_X D_Y \mathrm{Ric})(\bar{Z}, W)$$

$$+ \mathrm{Ric}\left(((D_Y D_X F)(Z) - (D_X D_Y F)(Z)), W\right)\}], \qquad (3.3\,c)$$

which proves the statement.

Theorem (3.4).—The necessary and sufficient condition that the almost Hermite manifold V_{2n} be generalized symmetric in *W of the first order and second kind is:

$$(D_Y D_X K)(\bar{Z}, T, W) - (D_X D_Y K)(\bar{Z}, T, W) - (D_{[Y,X]} K)(\bar{Z}, T, W)$$

$$+ K\left(((D_Y D_X F)(Z) - (D_{[Y,X]} F)(Z) - (D_X D_Y F)(Z)), T, W\right)$$

$$- \frac{1}{(2n-1)}\left[((D_Y D_X F)(Z) - (D_{[Y,X]} F)(Z)\right.$$

$$- (D_X D_Y F)(Z))\, \mathrm{Ric}(T, W) + \bar{Z}\left((D_Y D_X \mathrm{Ric})(T, W)\right.$$

$$\left. - (D_{[Y,X]} \mathrm{Ric})(T, W) - (D_Y D_X \mathrm{Ric})(T, W)\right)$$

$$- T\{(D_Y D_X \mathrm{Ric})(\bar{Z}, W) - (D_{[Y,X]} \mathrm{Ric})(\bar{Z}, W)$$

$$- (D_X D_Y \mathrm{Ric})(\bar{Z}, W) + \mathrm{Ric}((D_Y D_X F)(Z) - (D_{[Y,X]} F)(Z)$$

$$\left. - (D_X D_Y F)(Z), W)\}\right] = 0. \qquad (3.4)$$

Proof.—Now using the fact that V_{2n} is said to be generalized symmetric in *W of the first order and second if $A(X, Y) = 0$. Hence, from Theorem (3.3), we get the result.

Theorem (3.5).—In order that K*b*I be WK*b*I, we must have

$$\bar{Z}\, \mathrm{Ric}\left(((D_Y D_X F)(\bar{T}) - (D_X D_Y F)(\bar{T}) - (D_{[Y,X]} F)(\bar{T})), W\right)$$

$$+ \left((D_Y D_X F)(Z) - (D_{[Y,X]} F)(Z) - (D_X D_Y F)(Z)\right)$$

$$\times \mathrm{Ric}(T, W) = 0.$$

Proof.—From Theorem (3.2 *a*) and (3.3), we have

$$A(X, Y)\left(^*W(\bar{Z}, T, W) - K(\bar{Z}, T, W)\right)$$

$$= - \frac{1}{(2n-1)}\left[((D_Y D_X F)(Z) - (D_X D_Y F)(Z)\right.$$

$$- (D_{[Y,X]} F)(Z)\big) \, Ric(T, W) + \bar{Z}\big((D_Y D_X Ric)(T, W)$$
$$- (D_X D_Y Ric)(T, W) - (D_{[Y,X]} Ric)(T, W)\big)$$
$$- T\{(D_Y D_X Ric)(\bar{Z}, W) - (D_X D_Y Ric)(\bar{Z}, W)$$
$$- (D_{[Y,X]} Ric)(\bar{Z}, W) + Ric\big(((D_Y D_X F)(Z)$$
$$- (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z)), W\big)\}]. \qquad (3.5\,a)$$

Substituting the value of $^*W(\bar{Z}, T, W) - K(\bar{Z}, T, W)$ from (3.3 *a*) in (3.5 *a*) and using Th (3.2 *b*) and (3.2 *c*), we have

$$[\bar{Z}\big((D_Y D_X Ric)(T, W) - (D_X D_Y Ric)(T, W) - (D_{[Y,X]} Ric)(T, W)\big)$$
$$- Ric\big((D_Y D_X F)(\bar{T}) - (D_{[Y,X]} F)(\bar{T}) - (D_X D_Y F)(\bar{T}), W)\big)$$
$$- T(D_Y D_X Ric)(\bar{Z}, W) - D_{[Y,X]} Ric)(\bar{Z}, W)$$
$$- (D_X D_Y Ric)(\bar{Z}, W) + Ric\big((D_Y D_X F)(Z)$$
$$- (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z), W\big)$$
$$= \big((D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z)\big) \, Ric(T, W)$$
$$+ Z\big((D_Y D_X Ric)(T, W) - (D_X D_Y Ric)(T, W)$$
$$- (D_{[Y,X]} Ric)(T, W)\big) - T\{(D_Y D_X Ric)(\bar{Z}, W)$$
$$- (D_X D_Y Ric)(\bar{Z}, W) - (D_{[Y,X]} Ric)(\bar{Z}, W)$$
$$+ Ric\big((D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z), W\big)\}]$$

i.e.,

$$- \bar{Z} \, Ric\big(((D_Y D_X F)(\bar{T}) - (D_X D_Y F)(\bar{T}) - (D_{[Y,X]} F)(\bar{T})), W\big)$$
$$= \big((D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z)\big) \, Ric(T, W),$$

which proves the statement.

Theorem (3.6).—If an almost Hermite manifold is a projective birecurrent manifold of the first order and a Ricci birecurrent manifold for the same recurrence parameter, then it is a birecurrent manifold of the first order provided

$$(D_Y D_X F)(Z) \, Ric(T, W) + (D_X F)(Z)(D_Y Ric)(T, W)$$
$$- (D_{D_Y X} F)(Z) \, Ric(T, W) + (D_Y F)(Z)(D_X Ric)(T, W) = 0. \qquad (3.6)$$

or

$$\big((D_Y D_X F)(Z) - (D_X D_Y F)(Z) - (D_{[Y,X]} F)(Z)\big) \operatorname{Ric}(T, W) = 0.$$

Proof.—We have

$$\begin{aligned}
&(D_Y D_X {}^*W)(\bar{Z}, T, W) + (D_X {}^*W)\big((D_Y F)(Z), T, W\big) \\
&\quad + (D_Y {}^*W)(D_X F)(Z), T, W) + {}^*W\big((D_Y D_X F)(Z), T, W\big) \\
&= (D_X D_Y K)(\bar{Z}, T, W) + (D_X K)\big((D_Y F)(Z), T, W\big) \\
&\quad + (D_Y K)(D_X F)(Z), T, W) + K\big((D_Y D_X F)(Z), T, W\big) \\
&\quad - \frac{1}{(2n-1)} [(D_Y D_X F)(Z) \operatorname{Ric}(T, W) \\
&\quad + (D_X F)(Z)(D_Y \operatorname{Ric})(T, W) + (D_Y F)(Z)(D_X \operatorname{Ric})(T, W) \\
&\quad + \bar{Z}(D_Y D_X \operatorname{Ric})(T, W) - T\{(D_Y D_X \operatorname{Ric})(\bar{Z}, W) \\
&\quad + (D_X \operatorname{Ric})\big((D_Y F)(Z), W\big) + (D_Y \operatorname{Ric})((D_X F)(Z), W\big) \\
&\quad + \operatorname{Ric}\big((D_Y D_X F)(Z), W\big)\}].
\end{aligned} \tag{3.6 a}$$

Let the manifold be (1) projective birecurrent and Ricci-birecurrent for the same recurrence parameter, then we have:

$$\begin{aligned}
&a(X, Y)\Big({}^*W(\bar{Z}, T, W) + \frac{1}{(2n-1)}\big(\bar{Z} \operatorname{Ric}(T, W) - T \operatorname{Ric}(\bar{Z}, W)\big)\Big) \\
&\quad + (D_{D_Y X} K)(\bar{Z}, T, W) + K(D_{D_Y X} F)(Z), T, W) \\
&= (D_Y D_X K)(\bar{Z}, T, W) + (D_X K)\big((D_Y F)(Z), T, W\big) \\
&\quad + (D_Y K)\big((D_X F)(Z), T, W\big) + K\big((D_Y D_X F)(Z), T, W\big) \\
&\quad - \frac{1}{(2n-1)} \{(D_Y D_X F)(Z) \operatorname{Ric}(T, W) \\
&\quad - (D_{D_Y X} F)(Z) \operatorname{Ric}(T, W) + (D_X F)(Z)(D_Y \operatorname{Ric})(T, W) \\
&\quad + (D_Y F)(Z)(D_X \operatorname{Ric})(T, W) - T\big((D_X \operatorname{Ric})(D_Y F)(Z), W\big) \\
&\quad + (D_Y \operatorname{Ric})\big((D_X F)(Z), W\big) + \operatorname{Ric}\big((D_Y D_X F)(Z), W\big)\}.
\end{aligned} \tag{3.6 b}$$

Using (3.6) and (3.3 *a*), we get

$$(D_Y D_X K)(\bar{Z}, T, W) + (D_X K)\big((D_Y F)(Z), T, W\big) - K\big((D_{D_Y X} F)(Z), T, W\big) + (D_Y K)\big((D_X F)(Z), T, W\big) + K\big((D_Y D_X F)(Z), T, W\big)$$

$$= a(X, Y) K(\bar{Z}, T, W) + (D_{D_Y X} K)(\bar{Z}, T, W) \qquad (3\cdot 6\ c)$$

i.e., An almost Hermite manifold is (1) birecurrent manifold which proves the statement.

Similarly we can prove that if an almost Hermite manifold is (1) bire-eurrent manifold and either Ricci birecurrent or (1) projective birecurrent then it is also either (1) projective birecurrent or Ricci birecurrent manifold provided (3.6) is satisfied.

Note (1.1).—Theorem (3.2) to (3.6) can be put in an alternative form by making use of Ricci identities (2.1) to (2.5).

Note (1.2).—We can also prove theorems for the curvature tensors V, L and C similarly as theorem 3.3 to 3.6, for the projective curvature tensor *W.

Note (1.3).—If we put $(D_X F)(Y) = 0$ or $(D_X' F)(Y, Z) = 0$, then all the results holds good in a Kähler manifold.

References

1. Mishra, R. S. .. "An almost Hermite manifold—III," *The Proceedings of the Nat. Inst. of Sc. of India*, 1969. **35**, 140–48.
2. ———— .. *A Course in Tensor witb Application to Riemannian Geometes*, Pothishala Pıivate Ltd., Allahabad, 1965.
3. ———— .. "On Hermite space—II," *Tensor, N.S.*, 1968, **19**, 27–32.
4. Takano, K. .. "On a space with birecurrent curvature," *Tensor, N.S.*. 1971, **22**, 329–332,