EQUIDISTANCE AND PARALLELISM OF THE CONGRUENCES OF CURVES THROUGH POINTS OF A SUBSPACE

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Received June 11, 1952

(Communicated by Dr. Ram Behari, F.A.SC.)

1. INTRODUCTION

IN this paper I have considered a set of $(m-n)$ congruences of curves which are such that one curve of each congruence passes through each point of a subspace V_n of n dimensions immersed in a Riemannian manifold of m dimensions. Fxpressions for the angular and distantial spread vectors of one curve of a congruence with respect to the other have been found out and some theorems relating to these have been developed. Conditions necessary and sufficient that these congruences be normal have been established.

2. CONGRUENCES OF CURVES

Consider a V_n of co-ordinates x^i $(i = 1, \ldots, n)$ and metric $g_{ij}\, dx^i\, dx^j$ imbedded in a Riemannian V_m of co-ordinates y^α $(\alpha = 1, \ldots, m)$ and metric $a_{\alpha\beta}\, dy^\alpha\, dy^\beta$, where $m > n$. In the following the letters $\alpha, \beta, \gamma, \delta, \phi, \psi$ of Greek alphabet take the values 1 to m and the letters $\mu, \nu, \sigma, \rho, \tau$ the values $n+1$ to m; while the Latin indices range from 1 to n. Assuming the metrices to be positive, we have the relations

$$g_{ij} = a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;j}, \tag{2.1}$$

semi-colon (;) followed by indices denoting tensor derivatives with regard to y's or x's according as the following indices are Greek or Latin (McConnel, 1931).

If $N_{\nu|}{}^\alpha$ are the contravariant components in the y's of a system of unit vectors normal to V_n (Weatherburn, 1950, pp. 162–3),

$$a_{\alpha\beta}\, N_{\nu|}{}^\alpha\, N_{\mu|}{}^\beta = \delta_\nu^\mu, \tag{2.2}$$

and

$$a_{\alpha\beta}\, y^\alpha_{;i}\, N_{\mu|}{}^\beta = 0. \tag{2.3}$$

Also

$$\Omega_{\nu|ij} = a_{\alpha\beta}\, y^\alpha_{;ij}\, N_{\nu|}{}^\beta, \tag{2.4}$$

and

$$y^\alpha_{;ij} = \sum_\nu \Omega_{\nu|ij}\, N_{\nu|}{}^\alpha, \tag{2.5}$$

where $\Omega_{\nu|ij}$ are the components of a symmetric covariant tensor of the second order in x's.

Let us consider a set of $(m-n)$ congruences of curves which are such that a curve of each congruence passes through each point of the subspace. Suppose $\lambda_{\tau|}{}^{\alpha}$ are the contravariant components in the y's of a unit vector in the direction of a curve of a congruence, then (Mishra, 1951)

$$\lambda_{\tau|}{}^{\alpha} = t_{\tau|}{}^{l}\, y^{\alpha}{}_{;l} + \sum_{\nu} c_{\nu\tau|}\, N_{\nu|}{}^{\alpha}, \tag{2.6}$$

where $t_{\tau|}{}^{l}$ are the contravariant components of a vector in the subspace and $c_{\nu\tau|}$ is a scalar, such that if $\theta_{\nu\tau|}$ is the angle between the vectors $\lambda_{\tau|}{}^{\alpha}$ and $N_{\nu|}{}^{\alpha}$, then

$$\cos\theta_{\nu\tau|} = c_{\nu\tau|}, \tag{2.7}$$

and

$$1 - \sum_{\nu} \cos^2\theta_{\nu\tau|} = t_{\tau|}{}^{l}\, t_{\tau|l}. \tag{2.8}$$

Tensor derivative of (2.6) yields,

$$\lambda^{\alpha}{}_{\tau|;i} = \Big(t_{\tau|}{}^{l}{}_{;i} - \sum_{\nu} c_{\nu\tau|}\, \Omega_{\nu|ik}\, g^{kl}\Big)\, y^{\alpha}{}_{;l} +$$
$$\sum_{\nu} \Big(t_{\tau|}{}^{l}\, \Omega_{\nu|li} + c_{\nu\tau|;i} + \Sigma_{\mu}\, c_{\mu\tau|}\, \theta_{\nu\mu|i}\Big)\, N_{\nu|}{}^{\alpha}, \tag{2.9}$$

where

$$\theta_{\mu\nu|i} = a_{\alpha\beta}\, N^{\alpha}{}_{\nu|;i}\, N_{\mu|}{}^{\beta}. \tag{2.10}$$

If

$$\lambda_{\tau|;i}{}^{\alpha} = q_{\tau|i}{}^{l}\, y^{\alpha}{}_{;l} + \sum_{\nu} r_{\nu\tau|i}\, N_{\nu|}{}^{\alpha}, \tag{2.11}$$

then

$$q_{\tau|i}{}^{l} \equiv g^{lj} a_{\alpha\beta}\, \lambda_{\tau|;i}{}^{\alpha}\, y_{;j}{}^{\beta} = t_{\tau|}{}^{l}{}_{;i} - \Sigma_{\nu}\, c_{\nu\tau|}\, \Omega_{\nu|ik}\, g^{kl}, \tag{2.12}$$

and

$$r_{\nu\tau|i} \equiv a_{\alpha\beta}\, \lambda_{\tau|}{}^{\alpha}{}_{;i}\, N_{\nu|}{}^{\beta} = t_{\tau|}{}^{l}\, \Omega_{\nu|li} + c_{\nu\tau|;i} + \Sigma_{\mu}\, c_{\mu\tau|}\, \theta_{\nu\mu|i}. \tag{2.13}$$

3. ANGULAR SPREADS

We shall now consider the angular spreads (or associate curvatures, in Bianchi's terminology) of the curves $\lambda_{\tau|}{}^{\alpha}$ with respect to the curves $\lambda_{\nu|}{}^{\alpha}$, it being supposed that $\lambda_{\tau|}{}^{\alpha}$ is parallel along $N_{\nu|}{}^{\alpha}$ with respect to V_m. If their contravariant components be denoted by $S^{\lambda\lambda}_{\tau\nu}|^{\alpha}$, then (Graustein, 1934, p. 555).

$$S^{\lambda\lambda}_{\tau\nu}|^{\alpha} = \lambda_{\tau|}{}^{\alpha}{}_{;\beta}\, \lambda_{\nu|}{}^{\beta}. \tag{3.1}$$

Now,

$$\lambda_{\tau|}{}^{\alpha}{}_{;\beta}\, y^{\beta}{}_{;i} = \lambda_{\tau|}{}^{\alpha}{}_{;i}, \tag{3.2}$$

whence

$$\lambda_{\tau|}{}^{\alpha}{}_{;\beta} = \lambda_{\tau|}{}^{\alpha}{}_{;i}\, y_{;j}{}^{\gamma}\, a_{\beta\gamma}\, g^{ij}. \tag{3.3}$$

Use of (3.3) in (3.1) yields,

$$S^{\lambda\lambda}_{\tau\nu}|^{\alpha} = a_{\beta\gamma}\, \lambda^{\beta}_{\nu|}\, y^{\gamma}_{;j}\, \lambda_{\tau|}{}^{\alpha}{}_{;i}\, g^{ij},$$

$$= t_{\nu|}{}^{i}\, \lambda_{\tau|}{}^{\alpha}{}_{;i}, \tag{3.4}$$

by virtue of (2.6).

Thus the angular spread vector of the curve $\lambda_{\tau|}$ *with respect to* $\lambda_{\nu|}$ *is* $(1 - \Sigma_{\nu} \cos^2 \theta_{\nu\tau|})^{\frac{1}{2}}$ *times the derived vector of* $\lambda_{\tau|}$ *in the direction of the vector* $t_{\nu|}$ *of the subspace.*

If $\lambda_{\nu|}{}^{\beta}$ are the contravariant components of a normal to the subspace, $t_{\nu|}{}^{i} = 0$, and (3.4) assumes the form

$$S^{\lambda N}_{\tau\nu}|^{\alpha} = 0.$$

Hence as supposed the curves of the set of (m–n) congruences of curves such that one curve of each congruence passes through each point of the subspace V_n, *are parallel with respect to the normals to the subspace.*

Also since

$$S^{NN}_{\tau\nu}|^{\alpha} = 0,$$

and

$$S^{NN}_{\tau\tau}|^{\alpha} = 0,$$

the normals to a subspace of a Riemannian manifold are geodesics and are parallel with respect to each other.

Use of (2.11) in (3.4) yields,

$$S^{\lambda\lambda}_{\tau\nu}|^{\alpha} = t_{\nu|}{}^{i}\, (q_{\tau|i}{}^{l}\, y^{\alpha}_{;l} + \sum_{\rho} r_{\rho\tau|i}\, N_{\rho|}{}^{\alpha}), \tag{3.5}$$

or with the help of (2.12),

$$S^{\lambda\lambda}_{\tau\nu}|^{\alpha} = t_{\nu|}{}^{i}\, [y^{\alpha}_{;l}\, (t_{\tau|}{}^{l}{}_{;i} - \sum_{\mu} c_{\mu\tau|}\, \Omega_{\mu|ik}\, g^{kl}) +$$

$$\sum_{\rho} N_{\rho|}{}^{\alpha}\, (t_{\tau|}{}^{l}\, \Omega_{\rho|li} + c_{\rho\tau|;i} + \sum_{\mu} c_{\mu\tau|}\, \theta_{\rho\mu|i})]. \tag{3.6}$$

Thus (3·5) *and* (3·6) *are other expressions for the angular spread vector of the curve* $\lambda_{\tau|}$ *with respect to* $\lambda_{\nu|}$.

The resolved part of the angular spread vector of the curve $\lambda_{\tau|}$ with respect to $\lambda_{\nu|}$ along a normal $N_{\sigma|}$ to the subspace is given by

$$a_{\alpha\beta}\, S^{\lambda\lambda}_{\tau\nu}|^{\alpha}\, N_{\sigma|}{}^{\beta} = t_{\nu|}{}^{i}\, r_{\sigma\tau|i}. \tag{3.7}$$

The equation (3.7) *gives another interpretation for* $r_{\sigma\tau|i}$.

Similarly $$a_{\alpha\beta}\, S^{\lambda\lambda}_{\tau\nu}|^{\alpha}\, y^{\beta}_{;l} = t_{\nu|}{}^{i}\, q_{\tau|il}, \tag{3.8}$$

and (3·8) *gives another meaning for* $q_{\tau|il}$.

Putting $\nu=\tau$ in (3.4), (3.5) and (3.6) we get the contravariant components of the first curvature vector of the curve $\lambda_{\tau|}$. The square of the first curvature vector of the curve $\lambda_{\tau|}$ is given by

$$a_{\alpha\beta}\, S^{\lambda\lambda}_{\tau\tau}|^{\alpha}\, S^{\lambda\lambda}_{\tau\tau}|^{\beta}=a_{\alpha\beta}\,(q^{l}_{\tau|i}\, y^{\alpha}_{;l}+\sum_{\rho} r_{\rho\tau|i}\, N_{\rho|}{}^{\alpha})\,(q_{\tau|j}{}^{m}\, y^{\beta}_{;m}+\sum_{\sigma} r_{\sigma\tau|j}\, N_{\sigma|}{}^{\beta})\, t_{\tau|}{}^{i}\, t_{\tau|}{}^{j}$$

$$=(q^{l}_{\tau|i}\, q_{\tau|j}{}^{m}\, g_{lm}+\sum_{\rho} r_{\rho\tau|i}\, r_{\rho\tau|j})\, t_{\tau|}{}^{i}\, t_{\tau|}{}^{j}. \tag{3.9}$$

Substitutions from (2.12) and (2.13) give other expressions for the first curvature vector of the curve $\lambda_{\tau|}$.

The necessary and sufficient condition that the curve $\lambda_{\tau|}$ *is a geodesic is obtained by equating the right-hand member of* (3.9) *to zero.*

The contravariant components of angular spread vector of the normal $N_{\tau|}$ to the subspace with respect to $\lambda_{\nu|}$ is given by

$$S^{N\lambda}_{\tau\nu}|^{\alpha}=N_{\tau|}{}^{\alpha}{}_{;i}\, t_{\nu|}{}^{i}, \tag{3.10}$$

which cannot vanish unless the vector $\lambda_{\nu|}$ is normal to the subspace. Thus we see that *though the vector* $\lambda_{\nu|}$ *is parallel with respect to the normal* $N_{\tau|}$, *the converse is not true, in general.*

Since (Weatherburn, 1950, p. 170),

$$N_{\tau|}{}^{\alpha}{}_{;i}=-\,\Omega_{\tau|ik}\, g^{kj}\, y^{\alpha}_{;j}+\sum_{\mu}\theta_{\mu\tau|i}\, N_{\mu|}{}^{\alpha}, \tag{3.11}$$

the equation (3.10) assumes the form

$$S^{N\lambda}_{\tau\nu}|^{\alpha}=-\,\Omega_{\tau|ik}\, g^{kj}\, t_{\nu|}{}^{i}\, y^{\alpha}_{;j}+\sum_{\mu}\theta_{\mu\tau|i}\, N_{\mu|}{}^{\alpha}\, t_{\nu|}{}^{i}. \tag{3.12}$$

This is another expression for $S^{N\lambda}_{\tau\nu}|^{\alpha}$.

4. DISTANTIAL SPREAD VECTOR OF $\lambda_{\tau|}$ AND $\lambda_{\nu|}$

If we denote by $T^{\lambda\lambda}_{\tau\nu}|^{\alpha}$ the contravariant components of distantial spread vector (Graustein, 1934) of the curves $\lambda_{\tau|}$ and $\lambda_{\nu|}$, then

$$T^{\lambda\lambda}_{\tau\nu}|^{\alpha}=S^{\lambda\lambda}_{\tau\nu}|^{\alpha}-S^{\lambda\lambda}_{\nu\tau}|^{\alpha}, \tag{4.1}$$

$$=\lambda_{\tau}{}^{\alpha}{}_{|;\beta}\,\lambda_{\nu|}{}^{\beta}-\lambda_{\nu|}{}^{\alpha}{}_{;\beta}\,\lambda_{\tau|}{}^{\beta}. \tag{4.2}$$

Since $\quad S^{\lambda N}_{\nu\tau}|^{\alpha}=0,$

$$T^{N\lambda}_{\tau\nu}|^{\alpha}=S^{N\lambda}_{\tau\nu}|^{\alpha}, \tag{4.3}$$

and $\quad T^{\lambda N}_{\tau\nu}|^{\alpha}=-\,S^{N\lambda}_{\nu\tau}|^{\alpha}. \qquad (4.4)$

Hence *the distantial spread vector of a normal to the subspace* V_n *of a* V_m, *with respect to a curve* $\lambda_{\nu|}$ *of the congruence is the same as its angular spread vector with respect to the same curve* $\lambda_{\nu|}$.

The sum of the distantial spread vector of a curve $\lambda_{\tau|}$ of the congruence with respect to a normal $N_{\nu|}$ to the subspace, and the angular spread vector of the normal $N_{\nu|}$ with respect to the curve $\lambda_{\tau|}$ vanishes.

By virtue of (3.5) the equation (4.1) assumes the form

$$T^{\lambda\lambda}_{\tau\nu}|^{\alpha} = y^{\alpha}_{;l}\,(t_{\nu|}{}^{i}\,q_{\tau|i}{}^{l} - t_{\tau|}{}^{i}\,q_{\nu|i}{}^{l}) + \sum_{\mu}(t_{\nu|}{}^{i}{}_{;i}\,r_{\mu\tau|i} - t_{\tau|}{}^{i}\,r_{\mu\nu|i})\,N_{\mu|}{}^{\alpha}, \tag{4.5}$$

where $q^{l}_{\tau|i}$ and $r_{\mu\tau|i}$ are given by (2.12) and (2.13).

(4.5) *is another expression for* $T^{\lambda\lambda}_{\tau\nu}|^{\alpha}$.

From (4.5) it is obvious that

$$T^{NN}_{\tau\nu}|^{\alpha} = 0.$$

Hence *the distantial spread vector between any two normals of a subspace V_n immersed in V_m is a null vector.*

Also, a set of normals to a subspace V_n immersed in V_m are equidistant with regard to each other.

5. Condition For A Normal Congruence

We shall now find the condition that the congruence of curves $\lambda_{\tau|}$ be normal. We know that the necessary and sufficient condition for this is (Weatherburn, 1950, p. 103),

$$a_{\alpha\phi}\,\lambda_{\tau|}{}^{\phi}\,(a_{\beta\delta}\,\lambda_{\tau|}{}^{\delta}{}_{;\gamma} - a_{\gamma\delta}\,\lambda_{\tau|}{}^{\delta}{}_{;\beta}) + a_{\beta\phi}\,\lambda_{\tau|}{}^{\phi}\,(a_{\gamma\delta}\,\lambda_{\tau|}{}^{\delta}{}_{;\alpha} - a_{\alpha\delta}\,\lambda_{\tau|}{}^{\delta}{}_{;\gamma})$$
$$+ a_{\gamma\phi}\,\lambda_{\tau|}{}^{\phi}\,(a_{\alpha\delta}\,\lambda_{\tau|}{}^{\delta}{}_{;\beta} - a_{\beta\delta}\,\lambda_{\tau|}{}^{\delta}{}_{;\alpha}) = 0. \tag{5.1}$$

Use of (3.3) in this equation yields,

$$\lambda_{\tau|}{}^{\phi}\,\lambda_{\tau|}{}^{\delta}{}_{;i}\,y^{\psi}{}_{;j}\,g^{ij}\,[a_{\alpha\phi}\,(a_{\beta\delta}\,a_{\gamma\psi} - a_{\gamma\delta}\,a_{\beta\psi}) + \text{two other terms in which}$$
$$\alpha,\ \beta,\ \gamma \text{ take the values cyclically}] = 0. \tag{5.2}$$

By virtue of (2.6) and (2.12) the condition (5.2) can also be written as

$$(t_{\tau|}{}^{l}\,y^{\phi}{}_{;l} + \Sigma_{\nu}\,c_{\nu\tau|}\,N_{\nu|}{}^{\phi})\,(q_{\tau|i}{}^{l}\,y^{\delta}{}_{;l} + \sum_{\mu} r_{\mu\tau|i}\,N_{\mu|}{}^{\delta})\,y^{\psi}{}_{;j}\,g^{ij}\times$$
$$[a_{\alpha\phi}\,(a_{\beta\delta}\,a_{\gamma\psi} - a_{\gamma\delta}\,a_{\beta\psi}) + \text{two other similar terms with}$$
$$\alpha,\ \beta,\ \gamma \text{ occurring cyclically}], \tag{5.3}$$

where $q_{\tau|i}{}^{l}$ and $r_{\gamma\tau|i}$ are given by (2.12) and (2.13) respectively.

6. MUTUALLY ORTHOGONAL CURVES

Let $(m-n)$ curves of a congruence $\lambda_{\tau|}$ be mutually at right angles. Then the values of Ricci's coefficients of rotation $\gamma_{\rho\mu\nu|}$ are given by (Weatherburn, 1950, p. 99).

$$\gamma_{\rho\mu\nu|} = a_{\alpha\gamma}\, \lambda_{\rho|;\beta}{}^{\gamma}\, \lambda_{\mu|}{}^{\alpha}\, \lambda_{\nu|}{}^{\beta}. \tag{6.1}$$

Use of (3.3) in this equation yields,

$$\begin{aligned}\gamma_{\rho\mu\nu|} &= a_{\alpha\gamma}\, \lambda_{\mu|}{}^{\alpha}\, \lambda_{\nu|}{}^{\beta}\, \lambda_{\rho|}{}^{\gamma}{}_{;i}\, y^{\delta}{}_{;j}\, a_{\beta\delta}\, g^{ij},\\ &= a_{\alpha\gamma}\, \lambda_{\rho|}{}^{\gamma}{}_{;i}\, \lambda_{\mu|}{}^{\alpha}\, t_{\nu|j}\, g^{ij},\end{aligned} \tag{6.2}$$

by virtue of (2.6).

The equation (6.2) shows that if $\lambda_{\nu|}$ is a normal to the subspace

$$\gamma_{\rho\mu\nu} = 0.$$

It is to be noted that even if one of the curves $\lambda_{\nu|}$ is normal to the subspace and all other curves are mutually at right angles, the other curves may not be normal to the subspace.

The necessary condition that the curves of the congruence $\lambda_{\tau|}$ be geodesics is given by

$$\gamma_{\rho\tau\tau|} = 0. \quad (\rho = n+1, \ldots m). \tag{6.3}$$

By virtue of (6·1) and (6·2) this condition reduces to

$$a_{\alpha\gamma}\, \lambda_{\tau|}{}^{\alpha}\, \lambda_{\rho|}{}^{\gamma}{}_{;\beta}\, \lambda_{\tau|}{}^{\beta} = 0, \tag{6.4}$$

or

$$a_{\alpha\gamma}\, \lambda_{\tau|}{}^{\alpha}\, \lambda_{\rho|}{}^{\gamma}{}_{;i}\, t_{\tau|j}\, g^{ij} = 0, \tag{6.5}$$

or by virtue of (2.11),

$$a_{\alpha\gamma}\, (q_{\rho|i}{}^{l}\, y^{\gamma}{}_{;l} + \sum_{\nu} r_{\nu\rho|i}\, N_{\nu|}{}^{\gamma})\, \lambda_{\tau|}{}^{\alpha}\, t_{\tau|j}\, g^{ij} = 0, \tag{6.6}$$

or

$$(q_{\rho|i}{}^{l}\, t_{\tau|l} + \sum_{\nu} r_{\nu\rho|i}\, c_{\nu\tau|})\, t_{\tau|}{}^{i} = 0, \tag{6.7}$$

by virtue of (2.6).

(6.3) is clearly satisfied by the normals $N_{\tau|}$.

The condition (6.7) can further be written in the form

$$t_{\tau|}{}^{i}\, [t_{\tau|l}\, (t_{\rho|}{}^{l}{}_{;i} - \sum_{\nu} c_{\nu\rho|}\, \Omega_{\nu|ik}\, g^{kl}) + \sum_{\nu} (t_{\rho|}{}^{l}\, \Omega_{\nu|li} + c_{\nu\rho|;i} + \sum_{\mu} c_{\mu\rho|}\, \theta_{\nu\mu|i})\, c_{\nu\tau|}] = 0. \tag{6.8}$$

It may be noted in this connection that the conditions (6.3) to (6.8) are only the necessary conditions. They are not sufficient.

The expression on the left of (6·4) is the tendency of the vector $\lambda_{\rho|}$ in the direction of the vector $\lambda_{\tau|}$. *Hence if a curve of the congruence $\lambda_{\tau|}$ is geodesic the tendencies of all other mutually orthogonal curves of the congruence is zero.* This condition again is not sufficient.

If (5.1) is multiplied by $\lambda_{\mu|}{}^{\alpha}\,\lambda_{\nu|}{}^{\gamma}$ and summed for α and γ, the condition (5.1) that the congruence $\lambda_{\tau|}$ be normal, reduces to

$$\gamma_{\tau\mu\nu|} = \gamma_{\tau\nu\mu|}.$$

It may be noted here that τ, μ, ν are unequal.

If all the curves of the congruence are normal, we must have

$$\gamma_{\tau\mu\nu|} = 0.$$

In that case the curves of the congruence will be a set of normals to the subspace.

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