ON THE CONGRUENCES OF RIBAUCOUR

By Ram Behari and Ratan Shanker Mishra

(*Delhi University, Delhi*)

RIBAUCOUR CONGRUENCES[1] are the congruences formed by lines through points on one surface parallel to the normals to another surface, the two surfaces corresponding with orthogonality of linear elements. The latter surface is called the director surface and the former one is the surface of reference. The object of this paper is to obtain several properties of Ribaucour congruences.

1. Suppose S_1 is the surface of reference, the co-ordinates of any point on it being (x_1, y_1, z_1) and S is the director surface, the co-ordinates of any point on it being (x, y, z)

then
$$\Sigma \, dx \cdot dx_1 = 0 \quad [2]$$

or
$$\Sigma \frac{\partial x}{\partial u} \cdot \frac{\partial x_1}{\partial u} = 0; \quad \Sigma \frac{\partial x}{\partial v} \cdot \frac{\partial x_1}{\partial v} = 0; \quad \Sigma \frac{\partial x}{\partial u} \cdot \frac{\partial x_1}{\partial v} + \Sigma \frac{\partial x}{\partial v} \cdot \frac{\partial x_1}{\partial u} = 0 \tag{1.1}$$

let the last of these equations be replaced by

$$\Sigma \frac{\partial x}{\partial u} \cdot \frac{\partial x_1}{\partial v} = \phi H; \quad \Sigma \frac{\partial x}{\partial v} \cdot \frac{\partial x_1}{\partial u} = - \phi H \tag{1.2}$$

Then if X, Y, Z are the direction cosines of the normal to S; E, F, G, D, D$'$, D$''$ are the fundamental coefficients of the first and the second order and K is the total curvature of S,

$$\Sigma X \frac{\partial x_1}{\partial u} = \frac{D' \frac{\partial \phi}{\partial u} - D \frac{\partial \phi}{\partial v}}{KH}; \quad \Sigma X \frac{\partial x_1}{\partial v} = \frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{KH} \tag{1.3}$$

where

$$\frac{\partial x_1}{\partial u} = \frac{D \left(\phi \frac{\partial X}{\partial v} - X \frac{\partial \phi}{\partial v} \right) - D' \left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u} \right)}{KH} \tag{1.4}$$

and

$$\frac{\partial x_1}{\partial v} = \frac{D' \left(\phi \frac{\partial X}{\partial v} - X \frac{\partial \phi}{\partial v} \right) - D'' \left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u} \right)}{KH} \tag{1.5}$$

and similar expressions for y_1 and z_1.

Hence for the congruence of lines parallel to the normals to S, the coefficients e, f, f', g of Kummer's form are given by

$$e \equiv \Sigma \frac{\partial x_1}{\partial u} \cdot \frac{\partial X}{\partial u} = (D\mathscr{F} - D'\mathscr{E}) \frac{\phi}{\mathscr{H}} \tag{1·6}$$

$$f \equiv \Sigma \frac{\partial x_1}{\partial v} \cdot \frac{\partial X}{\partial u} = (D'\mathscr{F} - D''\mathscr{E}) \frac{\phi}{\mathscr{H}} \tag{1·7}$$

$$f' \equiv \Sigma \frac{\partial x_1}{\partial u} \cdot \frac{\partial X}{\partial v} = (D\mathscr{G} - D'\mathscr{F}) \frac{\phi}{\mathscr{H}} \tag{1·8}$$

$$g \equiv \Sigma \frac{\partial x_1}{\partial v} \cdot \frac{\partial X}{\partial v} = (D'\mathscr{G} - D''\mathscr{F}) \frac{\phi}{\mathscr{H}} \tag{1·9}$$

where $\mathscr{E}, \mathscr{F}, \mathscr{G}$ are the fundamental coefficients of the first order of the spherical representation of S.

Solving the equations (1·6) and (1·8) and the equations (1·7) and (1·9) we get

$$D = \frac{\mathscr{E}f' - \mathscr{F}e}{\phi \mathscr{H}} \tag{1·10}$$

$$D' = \frac{\mathscr{E}g - \mathscr{F}f}{\phi \mathscr{H}} = \frac{\mathscr{F}f' - \mathscr{G}e}{\phi \mathscr{H}} \tag{1·11}$$

$$D'' = \frac{\mathscr{F}g - \mathscr{G}f}{\phi \mathscr{H}} \tag{1·12}$$

The asymptotic lines on the director surface are given by

$$D du^2 + 2D' \, dudv + D'' dv^2 = 0 \tag{1·13}$$

or using the equations (1·10), (1·11) and (1·12) we get[3]

$$(\mathscr{E}f' - \mathscr{F}e) \, du^2 + (\mathscr{E}g - \mathscr{G}e + \mathscr{F}f' - \mathscr{F}f) \, dudv + (\mathscr{F}g - \mathscr{G}f) \, dv^2 = 0 \tag{1·14}$$

But this is also the equation of developable surfaces through the line of the congruence. Hence the *developable surfaces through a line of the congruence of Ribaucour cut the director surface in asymptotic lines.*

The lines of curvature on the director surface are given by

$$(ED' - FD) \, du^2 + (ED'' - GD) \, dudv + (FD'' - GD') \, dv^2 = 0 \tag{1·15}$$

But

$$E = \frac{1}{\mathscr{H}^2} [\mathscr{G}D^2 - 2\mathscr{F}DD' + \mathscr{E}D'^2] \tag{1·16}$$

$$F = \frac{1}{\mathscr{H}^2} [\mathscr{G}DD' - \mathscr{F}DD'' - \mathscr{F}D'^2 + \mathscr{E}D'D''] \tag{1·17}$$

$$G = \frac{1}{\mathscr{H}^2} [\mathscr{G}D'^2 - 2\mathscr{F}D'D'' + \mathscr{E}D''^2] \tag{1·18}$$

Writing down these values of E, F, G in (1·15) we get

$$(\mathcal{F}D - \mathcal{E}D')\,du^2 + (\mathcal{G}D - \mathcal{E}D'')\,dudv + (\mathcal{G}D' - \mathcal{F}D'')\,dv^2 = 0 \quad (1\cdot19)$$

Using the equations (1·6), (1·7), (1·8) and (1·9) equation (1·19) becomes

$$edu^2 + (f + f')\,dudv + gdv^2 = 0 \qquad (1\cdot20)$$

As in a congruence of Ribaucour the surface of reference is a middle surface, the equation (1·20) is the equation of the surfaces of distribution.[4] Hence the *surfaces of distribution through a line of a congruence of Ribaucour cut the director surface in the lines of curvature.*

The equations of characteristic surfaces are given by[5]

$$\begin{vmatrix} \delta du + \delta' dv & \delta' du + \delta'' dv \\ edu + \frac{1}{2}(f + f')\,dv & \frac{1}{2}(f + f')\,du + gdv \end{vmatrix} = 0 \quad (1\cdot21)$$

where δ, δ', δ'' are the coefficients of Sannia's second fundamental quadratic form. Using the equations (1·10), (1·11) and (1·12) we get

$$\begin{vmatrix} Ddu + D'dv & D'du + D''dv \\ edu + \frac{1}{2}(f + f')\,dv & \frac{1}{2}(f + f')\,du + gdv \end{vmatrix} = 0 \quad (1\cdot22)$$

With the help of equations (1·6), (1·7), (1·8) and (1·9) equation (1·22) becomes

$$\begin{vmatrix} Ddu + D'dv & D'du + D''dv \\ 2(D\mathcal{F} - D'\mathcal{E})\,du + (D\mathcal{G} - D''\mathcal{E})\,dv & (D\mathcal{G} - D''\mathcal{E})\,du + 2(D'\mathcal{G} - D''\mathcal{F})\,dv \end{vmatrix} = 0 \quad (1\cdot23)$$

or from the equations (1·16), (1·17) or (1·18) equation (1·23) becomes

$$\begin{vmatrix} Ddu + D'dv & D'du + D''dv \\ 2(ED' - FD)\,du + (ED'' - GD)\,dv & (ED'' - GD)\,du + 2(FD'' - GD')\,dv \end{vmatrix} = 0 \quad (1\cdot24)$$

But equation (1·24) is the equation of characteristic lines[6] of the director surface, hence *the characteristic surfaces of a congruence of Ribaucour cut the director surface in characteristic lines.*

Hence *the developable surfaces, surfaces of distribution and characteristic surfaces through a line of a congruence of Ribaucour cut the director surface in asymptotic lines, lines of curvature and characteristic lines respectively.*

From the above we see that if the spherical representations of developable surfaces, surfaces of distribution or characteristic surfaces are parametric, the spherical representations of asymptotic lines, lines of curvature or characteristic lines of director surface at the corresponding point are also parametric.

Hence *the spherical representations of developable surfaces, surfaces of distribution and characteristic surfaces through a line of the congruence of Ribaucour correspond to the spherical representations of asymptotic lines, lines of curvature and characteristic lines of the director surface at the corresponding point.*

It is to be noted that these three surfaces form a cycle in the sense that the central planes corresponding to any one of these three surfaces are the double planes of involution determined by the corresponding central planes of the other two,[7] also if $f = 0$ and $\phi = 0$ are the developable surfaces and surfaces of distribution, then the repeated applications of forming the Jacobian with these two forms lead to only these three surfaces. A corresponding result holds for the three families of curves of the director surface.

2. From the equation (1·14) we see that the developable surfaces are parametric if

$$\mathscr{E} f' - \mathscr{F} e = 0 \tag{2·1}$$

and

$$\mathscr{F} g - \mathscr{G} f = 0 \tag{2·2}$$

or from the equations (1·10) and (1·12) we get

$$D = D'' = 0 \tag{2·3}$$

Hence the equation (1·19) of the surfaces of distribution becomes

$$\mathscr{E} \, du^2 - \mathscr{G} \, dv^2 = 0 \tag{2·4}$$

Hence *the central planes for the spherical representation of the surfaces of distribution through a line of a rectilinear congruence of Ribaucour bisect the angles between the central planes of the spherical representations of developable surfaces.*

Also from the equation (1·21) we see that the characteristic surfaces are parametric if

$$\delta' = 0, \, f + f' = 0 \tag{2·5}$$

hence the equation (1·20) of the surfaces of distribution becomes

$$e du^2 + g dv^2 = 0 \tag{2·6}$$

In a congruence of Ribaucour the surface of reference is a middle surface

$$\therefore \quad e\mathscr{G} + g\mathscr{E} - \mathscr{F}(f + f') = 0 \tag{2·7}$$

With the help of equations (2·5) and (2·7) equation (2·6) becomes

$$\mathscr{E} \, du^2 - \mathscr{G} \, dv^2 = 0$$

Therefore *the central planes for the spherical representations of the surfaces of distribution through a line of a rectilinear congruence of Ribaucour*

bisect the angles between the central planes for the spherical representations of the characteristic surfaces also.

3. *Normal congruences of Ribaucour.*—We know that the necessary and sufficient condition for a normal rectilinear congruence is[8]

$$\mathscr{E}\delta'' + \mathscr{G}\delta - 2\mathscr{F}\delta' = 0 \tag{3.1}$$

With the help of equations (1·10), (1·11) and (1·12) equation (3·1) becomes

$$\mathscr{E}D'' + \mathscr{G}D - 2\mathscr{F}D' = 0 \tag{3.2}$$

Suppose the surfaces whose spherical representations are minimal lines are represented by the parametric curves on the unit sphere, then

$$\mathscr{E} = \mathscr{G} = 0 \tag{3.3}$$

With the help of equation (3·3) equation (3·2) becomes

$$D' = 0 \tag{3.4}$$

as the fundamental coefficients of the first and second order for the director surface are not proportional.

Hence *the necessary and sufficient condition that the congruence of Ribaucour be normal is that the surfaces whose spherical representations are minimal lines correspond to the conjugate systems at the corresponding points of the director surface.*

4. Let the parametric curves on the director surface of a normal rectilinear congruence be conjugate, then

$$D' = 0 \tag{4.1}$$

The equation (3·2) becomes with the help of equation (4·1)

$$\frac{D}{\mathscr{E}} = -\frac{D''}{\mathscr{G}} \tag{4.2}$$

Also the surfaces whose spherical representations on the unit sphere correspond to the parametric curves on the director surface have their parameters of distribution equal to

$$P_1 = \frac{\delta}{\mathscr{E}} = \frac{D}{\delta}\phi; \quad P_2 = -\frac{\delta''}{\mathscr{G}} = -\frac{D''}{\mathscr{G}}\phi \tag{4.3}$$

Hence, *the two surfaces through a ray of a normal congruence of Ribaucour which correspond to the conjugate lines on the director surface have their parameters of distribution equal in magnitude but opposite in sign.*

5. We shall now consider the normal congruences of Ribaucour in which the surface of reference is an orthogonal surface.

Here $\gamma \equiv \Sigma X \dfrac{\partial x_1}{\partial u} = 0$ $\qquad\qquad$ (5·1)

and $\gamma' \equiv \Sigma X \dfrac{\partial x_1}{\partial v} = 0$ $\qquad\qquad$ (5·2)

From the equations (1·3)

$$D' \frac{\partial \phi}{\partial u} - D \frac{\partial \phi}{\partial v} = 0 \qquad\qquad (5·3)$$

$$D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v} = 0 \qquad\qquad (5·4)$$

From the two equations (5·3) and (5·4)

either $\qquad DD'' - D'^2 = 0$ $\qquad\qquad$ (5·5)

or $\qquad \phi = \text{constant.}$ $\qquad\qquad$ (5·6)

If $\qquad DD'' - D'^2 = 0$

then $\qquad \gamma \equiv \Sigma X \dfrac{\partial x_1}{\partial u} = \dfrac{D' \frac{\partial \phi}{\partial u} - D \frac{\partial \phi}{\partial v}}{KH}$

and $\qquad \gamma' \equiv \Sigma X \dfrac{\partial x_1}{\partial v} = \dfrac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{KH}$

become indeterminate.

Hence $\phi = \text{constant} = C$ (say).

The necessary and sufficient condition that the congruences of Ribaucour should have the surface of reference as an orthogonal surface is that the Bianchi's characteristic function should be constant.

6. Suppose the surface S is referred to its lines of curvature, then

$$F = 0, \ D' = 0 \qquad\qquad (6·1)$$

With the help of equation (6·1), equations (1·6), (1·7), (1·8) and (1·9) become

$$e = \frac{D\mathscr{F}\phi}{\mathscr{H}} \ ; \ f = - \frac{D''\phi\mathscr{E}}{\mathscr{H}} \ ; \ f' = \frac{D\phi\mathscr{G}}{\mathscr{H}} \ ; \ g = - \frac{D''\phi\mathscr{F}}{\mathscr{H}} \qquad (6·2)$$

But since

$$\mathscr{F} = \frac{1}{H^2} (GDD' - FDD'' - FD'^2 + FD'D''), \qquad\qquad (6·3)$$

using equations (6·1) in the equation (6·3) we get

$$\mathscr{F} = 0 \qquad\qquad (6·4)$$

Hence using the equation (6·4) in the equation (6·2)

$$e = g = 0 \qquad\qquad (6·5)$$

Since in a congruence of Ribaucour the surface of reference is the middle surface, the equation (1·20) for the surfaces of distribution becomes

$$dv \cdot du = 0 \qquad (6 \cdot 6)$$

Using the equations (6·1) and (3·2) we see that the condition of a normal congruence of Ribaucour in which the lines of curvature at the corresponding points of the director surface are parametric is

$$\mathcal{E} D'' + \mathcal{G} D = 0 \qquad (6 \cdot 7)$$

From the equations (1·10), (1·11) and (1·12) we get

$$\delta = D\phi; \ \delta' = D'\phi; \ \delta'' = D''\phi. \qquad (6 \cdot 8)$$

Also[9]

$$- y \mathcal{H} = \delta_2 - \delta_1' - \Gamma'\delta + (\Gamma - \Delta')\delta' + \Delta\delta'' \qquad (6 \cdot 9)$$

$$y' \mathcal{H} = \delta_1'' - \delta_2' + \Gamma''\delta + (\Delta'' - \Gamma')\delta' - \Delta'\delta'' \qquad (6 \cdot 10)$$

where $y \left(\equiv \Sigma X \dfrac{\partial x_1}{\partial u} \right)$ and $y' \left(\equiv \Sigma X \dfrac{\partial x_1}{\partial v} \right)$ are given by the equations (1·3), the subscripts 1, and 2 in δ, δ', δ'' denoting differentiations with regard to 'u' and 'v' respectively. The letters Γ, Γ', Γ''; Δ, Δ', Δ'' have their usual meanings.[10]

With the help of equations (1·3) and (6·1) and (6·8) equations (6·9) and (6·10) become;

$$D \frac{\partial \phi}{\partial v} = \frac{\partial}{\partial v}(D\phi) - \Gamma' D\phi + \Delta D''\phi \qquad (6 \cdot 11)$$

$$D'' \frac{\partial \phi}{\partial u} = \frac{\partial}{\partial u}(D''\phi) + \Gamma'' D\phi - \Delta' D''\phi \qquad (6 \cdot 12)$$

But

$$\Gamma' \equiv \frac{\mathcal{G}\mathcal{E}_2 - \mathcal{F}\mathcal{G}_1}{2\mathcal{H}^2} = \frac{\mathcal{E}_2}{2\mathcal{E}}; \quad \Delta \equiv \frac{-\mathcal{F}\mathcal{E}_1 - \mathcal{E}\mathcal{E}_2 + 2\mathcal{E}\mathcal{F}_1}{2\mathcal{H}^2} = -\frac{\mathcal{E}_2}{2\mathcal{G}}$$

$$\Gamma'' \equiv \frac{-\mathcal{F}\mathcal{G}_2 - \mathcal{G}\mathcal{G}_1 + 2\mathcal{G}\mathcal{F}_2}{2\mathcal{H}^2} = -\frac{\mathcal{G}_1}{2\mathcal{E}}; \quad \Delta' \equiv \frac{\mathcal{E}\mathcal{G}_1 - \mathcal{F}\mathcal{E}_2}{2\mathcal{H}^2} = \frac{\mathcal{G}_1}{2\mathcal{G}}$$

Equations (6·11) and (6·12) reduce to

$$\frac{\partial}{\partial v} D - \frac{\mathcal{E}_2}{2} \left(\frac{D}{\mathcal{E}} + \frac{D''}{\mathcal{G}} \right) = 0$$

$$\frac{\partial}{\partial u} D'' - \frac{\mathcal{G}_1}{2} \left(\frac{D}{\mathcal{E}} + \frac{D''}{\mathcal{G}} \right) = 0$$

Hence D is a function of 'u' only and D'' is a function of 'v' only

$$\therefore \ \frac{D}{D''} = \frac{\text{a function of '}u\text{' only}}{\text{a function of '}v\text{' only}}$$

$$\therefore \ \frac{\partial^2}{\partial u \partial v} \log \frac{D}{D''} = 0 \qquad (6 \cdot 13)$$

and $\qquad\qquad$ D$' = 0.$ $\qquad\qquad\qquad$ (6·14)

which shows that in a normal congruence of Ribaucour the lines of curvature of the director surface are isothermal conjugate.

From the equations (6·4), (6·7) and (6·13) we see that

$$\frac{\partial^2}{\partial u \partial v} \log \frac{\mathscr{E}}{\mathscr{G}} = 0$$

and $\qquad\qquad\qquad\qquad \mathscr{F} = 0$

∴ in a normal congruence of Ribaucour the spherical representations of the surfaces of distribution are isometric.

This can be proved alternately as follows:—

From the equation (3·2) we see that the surface S is minimal.

But the spherical representations of the lines of curvature on a minimal surface are isometric and isothermal conjugate. Therefore the spherical representations of the lines of curvature on the director surface are isometric and isothermal conjugate.

Since the spherical representations of the lines of curvature of the director surface correspond to the spherical representations of the surfaces of distribution, the spherical representations of the surfaces of distribution through a line of a normal congruence of Ribaucour are both isometric and isothermal conjugate.

To summarise:—

The spherical representations of the lines of curvature on the director surface of a normal congruence of Ribaucour are isometric and isothermal conjugate.

The spherical representations of the surfaces of distribution through a line of a normal congruence of Ribaucour are also isometric and isothermal conjugate.

We know that the spherical representations of asymptotic lines on a minimal surface are isometric.

Hence the spherical representations of asymptotic lines on the director surface of a normal congruence of Ribaucour are isometric.

As the spherical representations of asymptotic lines on the director surface correspond to the spherical representations of developable surfaces through a line of a congruence of Ribaucour we get the result:

The spherical representations of developable surfaces through a line of a normal congruence of Ribaucour are also isometric.

7. *Isotropic congruences of Ribaucour.*—We shall find the condition that the congruences of Ribaucour be isotropic.

Since in a congruence of Ribaucour the surface of reference is a middle surface, we must have[11]

$$e = f + f' = g = 0 \qquad (7 \cdot 1)$$

using the equations $(7 \cdot 1)$ in the equations $(1 \cdot 6)$, $(1 \cdot 7)$, $(1 \cdot 8)$ and $(1 \cdot 9)$ we have

$$\frac{D}{\mathcal{E}} = \frac{D'}{\mathcal{F}} = \frac{D''}{\mathcal{G}} = \frac{1}{\lambda} \text{ (say)} \qquad (7 \cdot 2)$$

where

$$\mathcal{E} = D\lambda, \quad \mathcal{F} = D'\lambda, \quad \mathcal{G} = D''\lambda \qquad (7 \cdot 3)$$

From the equations $(1 \cdot 16)$, $(1 \cdot 17)$ and $(1 \cdot 18)$ we have

$$\frac{D}{E} = \frac{D\mathcal{H}^2}{\mathcal{G}D^2 - 2\mathcal{F}DD' + \mathcal{E}D'^2} = \lambda$$

$$\frac{D'}{F} = \frac{D'\mathcal{H}^2}{\mathcal{G}DD' - \mathcal{F}DD'' - \mathcal{F}D'^2 + \mathcal{E}D'D''} = \lambda$$

$$\frac{D''}{\mathcal{G}} = \frac{D''\mathcal{H}^2}{\mathcal{G}D'^2 - 2\mathcal{F}D'D'' + \mathcal{E}D''^2} = \lambda \text{ using the equation } (7 \cdot 3)$$

$$i.e., \frac{D}{E} = \frac{D'}{F} = \frac{D''}{G} = \lambda$$

hence the corresponding points are umbilical.[12]

Therefore the necessary and sufficient condition that the congruence of Ribaucour be isotropic is that the corresponding points on the director surface are umbilical, i.e., that the director surface is a sphere since a sphere is the only real surface all of whose points are umbilics.

8. We shall now consider the case in which the characteristic function ϕ vanishes.

Putting $\phi = 0$ in the equations $(1 \cdot 4)$ and $(1 \cdot 5)$

$$\frac{\partial x_1}{\partial u} = \frac{\partial y_1}{\partial u} = \frac{\partial z_1}{\partial u} = 0 \qquad (8 \cdot 1)$$

and

$$\frac{\partial x_1}{\partial v} = \frac{\partial y_1}{\partial v} = \frac{\partial z_1}{\partial v} = 0 \qquad (8 \cdot 2)$$

i.e., the surface of reference reduces to a point.

Hence the congruence of Ribaucour in which the characteristic function is zero is a system of rays through a point.

From the equations $(8 \cdot 1)$ and $(8 \cdot 2)$ we see that

$$e = f = f' = g = 0$$

Hence if in the congruence of Ribaucour the characteristic function vanishes, then it is a normal isotropic congruence which is a sort of a converse to the known result that the only normal isotropic congruence is a system of rays through a point.

9. *Pseudospherical congruences of Ribaucour.*—We shall now find the condition that the congruence of Ribaucour be pseudospherical.

We know that in a Pseudospherical congruence the mean parameter is constant[13]

$$i.e., \quad \frac{\mathscr{E}\,\delta'' + \mathscr{G}\,\delta - 2\mathscr{F}\,\delta'}{2\mathscr{H}^2} = \text{constant}$$

$$\text{or} \quad \frac{\mathscr{E}\,\mathrm{D}'' + \mathscr{G}\,\mathrm{D} - 2\mathscr{F}\,\mathrm{D}'}{2\mathscr{H}^2}\,\phi = \text{constant} \quad \text{from} \quad (1\cdot10), \quad (1\cdot11) \quad \text{and}$$

$(1\cdot12)$

$$\text{or} \quad \rho_1 + \rho_2 = \frac{\text{constant}}{\phi} \quad \text{where} \quad \rho_1 \text{ and } \rho_2 \text{ are the principal radii of}$$

curvature at the corresponding point of the director surface.

Hence if the congruence of Ribaucour is Pseudospherical then the sum of the principal radii of curvature at the corresponding point of the director surface varies inversely as Bianchi's characteristic function.

REFERENCES

1. Eisenhart .. *Differential Geometry of Curves and Surfaces*, p. 420.
2. ———————— .. *loc. cit.*, pp. 374–76.
3. ———————— .. *loc. cit.*, p. 398.
4. Zindler .. *Liniengeometrie*, 2, 108.
   Eisenhart .. *loc. cit.*, p. 422.
   Burgatti .. *Atti dei Lincei*, 1899, p. 515.
   Cifarelli .. *Annali di Matematica*, 1899, pp. 139–54.
5. Ogura, K. .. *Differential Geometry of the Line Congruence, Science Reports of Tohoku Imperial University*, 1916, p. 114.
6. Eisenhart .. *loc. cit.*, p. 131.
7. Ogura, K. .. *loc. cit.*
8. Eisenhart .. *loc. cit.*
9. See also the equations in *Bianchi, Lezioni* 1, 497; Sannia della Accademia di Torino, 45, 58.
10. Eisenhart .. *loc. cit.*, p. 153.
11. ———————— .. *loc. cit.*, p. 413.
12. Weatherburn .. *loc. cit.*, p. 70.
13. Slotnik .. " A method of applying tensor analysis to the study of Rectilinear congruences," *Mathematische Zeitschrift*, 28, § 10.
    Bianchi, L. .. *Annali di Matematica* (2), 15, 188; 71.