ON GF STRUCTURE MANIFOLD

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1. INTRODUCTION

LET us consider a differentiable manifold M_n of class C^∞. Let there exist on M_n a vector valued linear function F of class C^∞ such that

$$\bar{\bar{X}} = a^2 X \tag{1.1}$$

where

$$\bar{X} = F(X) \tag{1.2}$$

and a is any number. Then F is said to give a differentiable structure briefly GF structure to M_n defined by (1.1). It is well known that M_n is endowed with an almost product structure or an almost complex structure or an almost tangent structure according as $a = 1, -1$ or $a = i, -i$ or $a = 0, 1$.

If the given GF structure is endowed with a Hermite metric g such that

$$g(\bar{X}, \bar{Y}) + a^2 g(X, Y) = 0 \tag{1.3}$$

then we say that (F, g) gives to M_n a Hermite structure briefly H-structure subordinate to GF structure[1].

Note (1.1).—In this paper we will take 'a' such that $a^4 = 1$.

Definition (1.1).—A connection D in M_n equipped with a GF structure will be called a F-connection[3] if

$$(D_X F)(Y) = 0. \tag{1.4}$$

In view of (1.1) and (1.4) this equation is equivalent to

$$D_X \bar{Y} = \overline{D_X Y}, \quad D_{\bar{X}} \bar{Y} = \overline{D_{\bar{X}} Y}$$

$$\overline{D_{\bar{X}} \bar{Y}} = a^2 D_{\bar{X}} Y, \quad \overline{D_X \bar{Y}} = a^2 D_X Y.$$

Definition (1.2).—A connection D in M_n equipped with a GF structure will be called a semi F-connection if[3]

$$(\operatorname{div} F)(X) = 0 \quad \text{or} \quad (\operatorname{div} F)(\bar{X}) = 0. \tag{1.5}$$

Definition (1.3).—A connection D in M_n equipped with an H structure[3] will be called an almost F connection if

$$(D_X{}' F)(Y, Z) + (D_Y{}' F)(Z, X) + (D_Z{}' F)(X, Y) = 0$$

where 'F is a tensor of type (0, 2)

$$'F(X, Y) = g(\bar{X}, Y). \tag{1.6}$$

Let S be the torsion tensor of D

$$S(X, Y) = D_X Y - D_Y X - [X, Y] \tag{1.7}$$

then D is called half symmetric if[3]

$$a^2 S(X, Y) + S(\bar{X}, \bar{Y}) + \overline{S(\bar{X}, Y)} + \overline{S(X, \bar{Y})} = 0 \tag{1.8}$$

or equivalently

$$a^2 \overline{S(X, Y)} + \overline{S(\bar{X}, \bar{Y})} + a^2 S(\bar{X}, Y) + a^2 S(X, \bar{Y}) = 0.$$

The torsion tensors S and s of the respective connections D and E are related by

$$S(X, Y) - s(X, Y) = D_X Y - D_Y X - E_X Y + E_Y X. \tag{1.9}$$

Remark (1.1).—Taking $a = i$ or $-i$ in (1.8) we get the definition of half symmetric connection D in almost complex manifold.

Definition (1.4).—In GF structure manifold a connection D is called 0* connection if

$$(D_X F)(Y) + (D_{\bar{X}} F)(Y) = 0. \tag{1.10}$$

In view of (1.1) and (1.10) we get

$$D_X \bar{Y} - \overline{D_X Y} + a^2 D_{\bar{X}} Y - \overline{D_{\bar{X}} \bar{Y}} = 0$$

or equivalently

$$D_{\bar{X}}\bar{Y} - \overline{D_{\bar{X}}Y} + D_X Y - a^2\,\overline{D_X\bar{Y}} = 0.$$

THEOREM (1.1).—If D and E are related as

$$D_X Y \overset{\text{def.}}{=} \phi E_X Y + \theta E_X\bar{Y} + \sigma E_{\bar{X}}Y + \rho E_{\bar{X}}\bar{Y} + \alpha\overline{E_X\bar{Y}} + \beta\overline{E_X Y} + \gamma\overline{E_{\bar{X}}\bar{Y}} + \delta\overline{E_{\bar{X}}Y} \tag{1.11}$$

then the connection D defined by

$$D_X Y = (a^2\alpha + \delta - \rho)\,E_X Y + (\beta + \gamma - a^2\sigma)\,E_X\bar{Y} + \sigma E_{\bar{X}}Y + \rho E_{\bar{X}}\bar{Y} + \alpha\overline{E_X\bar{Y}} + \beta\overline{E_X Y} + \gamma\overline{E_{\bar{X}}\bar{Y}} + \delta\overline{E_{\bar{X}}Y} \tag{1.12}$$

is 0^* connection.

Proof.—From (1.11) we get

$$D_X\bar{Y} = \phi E_X\bar{Y} + a^2\,\theta E_X Y + \sigma E_{\bar{X}}\bar{Y} + a^2\,\rho E_{\bar{X}}Y + a^2\,\alpha\overline{E_X Y} + \beta\overline{E_X\bar{Y}} + a^2\,\gamma\overline{E_{\bar{X}}Y} + \delta\overline{E_{\bar{X}}\bar{Y}}. \tag{1.13}$$

$$\overline{D_X Y} = \phi\overline{E_X Y} + \theta\overline{E_X\bar{Y}} + \sigma\overline{E_{\bar{X}}Y} + \rho\overline{E_{\bar{X}}\bar{Y}} + a^2\,\alpha E_X\bar{Y} + a^2\,\beta E_X Y + a^2\,\gamma E_{\bar{X}}\bar{Y} + a^2\,\delta E_{\bar{X}}Y \tag{1.14}$$

$$(D_X F)(Y) = (\phi - a^2\alpha)(E_X\bar{Y} - \overline{E_X Y}) + (\theta - \beta)(a^2 E_X Y - \overline{E_X\bar{Y}}) + (\sigma - a^2\gamma)(E_{\bar{X}}\bar{Y} - \overline{E_{\bar{X}}Y}) + (\rho - \delta)(a^2 E_{\bar{X}}Y - \overline{E_{\bar{X}}\bar{Y}}). \tag{1.15}$$

Barring X and Y in (1.15) we get

$$(D_{\bar{X}}F)(Y) = (\phi - a^2\alpha)(a^2 E_{\bar{X}}Y - \overline{E_{\bar{X}}\bar{Y}}) + (\theta - \beta)(a^2 E_{\bar{X}}\bar{Y} - a^2\,\overline{E_{\bar{X}}Y}) + (\sigma - a^2\gamma)(E_X\bar{Y} - a^2\,\overline{E_X Y}) + (\rho - \delta) \times (E_X\bar{Y} - \overline{E_X Y}). \tag{1.16}$$

Adding (1.15) and (1.16) we get

$$(D_X F)(Y) + (D_{\bar{X}} F)(\bar{Y})$$

$$= (\phi - a^2 \alpha + \rho - \delta)(E_X \bar{Y} - \overline{E_X Y} + a^2 E_{\bar{X}} Y - \overline{E_{\bar{X}} \bar{Y}})$$

$$+ (\theta - \beta + a^2 \sigma - \gamma)(a^2 E_X Y - \overline{E_X \bar{Y}} + a^2 E_{\bar{X}} \bar{Y} - a^2 \overline{E_{\bar{X}} Y}) \tag{1.17}$$

If D is 0* connection then the left hand side is zero hence we get

$$\phi = a^2 \alpha - \rho + \delta \tag{1.18}$$

and

$$\theta = \beta + \gamma - a^2 \sigma. \tag{1.19}$$

Putting ϕ and θ in (1.11) we get (1.12).

THEOREM (1.2).—0* connection is semi F connection if and only if GF structure manifold is almost complex manifold.

Proof.—Let the connection D be 0* connection. We have[3]

$$(\text{Div } F)(Y) = (\phi - a^2 \alpha - a^2 \rho + a^2 \delta)(\text{div } F)(Y)$$

$$+ (\theta - \beta - \sigma + a^2 \gamma)(\text{div } F)(\bar{Y}). \tag{1.20}$$

Let GF structure manifold be almost complex manifold then in that case we have

$$a = \pm i. \tag{1.21}$$

For these values of 'a' we get in consequence of (1.18), (1.19) and (1.20)

$$(\text{Div } F)(Y) = 0$$

hence D is semi F-connection.

Conversely let D be semi F-connection, then we get

$$(\text{Div } F)(Y) = 0.$$

Hence from (1.20) we have

$$(a) \quad \phi = a^2 \alpha + a^2 \rho - a^2 \delta$$

$$(b) \quad = \beta + \sigma - a^2 \gamma. \tag{1.22}$$

Thus we see that for $a = \pm i$ only (1.22) (*a*) and (*b*), Coincide with (1.18) and (1.19). Hence if 0* connection is semi F-connection the manifold is almost complex.

THEOREM (1.3).—The condition for an 0* connection to be an F-connection or an M-conneetion or an almost F-connection is

$$\phi = a^2 \alpha, \quad \theta = \beta, \quad \sigma = a^2 \gamma, \quad \rho = \delta. \tag{1.23}$$

Proof.—The connection D is the most general F-connection if[3]

$$\phi = a^2 \alpha, \quad \theta = \beta, \quad \sigma = a^2 \gamma, \quad \rho = \delta. \tag{1.24}$$

The statement follows from (1.18), (1.19) and (1.24).

2. N-CONNECTION

Let us define a connection in M_n as follows:

Definition (2.1).—A connection D in M_n equipped with a GF structure will be called a N-connection it

$$(a)\ (D_{\bar X}F)(Y) + \overline{(D_X F)(Y)} = 0 \tag{2.1}$$

equivalently

$$(b)\ D_{\bar X}\bar Y + \overline{D_X \bar Y} - a^2 D_X Y - \overline{D_{\bar X} Y} = 0.$$

THEOREM (2.1).—If D and E are related as (1.11), then the connection D defined by

$$D_X Y = (a^2 \alpha + a^2 \rho - a^2 \delta) E_X Y + (\sigma - a^2 \gamma + \beta) E_X \bar Y + \sigma E_{\bar X} Y$$
$$+ \rho E_{\bar X} \bar Y + \alpha \overline{E_X Y} + \beta \overline{E_X \bar Y} + \gamma \overline{E_{\bar X} Y} + \delta \overline{E_{\bar X} \bar Y} \tag{2.2}$$

is N-connection.

Proof.—From (1.15) we have

$$(D_{\bar X}F)(Y) = (\phi - a^2 \alpha)(E_{\bar X}\bar Y - \overline{E_{\bar X} Y}) + (\theta - \beta)(a^2 E_{\bar X} Y - \overline{E_{\bar X}\bar Y})$$
$$+ (\sigma - a^2 \gamma)(a^2 E_X \bar Y - a^2 \overline{E_X Y}) + (\rho - \delta)$$
$$\times (E_X Y - a^2 \overline{E_X \bar Y}) \tag{2.3}$$

and

$$\overline{(D_X F)}(Y) = (\phi - a^2 \alpha)(\overline{E_X \bar{Y}} - a^2 E_X Y) + (\theta - \beta)$$
$$\times (a^2 \overline{E_X Y} - a^2 E_X \bar{Y}) + (\sigma - a^2 \gamma)$$
$$\times (\overline{E_{\bar{X}} \bar{Y}} - a^2 \overline{E_{\bar{X}} Y}) + (\rho - \delta)(a^2 \overline{E_{\bar{X}} Y} - a^2 E_{\bar{X}} \bar{Y}). \tag{2.4}$$

Adding (2.3) and (2.4) we get

$$(D_{\bar{X}} F)(Y) + \overline{(D_X F)(Y)}$$
$$= (\phi - a^2 \alpha - a^2 \rho + a^2 \delta)(E_{\bar{X}} \bar{Y} - \overline{E_{\bar{X}} Y} + \overline{E_X \bar{Y}} - a^2 E_X Y)$$
$$+ (\sigma - a^2 \gamma - \theta + \beta)(\overline{E_{\bar{X}} \bar{Y}} - a^2 E_{\bar{X}} Y + a^2 E_X \bar{Y} + a^2 \overline{E_X Y}) \tag{2.5}$$

D is N connection if and only if

$$\phi - a^2 \alpha - a^2 \rho + a^2 \delta = 0$$
$$\sigma - a^2 \gamma - \theta + \beta = 0. \tag{2.6}$$

Hence

$$(a)\ \phi = a^2 \alpha + a^2 \rho - a^2 \delta$$
$$(b)\ \theta = \sigma - a^2 \gamma + \beta. \tag{2.7}$$

Putting (2.7) in (1.11) we get (2.2).

THEOREM (2.2).– In M_n equipped with GF structure the N connection is semi F-connection.

Proof.– Putting the values of ϕ and θ from (2.7) (*a*) and (*b*) in (1.20) we get

$$(\text{Div } F)(Y) = 0. \tag{2.8}$$

Hence the connection is semi F-connection.

3. RIEMANNIAN CONNECTION

Let us now suppose that D is Riemannian connection. Barring X in (1.3) we get

$$a^2 g(X, \bar{Y}) + a^2 g(\bar{X}, Y) = 0,$$

Hence

$$'F(X, Y) + 'F(Y, X) = 0. \tag{3.1}$$

Thus $'F$ is skew symmetric bilinear tensor. Differentiating (1.3) covariantly we get

$$(D_X g)\big(F(Y), F(Z)\big) + g\big((D_X F)(Y), F(Z)\big) + g\big(F(D_X Y), (FZ)\big) + g\big(F(Y), (D_X F)(Z)\big) + g\big(F(Y), F(D_X Z)\big) + a^2 D_X(g)(Y, Z) + a^2 g(D_X Y, Z) + a^2 g(Y, D_X Z) = 0. \tag{3.2}$$

In consequence of (1.3) and (3.2) we get

$$g\big((D_X F)(Y), F(Z)\big) + g\big(F(Y), (D_X F)(Z)\big) = 0 \tag{3.3}$$

or

$$(D_X F)(Y, \bar{Z}) + (D_X 'F)(Z, \bar{Y}) = 0. \tag{3.4}$$

Thus we have

THEOREM (3.1).—In H structure manifold we have

(*a*) $(D_X 'F)(Y, \bar{Z}) = (D_X 'F)(\bar{Y}, Z)$

(*b*) $(D_X 'F)(\bar{Y}, \bar{Z}) = a^2 (D_X 'F)(Y, Z).$ (3.5)

Proof.—(3.5) (*a*) follows from (3.4) and (3.5) (*b*) follows from (3.5) (*a*) by barring Y.

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