ELLIPSOIDAL HARMONICS OF LARGE ORDERS.

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IN a previous memoir,[1] the author obtained various expressions for Ellipsoidal Wave-functions. During the course of that work, it appeared that the methods adopted there would give useful results in the case of Ellipsoidal Harmonics also. Owing to other work, the publication of these results has been very much delayed.

In this paper asymptotic expressions for large orders of the functions and the characteristic constants are obtained.

The fundamental form of Lamé equation adopted in this paper is

$$d^2E/d\xi^2 - \{n(n+1)k^2 sn^2\xi + A\}E = 0 \tag{1}$$

using Jacobian elliptic functions. n is an integer and A is one of the characteristic constants that will lead on solution to an Ellipsoidal Harmonic.

It has been shown by Whittaker[2] that the Ellipsoidal Harmonics satisfy elegant integral equations.

$$E_n(\xi) = i\lambda \int_{K-2iK'}^{K+2iK'} P_n\left(\frac{ik}{k'} cn\,\xi\,cn\,\eta\right) E_n(\eta)\,d\eta \tag{2}$$

$$E_n(\xi) = \lambda \int_{-2K}^{2K} P_n\left(\frac{1}{k'} dn\,\xi\,dn\,\eta\right) E_n(\eta)\,d\eta \tag{3}$$

$$E_n(\xi) = \lambda \int_{-2K}^{2K} P_n(k\,sn\,\xi\,sn\,\eta)\,E_n(\eta)\,d\eta. \tag{4}$$

The limits of integration for (2) have been changed to suit the conditions of the problem.

$$i\lambda \int_{K-2iK'}^{K+2iK'} P_n\left(\frac{ik}{k'} cn\,\xi\,cn\,\eta\right) E_n(\eta)\,d\eta$$

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[1] See *The Ind. Jour. Phys.*, 1934, **9**, 45.

[2] *Proc. Lond. Math. Soc.*, 1915, **14**, (2), 260, and also Whittaker and Watson, *Modern Analysis*, 564.

can be written as

$$4i\lambda \int_{K}^{K+iK'} P_n \left(\frac{ik}{k'} cn\, \xi\, cn\, \eta \right) \psi(\eta)\, d\eta \qquad \text{where}$$

$$4\psi(\eta) = E_n(2iK' + 2K - \eta) + E_n(\eta) + (-)^n E_n(2K - \eta)$$
$$+ (-)^n E_n(\eta - 2iK')$$

$\psi(\eta)$ is a polynomial in $cn\,\eta$ and hence in the range of integration, it is continuous, bounded function with limited total fluctuation. According as

$$ik/k'\, cn\, \xi\, cn\, \eta \text{ is } > \text{ or } < 1$$

$$P_n \left(\frac{ik}{k'} cn\, \xi\, cn\, \eta \right) \text{ is asymptotically equivalent to}$$

$$\frac{\exp(n + \tfrac{1}{2}) \left\{ \cosh^{-1} \frac{ik}{k'} cn\, \xi\, cn\, \eta \right\}}{\sqrt{2\pi n} \left\{ -\frac{k^2}{k'^2} cn^2\, \xi\, cn^2\, \eta - 1 \right\}^{\frac{1}{4}}}$$

or $$\sqrt{\frac{2}{n\pi}} \; \frac{\cos \left\{ (n + \tfrac{1}{2}) \arccos \frac{ik}{k'} cn\, \xi\, cn\, \eta - \pi/4 \right\}}{\left(1 + \frac{k^2}{k'^2} cn^2\, \xi\, cn^2\, \eta \right)^{\frac{1}{4}}}$$

if n is large.

These forms show that the method of steepest descents could profitably be applied to obtain the dominant terms for the asymptotic expressions for $E_n(\xi)$. Writing the first expression as

$$\frac{\exp(n + \tfrac{1}{2}) \cosh^{-1} cn\, \xi \cdot \exp - (n + \tfrac{1}{2}) \left\{ \cosh^{-1} cn\, \xi - \cosh^{-1} \frac{ik}{k'} cn\, \xi\, cn\, \eta \right\}}{\sqrt{2\pi n} \left\{ -\frac{k^2}{k'^2} cn^2\, \xi\, cn^2\, \eta - 1 \right\}^{\frac{1}{4}}}$$

it is seen that the main contribution to the integral is obtained in the neighbourhood of the point $\eta = K + iK'$. From the method of stationary phase, it is seen that if even the second form of $P_n \left(\frac{ik}{k'} cn\, \xi\, cn\, \eta \right)$ is used the main contribution is from the same point $\eta = K + iK'$. Hence it is sufficient to consider the asymptotic value for P_n when η is nearly $K + iK'$. In the present investigation it may be assumed that $cn\,\xi \gg 1$. Then $E_n(\xi) \sim$

$$\frac{4i\lambda}{\sqrt{2\pi n}} \cdot \exp(n + \tfrac{1}{2}) \cosh^{-1} cn\, \xi$$

$$\times \int_{K}^{K+iK'} \frac{\exp - (n + \tfrac{1}{2}) \left\{ \cosh^{-1} cn\, \xi - \cosh^{-1} \frac{ik}{k'} cn\, \xi\, cn\, \eta \right\}}{\left\{ -\frac{k^2}{k'^2} cn^2\, \xi\, cn^2\, \eta - 1 \right\}^{\frac{1}{4}}} \psi(\eta)\, d\eta. \qquad (5)$$

To obtain the dominant term, it is sufficient if η is replaced by $K + iK'$ in the expression $\psi(\eta) \left\{ -\dfrac{k^2}{k'^2} cn^2 \xi \, cn^2 \eta - 1 \right\}^{-\frac{1}{4}}$ as it is bounded and has limited total fluctuation in the range of integration and the argument of the exponential term is expanded in terms of $K + iK' - \eta$ and terms higher than the square are neglected.

Putting $\eta = K + iK' - i\sigma$ it follows that

$$E_n(\xi) \sim$$

$$- \frac{4\lambda}{\sqrt{2\pi n}} \{cn\, \xi + \sqrt{(cn^2 \xi - 1)}\}^{n + \frac{1}{2}} \psi(K + iK')(cn^2 \xi - 1)^{-\frac{1}{4}}$$

$$\times \int_0^{K'} \exp - \tfrac{1}{2}(n + \tfrac{1}{2}) \frac{\sigma^2 cn\, \xi}{\sqrt{(cn^2 \xi - 1)}} \cdot d\sigma$$

$$\sim - \frac{2\lambda \psi(K + iK')}{n \sqrt{cn\, \xi}} \{cn\, \xi + \sqrt{(cn^2 \xi - 1)}\}^{n + \frac{1}{2}}. \tag{6}$$

As $E_n(\xi)$ is a polynomial in $cn\, \xi$ it follows that

$$E_n(\xi) \sim$$

$$- \frac{\lambda}{n} \{E_n(K + iK') + (-)^n E_n(K - iK')\} \frac{\{cn\, \xi + \sqrt{(cn^2 \xi - 1)}\}^{n+\frac{1}{2}}}{\sqrt{cn\, \xi}}. \tag{7}$$

When the Ellipsoidal Harmonic is a polynomial in $dn\, \xi$ it follows from (3)

$$E_n(\xi) \sim$$

$$\frac{\lambda}{nk} \{E_n(0) + E_n(2K)\} \left(\frac{dn\, \xi + kcn\, \xi}{k'} \right)^{n+\frac{1}{2}} \left(\frac{k'}{dn\, \xi} \right)^{\frac{1}{2}}. \tag{8}$$

Corresponding to (4)

$$E_n(\xi) \sim$$

$$\frac{\lambda}{nk'} \{E_n(K) + (-)^n E_n(-K)\} \frac{\{k\, sn\, \xi + \sqrt{(k^2 sn^2 \xi - 1)}\}^{n+\frac{1}{2}}}{\sqrt{k\, sn\, \xi}}. \tag{9}$$

The arguments $cn\, \xi + i\, sn\, \xi$; $dn\, \xi + k\, cn\, \xi$ and $k\, sn\, \xi + i\, dn\, \xi$ are doubly periodic functions of ξ. It follows that if equation (1) has asymptotic expansions of Horn and Jeffreys type in the form $\psi\, \epsilon^{(n+\frac{1}{2})\phi}$ the choice has to be limited by the condition that ϵ^ϕ is a doubly periodic function.

Let the following expansion be assumed :

$$E(\xi) = \psi\, \epsilon^{(n+\frac{1}{2})\phi} \{1 + f_1/(n+\tfrac{1}{2}) + f_2/(n+\tfrac{1}{2})^2 + \cdots\}$$

$$A = a_{-2}(n+\tfrac{1}{2})^2 + a_{-1}(n+\tfrac{1}{2}) + a_0 + a_1/(n+\tfrac{1}{2}) + \cdots$$

where $\psi, \phi, f_1, f_2 \cdots$ are functions of ξ only, $a_{-2}, a_{-1}, a_0 \cdots$ are constants and independent of n. Substituting the above in equation (1) and equating

coefficients of powers of $(n + \frac{1}{2})$ it is seen that

$$\phi'^2 - (k^2 \, sn^2 \, \xi + a_{-2}) = 0$$

$$\phi'' + \frac{2\phi'\psi'}{\psi} - a_{-1} = 0$$

$$2\phi'f_1' + \psi''/\psi - (a_0 - \tfrac{1}{4} k^2 \, sn^2 \, \xi) = 0$$

$$2\phi'f_2' + \psi''f_1/\psi + 2\psi'f_1'/\psi + f_1'' - (a_0 - \tfrac{1}{4}k^2 \, sn^2 \, \xi) f_1 - a_1 = 0$$

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where dashes denote differentiation with respect to ξ ;
or

$$\psi = \frac{\text{const } \epsilon^{\pm \int a_{-1} \, d\xi / \sqrt{(a_{-2} + k^2 \, sn^2 \, \xi)}}}{(a_{-2} + k^2 \, sn^2 \, \xi)^{\frac{1}{4}}}$$

$$2f_1 + \frac{\psi'}{\psi\phi'} + \int \left\{ \frac{\psi'}{\psi} \left(\frac{\psi'}{\psi} + \frac{\phi''}{\phi'} \right) - a_0 + \frac{k^2}{4} \, sn^2 \, \xi \right\} \frac{d\xi}{\phi'} = 0$$

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If ϵ^ϕ has to be a doubly-periodic function there are only three types of possible values for ϕ', namely, $\pm k \, sn \, \xi$; $\pm ik \, cn \, \xi$; $\pm i \, dn \, \xi$ corresponding to the following values of a_{-2} : 0, $-k^2$ and -1 respectively.

When $\phi' = \pm k \, sn \, \xi$ the value of ψ is

$$\frac{1}{\sqrt{sn \, \xi}} \left(\frac{dn \, \xi - cn \, \xi}{k' \, sn \, \xi} \right)^{\pm a_{-1}/2k}.$$

Hence to a first approximation

$$E_n(\xi) = \frac{1}{\sqrt{sn \, \xi}} \left[A \left(\frac{dn \, \xi - k \, cn \, \xi}{k'} \right)^{n + \frac{1}{2}} \left(\frac{dn \, \xi - cn \, \xi}{k' \, sn \, \xi} \right)^{a_{-1}/2k} \right.$$

$$\left. + B \left(\frac{dn \, \xi - k \, cn \, \xi}{k'} \right)^{-n - \frac{1}{2}} \left(\frac{dn \, \xi - cn \, \xi}{k' \, sn \, \xi} \right)^{-a_{-1}/2k} \right]$$

where A and B are constants. As $dn^2\xi - k^2 cn^2\xi = k'^2$ and $dn^2\xi - cn^2\xi = k'^2 sn^2\xi$ it follows that the function $E_n(\xi)$ would be doubly-periodic and symmetrical about the usual points if and only if $A = \pm B$ and $a_{-1}/2k$ is $l + \frac{1}{2}$ where l is an integer. And

$$2f_1 - \left\{ \frac{-(2l + 1) \pm (l^2 + l + 1) \, cn \, \xi \, dn \, \xi}{2k \, sn^2 \, \xi} \right\}$$

$$\pm \frac{1}{k} \left\{ a_0 - \tfrac{1}{2} (1 + k^2)(l^2 + l + \tfrac{1}{2}) \right\} \log \frac{dn \, \xi - cn \, \xi}{k \, sn \, \xi} = 0$$

the constant of integration may be omitted as it will finally multiply the whole function by a constant. If f_{1+} and f_{1-} are the two values of f_1 corresponding to $\pm\ k\ sn\ \xi\ \pm\ \phi'$ then to a second approximation

$$E_n\ (\xi)\ = A\ \epsilon^{(n+\frac{1}{2})\phi}\ \psi_1\left(1\ +\ \frac{f_{1+}}{n\ +\ \frac{1}{2}}\right)$$

$$+\ B\ \epsilon^{-(n+\frac{1}{2})\phi}\ \psi_2\left(1\ +\ \frac{f_{1-}}{n\ +\ \frac{1}{2}}\right).$$

But this function is not symmetrical or anti-symmetrical about $\xi = K$ with the condition $A = \pm B$ unless the logarithmic terms disappear. Hence

$$a_0\ =\ \tfrac{1}{2}\ (1\ +\ k^2)\ (l^2\ +\ l\ +\ \tfrac{1}{2}).$$

Similarly at all stages of working it will be necessary to equate the coefficients of logarithmic terms to zero.

A statement of the values of the first three functions and the corresponding constants are given below :—

a_{-2}	ϕ'	$\epsilon^{(n+\frac{1}{2})\phi}$
0	$\pm\ k\ sn\ \xi$	$\left(\dfrac{dn\ \xi\ -\ k\ cn\ \xi}{k'}\right)^{\pm\ (n+\frac{1}{2})}$
$-\ k^2$	$\pm\ ik\ cn\ \xi$	$(dn\ \xi\ +\ ik\ sn\ \xi)^{\pm\ (n+\frac{1}{2})}$
$-\ 1$	$\pm\ i\ dn\ \xi$	$(cn\ \xi\ +\ i\ sn\ \xi)^{\pm\ (n+\frac{1}{2})}$

a_{-1}	ψ
$2k\ (l\ +\ \frac{1}{2})$	$\dfrac{1}{\sqrt{sn\ \xi}}\left(\dfrac{dn\ \xi\ -\ cn\ \xi}{k'\ sn\ \xi}\right)^{\pm\ (l+\frac{1}{2})}$
$2ikk'\ (l\ +\ \frac{1}{2})$	$\dfrac{1}{\sqrt{cn\ \xi}}\left(\dfrac{dn\ \xi\ +\ k'\ sn\ \xi}{cn\ \xi}\right)^{\pm\ (l+\frac{1}{2})}$
$2k'\ (l\ +\ \frac{1}{2})$	$\dfrac{1}{\sqrt{dn\ \xi}}\left(\dfrac{cn\ \xi\ +\ ik'\ sn\ \xi}{dn\ \xi}\right)^{\pm\ (l+\frac{1}{2})}$

a_0	f_1
$\frac{1}{2}(1 + k^2)(l^2 + l + \frac{1}{2})$	$\dfrac{-(2l+1) \pm (l^2+l+1)\,cn\,\xi\,dn\,\xi}{4k\,sn^2\,\xi}$
$k'^2/4 - l(l+1)(k^2 - \frac{1}{2})$	$\dfrac{-(2l+1)\,k' \mp (l^2+l+1)\,sn\,\xi\,dn\,\xi}{4ik\,cn^2\,\xi}$
$\frac{1}{4} + \frac{1}{2}l(l+1)(1 + k'^2)$	$\dfrac{-(2l+1)\,k' \pm l(l+1)\,k^2\,sn\,\xi\,cn\,\xi}{4i\,dn^2\,\xi}$

As usual, this last method of obtaining the asymptotic expansion is less cumbersome than the earlier adopted in this paper.

There are three distinct asymptotic expansions for the value of the characteristic constant as in the case of the Ellipsoidal Wave-functions. The first and the third lead to real values of A while the second expression gives a complex value of A. The exact significance of these complex values is not known.

The constants of the asymptotic expansions may easily be obtained by comparing them with the expressions obtained earlier by the method of steepest descents in (7), (8) and (9).