ON A METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS IN SERIES.

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Given a linear differential equation

$$f(\mathrm{D})\, y + \mathrm{I}y = 0$$

where $\mathrm{D} = d/dx$; and I is a function of x only, it is a common method of solution to substitute for y in the above equation a series of form $\Sigma\, a_n\, u_n\,(x)$ where the a's are constants and u's are linearly independent functions of x only and obtain the values of the constant coefficients. There occur several problems in applied mathematics where the above method is insufficient to determine the mutual relation between the coefficients. Also the usually known methods of solving the differential equations do not lead to solutions which would be readily interpretable for the actual problems. As a particular example one may notice

$$(\mathrm{D}^2 - a^2)^m\, y = \mathrm{I}y$$

which type of equations occur in stability problems. As the method is capable of application to other problems also, a brief account of the method of solution that has been introduced would perhaps be of interest.*

Take the very simple example

$$\mathrm{D}^2 y - p^2 y = 0$$

and assume, in the interval $-\pi < x < \pi$, a fourier series $\frac{1}{2}\,\mathrm{A}_0 + \Sigma\, \mathrm{A}_n \cos nx + \Sigma\, \mathrm{B}_n \sin nx$ and substitute this in the differential equation and equate the coefficients of $\cos nx$ and $\sin nx$ to zero. The only relation that is obtained is $\mathrm{A}_0 = 0 = \mathrm{A}_n = \mathrm{B}_n$ unless ip is a real integer. This null solution is obviously incorrect as it is well known that $\cosh px$ and $\sinh px$ have valid fourier expansions in the interval. However in this particular example, the solution could have been obtained otherwise.

Returning to the general problem let $f(\mathrm{D})$ be a polynomial in D and let it be assumed that the differential equation $f(\mathrm{D})\, y = 0$ can be completely

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* See *e.g.* "On the Differential Equation of the Instability Problems." S. L. Malurkar and M. P. Srivastava. *Proc. Ind. Acad. Sci.*, 1937, 5, 34.

solved in terms of known functions. With the knowledge of this solution the complete solution of the equation

$$f(D)\, y = Iy \tag{1}$$

is attempted here.

Assume that in the interval $-\pi < x < \pi$

$$y = \tfrac{1}{2} A_0 + \Sigma A_n \cos nx + \Sigma B_n \sin nx$$

Substitute this value of y only on the right side of the equation in (1). Then it can be written as

$$f(D)\, y = \{\tfrac{1}{2} A_0 + \Sigma A_n \cos nx + \Sigma B_n \sin nx\}\, I \tag{2}$$

The solution of equation (2) is

$$y = Y + P(x)$$

where Y is the complementary solution involving the appropriate number of arbitrary constants and P (x) is a particular solution of the modified differential equation (2). Let it be assumed that Y and P (x) can be developed as fourier series. Let P (x) be written as

$$\tfrac{1}{2} C_0 + \Sigma C_n \cos nx + \Sigma D_n \sin nx.$$

The C's and D's are obviously functions of the A's and B's. Comparing the fourier series for Y + P (x) with the original assumption for y, it easily follows that

$$\begin{aligned} A_0 - C_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} Y\, dx \\ A_n - C_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} Y \cos nx\, dx \\ B_n - D_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} Y \sin nx\, dx \end{aligned} \tag{3}$$

Hence the C's and D's and therefore the A's and B's are obtained as functions of the arbitrary constants involved in Y. The function P (x) can be expressed free from unknown constants apart from those involved in Y. The arbitrary constants can be evaluated by applying the boundary conditions to the function Y + P (x). Then the values of the A's and B's are determinable completely.

In this method it may be pointed out that the equation (1) is solved in two stages. An assumption for y in a series form is made only for a portion of the terms of the differential equation. Then the modified differential equation is solved completely. The solution now obtained is compared or equated to the original assumption.

A simple example of the application of the method to a differential equation whose solution can also be found out by other simpler methods is given here. Take

$$d^2y/dx^2 - a^2y = \lambda^2 y \tag{4}$$

In the interval $0 < x < \pi$ it is sufficient to assume a fourier sine series in the form

$$y = \Sigma \, A_n \sin nx$$

Then the equation (4) is transformed to

$$d^2y/dx^2 - a^2y = \lambda^2 \, \Sigma \, A_n \sin nx \tag{5}$$

and this modified equation is solved.

It is easy to show that the solution of (5) is

$$y = B \cosh a \, (\pi/2 - x) + C \sinh a \, (\pi/2 - x)$$
$$- \lambda^2 \, \Sigma \, A_n \sin nx/(n^2 + a^2)$$

The forms $\cosh a \, (\pi/2 - x)$ and $\sinh a \, (\pi/2 - x)$ are better as the fourier expansions of these need contain either only the odd or even multiples of x. As in the interval $0 < x < \pi$

$$\cosh a \, (\pi/2 - x) = \frac{2}{\pi} \sum_{1}^{\infty} n \, (1 - \cos n\pi) \cosh a\pi/2 \sin nx/(n^2 + a^2)$$

and

$$\sinh a \, (\pi/2 - x) = \frac{2}{\pi} \sum_{1}^{\infty} n \, (1 + \cos n\pi) \sinh a\pi/2 \sin nx/(n^2 + a^2)$$

it follows from a comparison of the solution of the (5) with the assumed series for y before the modification was made that

$$A_n = \frac{2B \cosh a\pi/2}{\pi(n^2 + a^2)} (1 - \cos n\pi) + \frac{2C \sinh a\pi/2}{\pi(n^2 + a^2)} (1 + \cos n\pi)$$
$$- \lambda^2 A_n/(n^2 + a^2)$$

or

$$A_n = \frac{2n}{\pi(n^2 + a^2 + \lambda^2)} \{B \cosh a\pi/2 \, (1 - \cos n\pi) + C \sinh a\pi/2 \, (1 + \cos n\pi)\}$$

Hence comparing with the expansions above it is seen that

$$y = \frac{B \cosh a\pi/2}{\cosh b\pi/2} \cosh b(\pi/2 - x) + \frac{c \sinh a\pi/2}{\sinh b\pi/2} \sinh b(\pi/2 - x)$$

where $b^2 = a^2 + \lambda^2$ which would have been the solution had the differential equation been solved in the form

$$(d^2/dx^2 - b^2) \, y = 0$$

Take another example

$$d^2y/dx^2 + p^2y = 0 \tag{6}$$

where p is not an integer and the solution which vanishes at $x = 0$ and $x = \pi$ is required. The equation might be split up into two portions as

$$d^2y/dx^2 = - p^2y$$

Assume that in the interval $0 < x < \pi$ y has the expansion $\Sigma A_n \sin nx$. The modified differential equation which has to be solved is

$$d^2y/dx^2 = - p^2 \Sigma A_n \sin nx$$

whose solution is obviously

$$L + Mx - p^2 \Sigma \frac{A_n}{n^2} \sin nx$$

Comparing this with the original assumption it follows that

$$2 (1 - \cos n\pi) L - 2M \cos n\pi = \pi n A_n (1 - p^2/n^2)$$

Applying the boundary conditions to the form of the solution obtained after integration it follows that

$$L = 0 = M = A_n \text{ and hence } y = 0$$

In any practicable method of solution of any differential equation in series, term by term differentiation enters invariably and the justification of the process can be made mostly after the solution has been obtained in some series form. If the resulting series lead to divergent types the process is given up as not legitimate. In the present method, *i.e.*, in the determination of the particular solution of the modified differential equation term by term integration is mostly involved. The necessary justification of convergence of the solution obtained will be less than in the case of term by term differentiation.

The method given above can be put slightly in a more general symbolic form. If the differential equation $f(D) y = 0$ can be split up into two terms and the equation is capable of being written as

$$f_1 (D) y + f_2 (D) y = 0 \tag{7}$$

where f_1 (D) and f_2 (D) are two polynomials in D and the order of the polynomial f_1 is greater than that of the polynomial f_2; it may be assumed that

$$y = \tfrac{1}{2} A_0 + \Sigma (A_n \cos nx + B_n \sin nx)$$

and the differential equation may be solved in the modified form

$$f_1 (D) y = - f_2 (D) \{\tfrac{1}{2} A_0 + \Sigma (A_n \cos nx + B_n \sin nx)\} \tag{8}$$

This equation may be solved as before with the knowledge of the solution of $f_1 (D) y = 0$ in the form $y = Y + P(x)$. Comparing the fourier coefficients of $Y + P(x)$ with the A's and B's, the mutual relations are obtained. By applying the boundary conditions to $Y + P(x)$ as before the complete solution is obtained.

Instead of assuming fourier type of series, it is often possible and sometimes preferable to assume expansions involving other functions which form a complete orthogonal set, *e.g.*, the Legendre functions, the normal Mathieu functions or the many wave-functions that have come into prominence in quantum mechanics. Of course in these cases f (D) need not be purely a polynomial in D. The differential equation is broken up into two suitable parts in one of which the series in terms of the orthogonal functions is substituted. The modified differential equation is then completely solved and this solution is compared with the original assumed expansion and the coefficients determined.

In subsequent papers the author hopes to publish the utility of these methods in solving actual problems in applied mathematics.