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A submersion principle and its applications

By

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1. Introduction

Let G be a real reductive group and π an irreducible admissible representation of G. Let Θ denote the character of π. We recall that Θ is a distribution on G defined by

$$\Theta(f) = \operatorname{tr} \pi(f) \quad (f \in C_c^\infty(G)).$$

It is well known that Θ is a locally summable function on G which is analytic on the set G' of regular elements. Fix $\gamma_0 \in G'$ and let Γ be the Cartan subgroup of G containing γ_0. Put

$$\Gamma' = G' \cap \Gamma \text{ and } G_\Gamma = \bigcup_{x \in G} x\Gamma' x^{-1}.$$

The mapping $(x, \gamma) \mapsto x\gamma x^{-1}$ of $G \times \Gamma'$ into G is everywhere submersive. Since Θ is invariant under all inner automorphisms of G, one proves easily that Θ defines a distribution θ on Γ'. Let $\mathfrak{Z}$ be the algebra of all differential operators on G which commute with both left and right translations. Then Θ satisfies the differential equations

$$z\Theta = \chi(z)\Theta \quad (z \in \mathfrak{Z}),$$

where χ is the infinitesimal character of π. We can transcribe these equations in terms of θ. In this way we obtain a system of differential equations for θ on Γ'. It turns out that this system is elliptic and therefore θ is an analytic function on Γ'. But this implies that Θ coincides with an analytic function on G_Γ.

We would like to prove a similar result in the p-adic case. Let Ω be a $\mathfrak{p}$-adic field and G the group of all Ω-rational points of a connected reductive Ω-group $\mathbf{G}$ [3]. Then G, with its usual topology, is a locally compact, totally disconnected, unimodular group. Let dx denote its Haar measure. We shall use the terminology of [3] without further comment.

Let π be an admissible irreducible representation of G. Then for every $f \in C_c^\infty(G)$, the operator

$$\pi(f) = \int_G f(x)\pi(x)\,dx$$

has finite rank. Put

$$\Theta(f) = \operatorname{tr} \pi(f).$$

Then Θ is a distribution on G. Let G' be the set of all points $x \in G$ where $D_G(x) \neq 0$ ([3], § 15). Then G' is an open, dense subset of G whose complement has measure zero. We intend to show that Θ coincides with a locally constant function F on G'. This means that

$$\Theta(f) = \int f(x) F(x)\, dx,$$

for all $f \in C_c^\infty(G')$.

In case char $\Omega = 0$, this fact was first proved by Howe by making use of his Kirillov theory for p-adic groups. Moreover when char $\Omega = 0$, it is known that Θ is a locally summable function on G [4, 5]. But these methods, which make extensive use of Lie algebras and the exponential mapping, do not seem to work in characteristic p. We shall therefore construct a proof on a totally different principle.

2. The submersion principle

Let P be a parabolic subgroup of G and $x \mapsto x^*$ the projection of G on $G^* = G/P$.

Theorem 1 (the submersion principle). *Fix $\gamma \in G'$. Then the mapping*

$$x \mapsto (x\gamma x^{-1})^*$$

from G to G^* is everywhere submersive.

When char $\Omega = 0$, the proof is very easy. Moreover Borel assures me that this principle is actually true over an arbitrary field.

Let us verify it when char $\Omega = 0$. Put

$$\phi_\gamma : x \mapsto (x\gamma x^{-1})^*.$$

Then $\quad \phi_\gamma(xy) = \phi_{y\gamma y^{-1}}(x) \qquad (x, y \in G).$

Hence it is enough to prove that ϕ_γ is submersive at $x = 1$.

Fix a split component A of P and let $P = MN$ be the corresponding Levi decomposition of P. Let $(\bar{P}, A)$ denote the p-pair opposite to (P, A). Then $\bar{P} = M\bar{N}$. We denote the Lie algebras of $G, P, M, N, \bar{N}$ by $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$ respectively. Then

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{n}, \; \mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n},$$

and we have to verify that

$$(\operatorname{Ad}(\gamma^{-1}) - 1)\,\mathfrak{g} + \mathfrak{p} = \mathfrak{g}.$$

Fix a symmetric, nondegenerate, G-invariant bilinear form B on $\mathfrak{g}$ with values in Ω. Since G is reductive, this is possible. Let Γ be the Cartan subgroup of G determined by γ and $\mathfrak{c}$ the Lie algebra of Γ. Then $\mathfrak{c} = \ker\,(\operatorname{Ad}(\gamma) - 1)$. For any linear subspace $\mathfrak{q}$ of $\mathfrak{g}$, let $\mathfrak{q}^{\perp}$ denote the space of all $Y \in \mathfrak{g}$ such that $B(X, Y) = 0$ for all $X \in \mathfrak{q}$. Then

$$\mathfrak{p}^{\perp} = \mathfrak{n}, \quad ((\mathrm{Ad}(\gamma^{-1}) - 1)\,\mathfrak{g})^{\perp} = \ker\ (\mathrm{Ad}(\gamma) - 1) = \mathfrak{c}.$$

Therefore $((\mathrm{Ad}(\gamma^{-1}) - 1)\,\mathfrak{g} + \mathfrak{p})^{\perp} \subset \mathfrak{c} \cap \mathfrak{n} = \{0\}$,

and this implies the desired result.

3. The function $f_{\alpha,\,\gamma}$

Fix γ and P as in theorem 1. Then the mapping*

$$(x, p) \mapsto {}^{x}\gamma \cdot p \qquad (x \in G, p \in P)$$

of $G \times P$ into G is everywhere submersive. Hence ([2], p. 49) there exists a unique linear mapping

$$\alpha \mapsto f_{\alpha,\,\gamma} \qquad (\alpha \in C_c^{\infty}(G \times P))$$

from $C_c^{\infty}(G \times P)$ to $C_c^{\infty}(G)$ such that

$$\int_{G \times P} \alpha(x : p)\, F({}^{x}\gamma \cdot p)\, dx d_l p = \int_G f_{\alpha,\,\gamma}(x)\, F(x)\, dx,$$

for all $F \in C_c^{\infty}(G)$. Here $d_l p$ is the left Haar measure on P.

Lemma 1. *Fix* $\alpha \in C_c^{\infty}(G \times P)$. *Then the mapping*

$$y \mapsto f_{\alpha,\,y} \qquad (y \in G')$$

from G' *to* $C_c^{\infty}(G)$ *is locally constant.*

The mapping

$$(y, x, p) \mapsto (y, {}^{x}y \cdot p)$$

from $G' \times G \times P$ to $G' \times G$ is submersive. Hence there exists a unique linear mapping

$$\beta \mapsto \phi_{\beta} \qquad (\beta \in C_c^{\infty}(G' \times G \times P))$$

from $C_c^{\infty}(G' \times G \times P)$ to $C_c^{\infty}(G' \times G)$ such that

$$\int \beta(y : x : p)\, \Phi(y : {}^{x}y \cdot p)\, dy dx d_l p = \int \phi_{\beta}(y : x)\, \Phi(y : x)\, dy dx$$

for all $\Phi \in C_c^{\infty}(G' \times G)$. Now let

$$\Phi(y : x) = \lambda(y)\, F(x),$$

where $\lambda \in C_c^{\infty}(G')$ and $F \in C_c^{\infty}(G)$. Then

$$\int_G \lambda(y)\, dy \int_{G \times P} \beta(y : x : p)\, F({}^{x}y \cdot p)\, dx d_l p = \int_G \lambda(y)\, dy \int_G \phi_{\beta}(y : x)\, F(x)\, dx.$$

* We write ${}^{x}y = xyx^{-1}$ for $x, y \in G$.

This being true for all λ, we conclude that

$$\int_{G\times P} \beta(y:x:p)\, F({}^{x}y\cdot p)\, dx d_l p = \int \phi_\beta(y:x)\, F(x)\, dx$$

for all $y \in G'$ and $F \in C_c^\infty(G)$.

Now fix $y_0 \in G'$ and put

$$\beta(y:x:p) = \mu(y)\, \alpha(x:p)$$

where $\alpha \in C_c^\infty(G \times P)$, $\mu \in C_c^\infty(G')$ and $\mu(y_0) = 1$. Let G_0 be a neighbourhood of y_0 in G' such that $\mu = 1$ on G_0. Then $\beta(y:x:p) = \alpha(x:p)$ for $y \in G_0$ and therefore

$$\int_{G\times P} \beta(y:x:p)\; F({}^{x}y\cdot p)\, dx d_l p = \int \alpha(x:p)\, F({}^{x}y\cdot p)\, dx d_l p$$

$$= \int f_{\alpha,y}(x)\, F(x)\, dx.$$

Hence $\int f_{\alpha,y}(x)\; F(x)\, dx = \int \phi_\beta(y:x)\, F(x)\, dx$

for $y \in G_0$ and $F \in C_c^\infty(G)$. This shows that

$$f_{\alpha,y}(x) = \phi_\beta(y:x) \quad (y \in G_0, x \in G).$$

Since $\phi_\beta \in C_c^\infty(G' \times G)$, our assertion is now obvious.

4. The operator T_x

Fix a minimal p-pair (P, A) in $G\,(P = MN)$ and let K be an open compact subgroup of G of Bruhat-Tits ([2], p. 16) corresponding to A. Let π be an admissible representation of G on V. We recall that $\mathrm{End}^0 V$ is defined to be the subspace of all $T \in \mathrm{End}\, V$ such that the mappings

$$x \mapsto \pi(x)\, T,\; x \mapsto T\pi(x),$$

of G into End V are both smooth.

Let dk denote the normalised Haar measure on K.

Theorem 2. *Let π be an admissible representation of G on V such that V is a finite G-module under π. For $x \in G'$, define*

$$T_x = \int_K \pi(kxk^{-1})\, dk.$$

Then $T_x \in \mathrm{End}^0 V$ and $x \mapsto T_x$ is a smooth mapping from G' to End ${}^0V$.

Let K_0 be an open and normal subgroup of K and V_0 the subspace of all vectors in V, which are left fixed by K_0. Then dim $V_0 < \infty$. By choosing K_0 sufficiently small, we may assume that V is spanned by elements of the form $\pi(x)\, v$ $(x \in G, v \in V_0)$.

Let M^+ denote the set of all $m \in M$ such that $\langle \alpha, H_M(m)\rangle \geqslant 0$ for every root α of (P, A) (see [3], § 7) for the definition of H_M). Then $G = KM^+ \cdot K$. Put $A^+ = A \cap M^+$. Since M/A is compact, we conclude that $M^+ \subset CA^+$ where C is a compact subset of M. Hence it is clear that we can choose an open compact subgroup P_0 of P such that $m^{-1}P_0 m \subset K_0$ for all $m \in M^+$.

Let α denote the characteristic function of $K \times P_0$ and put $f_y = f_{\alpha, y}$ $(y \in G')$ in the notation of lemma 1. Then $y \mapsto f_y$ is a smooth mapping from G' to $C_c^\infty(G)$ and

$$\int_{K \times P_0} F({}^k y \cdot p)\, dk dp = \int_G f_y(x) F(x)\, dx,$$

for all locally summable functions F on G. (Here dp is the normalised Haar measure on P_0.) From this we deduce immediately that

$$\int_{K \times P_0} \pi({}^k y \cdot p)\, dk dp = \int_G f_y(x)\pi(x)\, dx = \pi(f_y).$$

Let V_y be the smallest K-invariant subspace of V containing $\pi(f_y) V$. Since $f_y \in C_c^\infty(G)$, it is clear that $\dim V_y < \infty$.

Since K_0 is normal in K, V_0 is stable under $\pi(K)$. Therefore since $G = KM^+K$ V is spanned by $\pi(KM^+) V_0$. Fix $k \in K$, $m \in M^+$ and $v \in V_0$. Then

$$T_y \pi(km) v = \pi(k) T_y \pi(m) v \qquad (y \in G').$$

But $\quad \pi(f_y)\pi(m) v = T_y \int_{P_0} \pi(pm)\, v dp = T_y \pi(m) \int_{P_0} \pi(m^{-1}pm)\, v dp = T_y \pi(m) v$

since $m^{-1} P_0 m \subset K_0$. Therefore

$$T_y \pi(km) v = \pi(k)\pi(f_y)\pi(m) v \in V_y.$$

This shows that $T_y V \subset V_y$. Since T_y commutes with $\pi(k)$ $(k \in k)$, $\dim V_y < \infty$ and π is admissible, it is now clear that $T_y \in \mathrm{End}^0 V$.

Fix k, m, v as above. Since the mapping $y \mapsto f_y$ is smooth, it follows from the result obtained above that the mapping

$$y \mapsto T_y \pi(km) v = \pi(k)\pi(f_y)\pi(m) v \ (y \in G'),$$

from G' to V is also smooth. The second statement of the theorem is now obvious if we recall that V is spanned by $\pi(KM^+) V_0$.

Corollary. Let Θ denote the character of π. Then Θ coincides on G' with the locally constant function

$$x \mapsto \operatorname{tr} T_x \qquad (x \in G').$$

Put $f^0(x) = \int_K f(kxk^{-1})\, dk \qquad (x \in G),$

for $f \in C_c^\infty(G)$. Then

$$\Theta(f) = \Theta(f^0) = \operatorname{tr} \pi(f^0).$$

But $$\pi(f^0) = \int_G f(x)\, T_x\, dx$$

for $f \in C_c^\infty(G')$. Hence

$$\Theta(f) = \int_G f(x) \operatorname{tr} T_x\, dx \qquad (f \in C_c^\infty(G')).$$

5. Some applications

Let Γ be a Cartan subgroup of G. For $\gamma \in \Gamma' = \Gamma \cap G'$ and $f \in C_c^\infty(G)$, define

$$F_f(\gamma) = |D(\gamma)|_p^{\frac{1}{2}} \int_{G/A_\Gamma} f(x\gamma x^{-1})\, dx^*$$

where $D = D_G$, A_Γ is the split component of Γ and dx^* is an invariant measure on G/A_Γ.

Theorem 3. *Let K_0 be an open compact subgroup of G. Given $\gamma_0 \in \Gamma'$, we can choose a neighbourhood ω of γ_0 in Γ' such that F_f is constant on ω for every $f \in C_c(G/K_0)$.*

This result had been proved by Howe some years ago in the case char $\Omega = 0$.

Without loss of generality, we may assume that K_0 is a normal subgroup of K. Let us now use the notation of § 4. Then

$$\int_{K \times P_0} F({}^k\gamma \cdot p)\, dk dp = \int_G f_\gamma(x) F(x)\, dx,$$

for $\gamma \in \Gamma'$ and $F \in C_c^\infty(G)$. Fix $f \in C_c(G/K_0)$ and put

$$g(x) = \int_K f(kxk^{-1})\, dk \qquad (x \in G).$$

Since K_0 is normal in K, $g \in C_c(G//K_0)$. Fix $m \in M^+$ and let $F(x) = g(m^{-1}xm)$ $(x \in G)$. Then

$$\int_{K \times P_0} g(m^{-1} \cdot {}^k\gamma \cdot pm)\, dk dp = \int f_\gamma(x)\, g(m^{-1}xm)\, dx.$$

But $pm = m \cdot m^{-1}pm \in mK_0$. Hence

$$\int_K g(m^{-1} \cdot {}^k\gamma \cdot m)\, dk = \int f_\gamma(x)\, g(m^{-1}xm)\, dx \qquad (\gamma \in \Gamma').$$

Fix a neighbourhood Γ_0 of γ_0 in Γ' such that $f_\gamma = f_{\gamma_0}$ for $\gamma \in \Gamma_0$. This is possible by lemma 1. Then

$$\int_K g(m^{-1} \cdot {}^k\gamma \cdot m)\, dk = \int_K g(m^{-1} \cdot {}^k\gamma_0 \cdot m)\, dk \qquad (\gamma \in \Gamma_0).$$

By standard arguments, the proof of theorem 3 is reduced to the case when Γ is elliptic. Then $A_\Gamma = Z$ where Z is the maximal split torus lying in the centre of G. We know from the work of Bruhat and Tits that

$$K \backslash G/KZ \simeq M^+/{}^0MZ.$$

Therefore $F_f(\gamma) = |D(\gamma)|_{\mathfrak{p}}^{\frac{1}{2}} \int_{G/Z} f(x\gamma x^{-1})\, dx^*$

$$= |D(\gamma)|_{\mathfrak{p}}^{\frac{1}{2}} \sum_{m \in M^+/{}^0MZ} \mu(m) \int_K g(m^{-1} \cdot {}^k\gamma \cdot m)\, dk,$$

where $\mu(m)$ is the Haar measure of KmK. Let ω be a neighbourhood of γ_0 in Γ_0 such that $|D(\gamma)|_{\mathfrak{p}}$ is constant for $\gamma \in \omega$. Then

$$F_f(\gamma) = F_f(\gamma_0) \qquad (\gamma \in \omega).$$

Since ω is independent of $f \in C_c(G/K_0)$, the theorem is proved.

Theorem 3 makes it possible to prove Lemma 13 of ([3], § 16) without any restriction on char Ω.

I believe it is important, from the point of view of harmonic analysis, to obtain an analogue of theorem 5 of ([1], p. 32). We give below a result which may be regarded as a step in this direction.

Take $K_0 = K$ and define f_γ $(\gamma \in \Gamma')$ as in § 4. Then $f_\gamma \geq 0$. Put

$$\beta(\gamma) = \sup_x f_\gamma(x) \qquad (\gamma \in \Gamma')$$

and define Ξ as in ([3], § 14).

Theorem 4. *Let ω be a compact subset of Γ. Then we can choose a positive number c such that*

$$\int_K \Xi(m^{-1} \cdot {}^k\gamma \cdot m)\, dk \leq c\beta(\gamma)\, \Xi(m)^2,$$

for all $m \in M^+$ and $\gamma \in \omega' = \omega \cap \Gamma'$.

It is obvious that

$$\operatorname{Supp} f_\gamma \subset \bigcup_{k \in K} ({}^k\gamma \cdot P_0). \qquad (\gamma \in \Gamma').$$

Therefore $\operatorname{Supp} f_\gamma \subset {}^K\omega \cdot P_0 = C$ (say)

for $\gamma \in \omega'$. Since C is compact, we can choose a finite number of elements y_i $(1 \leq i \leq r)$ in G such that

$$\operatorname{Supp} f_\gamma \subset \bigcup_{1 \leq i \leq r} y_i K$$

for all $\gamma \in \omega'$.

Now fix $\gamma \in \omega'$, $m \in M^+$ and put

$$F(x) = \Xi(m^{-1} x m) \qquad (x \in G)$$

in the relation

$$\int_{K \times P_0} F({}^k\gamma \cdot p)\, dk\, dp = \int_G f_\gamma(x) F(x)\, dx.$$

Observe that

$$F({}^k\gamma \cdot p) = \Xi(m^{-1} \cdot {}^k\gamma \cdot pm) = \Xi(m^{-1} \cdot {}^k\gamma \cdot m) \qquad (p \in P_0)$$

since $m^{-1}\,pm \in K$. Therefore

$$\int_K \Xi(m^{-1}\cdot {}^k\gamma\cdot m)\,dk \leq \beta(\gamma)\sum_i \int_K \Xi(m^{-1}y_i km)\,dk$$

$$= \beta(\gamma)\sum_i \Xi(m^{-1}y_i)\,\Xi(m)$$

from the identity

$$\int_K \Xi(xky)\,dk = \Xi(x)\,\Xi(y) \qquad (x, y \in G).$$

We can choose a number $c_1 \geq 1$ such that

$$\Xi(xy_i) \leq c_1\,\Xi(x) \qquad (1 \leq i \leq r)$$

for all $x \in G$. Put $c = rc_1$ and observe that $\Xi(x^{-1}) = \Xi(x)$. Then we get

$$\int_K \Xi(m^{-1}\cdot {}^k\gamma\cdot m) \leq c\beta(\gamma)\,\Xi(m)^2$$

and this proves our assertion.

One would like to verify that

$$\sup_{\gamma\in\omega'} |D(\gamma)|_{\mathfrak{p}}^{\frac{1}{2}}\,\beta(\gamma) < \infty.$$

I believe this to be true but do not have a proof.

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