A FORMULA FOR SEMISIMPLE LIE GROUPS

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In order to save space we shall follow strictly the notation of an earlier paper.[1] Define $\mathfrak{k}_0$, $\mathfrak{p}_0$ as usual,[2] and suppose that $\mathfrak{h}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0 + \mathfrak{h}_0 \cap \mathfrak{p}_0$. Then we say that $\mathfrak{h}_0$ is fundamental if $\mathfrak{h}_0 \cap \mathfrak{k}_0$ is maximal Abelian in $\mathfrak{k}_0$. Any two fundamental Cartan subalgebras of $\mathfrak{g}_0$ are conjugate under G_0.

THEOREM 1. *Let $\mathfrak{h}_1$ be a connected component of $\mathfrak{h}_0'$. Then there exists a real number c such that*

$$\lim_{H \to 0} F_f(H; \partial(\pi)) = cf(0) \qquad (H \in \mathfrak{h}_1)$$

for every $f \in \mathcal{C}(\mathfrak{g}_0)$. Moreover, $c = 0$ if $\mathfrak{h}_0$ is not fundamental.

THEOREM 2. *Suppose that $\mathfrak{h}_0$ is fundamental and $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ are all the distinct connected components of $\mathfrak{h}_0'$. Let c_j be the real number of Theorem 1 corresponding to $\mathfrak{h}_j$ $(1 \leq j \leq r)$. Then $c_1 + c_2 + \ldots + c_r \neq 0$.*

The following result plays an essential role in the proofs of the above theorems. Let dH denote the Euclidean measure on $\mathfrak{h}_0$ and dk the Haar measure on the compact analytic subgroup K_0 of G_0 corresponding the $\mathfrak{k}_0$.

LEMMA 1. *Assume that $\mathfrak{h}_0 \subset \mathfrak{k}_0$, and for any $g \in \mathcal{C}(\mathfrak{h}_0)$ put*

$$\hat{g}(X) = \int_{K_0 \times \mathfrak{h}_0} \exp\,(iB(X, kH))\pi(H)^2 \sum_{s \in W} g(sH)\, dk\, dH \qquad (X \in \mathfrak{g}_0)$$

Then the integral

$$F_{\hat{g}}(H) = \pi(H) \int_{G^*} \hat{g}(x^*H)dx^*$$

is convergent for $H \in \mathfrak{h}_0'$, and there exists a constant $c \neq 0$ such that

$$\sum_{s \in W} \epsilon(s)\, F_{\hat{g}}(sH') = c \int_{\mathfrak{h}_0} \sum_{s \in W} \epsilon(s) \exp\,(iB(H', sH))\pi(H)g(H)\, dH$$

for all $H' \in \mathfrak{h}_0'$ and $g \in \mathcal{C}(\mathfrak{h}_0)$.

Let G be any connected Lie group whose Lie algebra is $\mathfrak{g}_0$, and let A be the Cartan subgroup of G corresponding to $\mathfrak{h}_0$.

LEMMA 2. *Let Ξ be the centralizer in G of an element $a_0 \in A$. Denote the natural mapping of G on $\bar{G} = G/\Xi$ by $x \to \bar{x}$ $(x \in G)$. Then we can find a neighborhood B of a_0 in A with the following property. Given any compact set ω in G, there exists a compact set $\bar{\Omega}$ in $\bar{G}$ satisfying the condition that if $xax^{-1} \in \omega$ for some $a \in B$ and $x \in G$, then $\bar{x} \in \bar{\Omega}$.*

By combining this with Theorem 2 of Paper 1, we can obtain certain results on G as follows. λ being an indeterminate, let $D(x)$ $(x \in G)$ denote the coefficient of λ^l in det $\{(\lambda + 1)I - \mathrm{Ad}(x)\}$, where l is the rank and I the identity mapping of $\mathfrak{g}$. Let A' be the set of those points $a \in A$ where $D(a) \neq 0$. We regard A' as an open submanifold of the Lie group A. Note that $G/A = G_0/A_0 = G^*$, and put $a^{x^*} = xax^{-1}$ $(a \in A, x \in G)$, where x^* denotes the coset xA in G^*. Let $C_c^\infty(G)$ be the set of all indefinitely differentiable functions on G which vanish outside a compact set. Then the integral

$$\phi_f(a) = |D(a)|^{1/2} \int_{G^*} f(a^{x^*})\, dx^*$$

converges for $f \in C_c^\infty(G)$ and $a \in A'$, and ϕ_f is of class C^∞ on A'. Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{H}$ the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{h})$. Then elements of $\mathfrak{B}$ and $\mathfrak{H}$ can be regarded as differential operators on G and A, respectively.[3] Let $\mathfrak{Z}$ denote the center of $\mathfrak{B}$ and γ the isomorphism of $\mathfrak{Z}$ into $\mathfrak{H}$ described in Paper 2 (p. 394). Then one can show that $\phi_{zf} = \gamma(z)\phi_f$ $(f \in C_c^\infty(G), z \in \mathfrak{Z})$. Let us call a subset of G *bounded* if its closure is compact. Consider the space $\mathfrak{C}_0(A')$ of all functions g on A' of class C^∞ satisfying the following two conditions: (1) g is zero (on A') outside some bounded set; (2) $\tau_v(g) = \sup_{a \in A'} |g(a; v)| < \infty$ for every $v \in \mathfrak{H}$. We topologize $C_c^\infty(G)$ in the usual way[4] and $\mathfrak{C}_0(A')$ by means of the seminorms τ_v $(v \in \mathfrak{H})$.

THEOREM 3. *$\phi_f \in \mathfrak{C}_0(A')$ for $f \in C_c^\infty(G)$ and $f \to \phi_f$ is a continuous mapping of $C_c^\infty(G)$ into $\mathfrak{C}_0(A')$. Moreover, for any compact set ω in G we can find a bounded subset B of A' with the following property. If the carrier of f is contained in ω, then $\phi_f = 0$ outside B.*

COROLLARY. *Let da denote the Haar measure on A. Then if g is a measurable function on A which is bounded on every compact set, the mapping T_g: $f \to \int_A g\phi_f da$ $(f \in C_c^\infty(G))$ is a distribution on G.*

The distributions of the form T_g, where g is a character of A, are closely related to the characters of G (see Paper 2).

$S(\mathfrak{h})$ is isomorphic to $\mathfrak{H}$ under a mapping which preserves every element of $\mathfrak{h}$, and so for any $q \in S(\mathfrak{h})$ we get a differential operator on A, which we shall denote by $\partial(q)$. Then Theorems 1 and 2 have the following analogues on G.

THEOREM 4. *Let A_1 be a connected component of A' whose closure contains 1. Then there exists a constant c such that*

$$\lim_{a \to 1} \phi_f(a; \partial(\pi)) = cf(1) \qquad (a \in A_1)$$

for all $f \in C_c^\infty(G)$. Moreover, $c = 0$ if $\mathfrak{h}_0$ is not fundamental.

THEOREM 5. *Now suppose that $\mathfrak{h}_0$ is fundamental and $A_1, \ldots, A_r$ are all the distinct components of A' whose closures contain 1. Let c_j be the constant of Theorem 4 corresponding to A_j $(1 \leq j \leq r)$. Then not every c_j can be zero.*

Theorem 5 gives a simple formula for $f(1)$ in terms of ϕ_f. In case G is compact, the proof of this formula is quite trivial. Moreover, I had proved it earlier[5] for (1) a complex semisimple group and (2) the 2×2 real unimodular group. More recently Gelfand and Graev[6] had extended it to the $n \times n$ real unimodular group.

[1] These PROCEEDINGS, 42, 252–253, 1956. This will be referred to as "Paper 1."

[2] See *Trans. Am. Math. Soc.*, 75, 187, 1953.

[3] See *Bull. Am. Math. Soc.*, **61**, 389–396, 1955. This will be referred to as "Paper 2."

[4] See L. Schwartz, *Théorie des distributions*, Vol. **1** (Paris: Hermann & Cie, 1950).

[5] See *Trans. Am. Math. Soc.*, **76**, 514, 1954, and these PROCEEDINGS, **38**, 339, 1952.

[6] *Doklady Akad. Nauk S.S.S.R.*, N.S., **92**, 461–464, 1953.