IN Variant DIFFERENTIAL OPERATORS ON A SEMISIMPLE LIE ALGEBRA

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Let $R$ and $C$ be the fields of real and complex numbers, respectively, and $E_0$ a vector space over $R$ of finite dimension. Then, $E$ being the complexification of $E_0$, we consider the symmetric algebra $S(E)$ and the algebra $Q(E)$ of polynomial functions on $E$. We regard $E_0$ as a differentiable manifold and, for any differential operator $D$ and indefinitely differentiable function $f$, denote by $f(X; D)$ the value of $Df$ at $X \in E_0$. Corresponding to any $X$ in $E_0$ we define the differential operator $\partial(X)$ by

$$f(Y; \partial(X)) = \left\{ \frac{d}{dt} f(Y + tX) \right\}_{t=0} (Y \in E_0, t \in R).$$

Then $\partial$ can be extended uniquely to an isomorphism of $S(E)$ into the algebra of differential operators on $E_0$. Let $U$ be an open subset of $E_0$. By $C(U)$ we mean the space of all functions $f$ on $U$ of class $C^\infty$ such that $\tau(q, \partial(p); f) = \sup_{x \in U} |q(X)f(X; \partial(p))| < \infty$ for all $q \in Q(E)$ and $p \in S(E)$. We define a topology in $C(U)$ by means of the collection of these seminorms $\tau(q, \partial(p)); (q \in Q(E), p \in S(E)).$

Let $g_0$ be a semisimple Lie algebra over $R$ and $h_0$ a Cartan subalgebra of $g_0$. Complexify $g_0$, $h_0$ to $g$ and $h$, respectively. We can identify $S(g)$ and $Q(g)$ by means of the fundamental bilinear form $B(X, Y) = sp(ad X ad Y)(X, Y \in g)$ on $g$. Let $G_0$ be the (connected) adjoint group of $g_0$. A function $f$ on $g_0$ will be called invariant if $f(xX) = f(X)$ for all $x \in G_0$ and $X \in g_0$. Let $I(g)$ be the subalgebra of $S(g)$ consisting of invariant polynomial functions. Similarly, let $I(h)$ denote the algebra of those elements in $S(h)$ which are invariant under the Weyl group $W$ (of $g$ with respect to $h$). For any $p \in I(g)$, let $\overline{p}$ denote the restriction of the polynomial function $p$ on $h$. Then $p \rightarrow \overline{p}$ is an isomorphism of $I(g)$ onto $I(h)$. Let $P$ denote the set of all positive roots (of $g$ with respect to $h$) under some fixed order. Put $\pi = \Pi_{\alpha \in P}$. Then $\pi \in S(h)$. Notice that if $q \in S(h), \partial(q)$ is a differential operator on $h_0$.

**Theorem** 1. Suppose that $f$ is an invariant function on $g_0$ of class $C^\infty$. Then, if $p \in I(g)$,

$$\pi(H) f(H; \partial(p)) = g(H; \partial(\overline{p})), \quad (H \in h_0),$$

where $g$ is the function on $h_0$ given by $g(H) = \pi(H) f(H)$.

Let $dx$ denote the Haar measure of $G_0$.

**Corollary.** Suppose that $G_0$ is compact. Then

$$\pi(H) \pi(H') \int_{G_0} \exp B(xH, H') dx = c \sum_{s \in W} \chi(s) \exp B(sH, H')$$

for all $H, H'$ in $h$. Here $c$ is a constant which is easily determined by the condition $\int_{G_0} dx = 1$.  

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Let \( A_0 \) be the Cartan subgroup of \( G_0 \) corresponding to \( \mathfrak{h}_0 \), and let \( dx^* \) denote the invariant measure on the factor space \( G^* = G_0/A_0 \). Define \( x^*H = xH \) \((H \in \mathfrak{h})\), where \( x \) is any element in the coset \( x^* \in G^* \). Let \( \mathfrak{b}_0' \) be the set of those elements \( H \in \mathfrak{h}_0 \), where \( \pi(H) \neq 0 \). Then, if \( f \in C(\mathfrak{h}_0) \), the integral

\[
F_f(H) = \pi(H) \int_{G^*} f(x^*H) \, dx^*
\]

is convergent for \( H \in \mathfrak{b}_0' \), and \( F_f \) is of class \( C^\infty \) on \( \mathfrak{b}_0' \). Moreover, it follows from Theorem 1 that \( F_{\phi(p)f} = \partial(\bar{p}) \, F_f \) \((p \in I(\mathfrak{g}))\). This relation, in its turn, implies the following result.

**Theorem 2.** The mapping \( f \mapsto F_f \) is a continuous mapping of \( C(\mathfrak{h}_0) \) into \( C(\mathfrak{b}_0') \).

For any \( f \in C(\mathfrak{h}_0) \), let \( \hat{f} \) denote the Fourier transform of \( f \), so that

\[
\hat{f}(X) = \int_{\mathfrak{g}_0} \exp(iB(X, Y)) \, f(Y) \, dY \quad (X \in \mathfrak{g}_0),
\]

where \( dY \) is the (suitably normalized) Euclidean measure on \( \mathfrak{g}_0 \). Then it is possible to obtain interesting relations between \( F_f \) and \( F_{\hat{f}} \). For example, if \( G_0 \) is either compact or complex,

\[
F_{\hat{f}}(H) = \int_{\mathfrak{b}_0} \exp(iB(H, H')) \, F_f(H') \, dH' \quad (H \in \mathfrak{b}_0')
\]

where \( dH' \) is the (suitably normalized) Euclidean measure on \( \mathfrak{b}_0' \). Similar but more complicated relations hold in other cases as well, under suitable restrictions on \( f \).

If \( G_0 \) is a complex semisimple group, the above relation between \( F_f \) and \( F_{\hat{f}} \) can be obtained directly without much difficulty. This, in fact, is the starting point of the proof of Theorem 1.

2. This theorem should be compared with Lemma 2 of *Bull. Am. Math. Soc.*, 61, 394, 1955.
3. \( \varepsilon(s) = 1 \) or \(-1\), according as \( s \) takes an even or odd number of positive roots into negative roots.