INVARIANT DIFFERENTIAL OPERATORS ON A SEMISIMPLE LIE ALGEBRA

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Let R and C be the fields of real and complex numbers, respectively, and E_0 a vector space over R of finite dimension. Then, E being the complexification of E_0, we consider the symmetric algebra $S(E)$ and the algebra $Q(E)$ of polynomial functions on E. We regard E_0 as a differentiable manifold and, for any differential operator D and indefinitely differentiable function f, denote by $f(X; D)$ the value of Df at $X \epsilon E_0$. Corresponding to any X in E_0 we define the differential operator $\partial(X)$ by

$$f(Y;\ \partial(X)) = \left\{\frac{d}{dt} f(Y + tX)\right\}_{t=0} \quad (Y \epsilon E_0,\ t \epsilon R).$$

Then ∂ can be extended uniquely to an isomorphism of $S(E)$ into the algebra of differential operators on E_0. Let U be an open subset of E_0. By $\mathcal{C}(U)$ we mean the space of all functions f on U of class C^∞ such that $\tau(q,\ \partial(p);\ f) = \sup_{X \epsilon U} |q(X) f(X;\ \partial(p))| < \infty$ for all $q \epsilon Q(E)$ and $p \epsilon S(E)$. We define a topology in $\mathcal{C}(U)$ by means of the collection of these seminorms $\tau(q,\ \partial(p))$ $(q \epsilon Q(E), p \epsilon S(E))$.

Let $\mathfrak{g}_0$ be a semisimple Lie algebra over R and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$. Complexify $\mathfrak{g}_0$, $\mathfrak{h}_0$ to $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We can identify[1] $S(\mathfrak{g})$ and $Q(\mathfrak{g})$ by means of the fundamental bilinear form $B(X, Y) = \mathrm{sp}(\mathrm{ad}\ X\ \mathrm{ad}\ Y)(X, Y \epsilon \mathfrak{g})$ on $\mathfrak{g}$. Let G_0 be the (connected) adjoint group of $\mathfrak{g}_0$. A function f on $\mathfrak{g}_0$ will be called invariant if $f(xX) = f(X)$ for all $x \epsilon G_0$ and $X \epsilon \mathfrak{g}_0$. Let $I(\mathfrak{g})$ be the subalgebra of $S(\mathfrak{g})$ consisting of invariant polynomial functions. Similarly, let $I(\mathfrak{h})$ denote the algebra of those elements in $S(\mathfrak{h})$ which are invariant under the Weyl group W (of $\mathfrak{g}$ with respect to $\mathfrak{h}$). For any $p \epsilon I(\mathfrak{g})$, let $\bar{p}$ denote the restriction of the polynomial function p on $\mathfrak{h}$. Then $p \rightarrow \bar{p}$ is an isomorphism of $I(\mathfrak{g})$ onto $I(\mathfrak{h})$. Let P denote the set of all positive roots (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) under some fixed order. Put $\pi = \prod_{\alpha \epsilon P} \alpha$. Then $\pi \epsilon S(\mathfrak{h})$. Notice that if $q \epsilon S(\mathfrak{h})$, $\partial(q)$ is a differential operator on $\mathfrak{h}_0$.

THEOREM[2] 1. *Suppose that f is an invariant function on $\mathfrak{g}_0$ of class C^∞. Then, if $p \epsilon I(\mathfrak{g})$,*

$$\pi(H) f(H;\ \partial(p) = g(H;\ \partial(\bar{p})) \qquad (H \epsilon \mathfrak{h}_0),$$

where g is the function on $\mathfrak{h}_0$ given by $g(H) = \pi(H) f(H)$.

Let dx denote the Haar measure of G_0.

COROLLARY. *Suppose that G_0 is compact. Then*[3]

$$\pi(H)\ \pi(H') \int_{G_0} \exp B(xH, H')\ dx = c \sum_{s \epsilon W} \varepsilon(s) \exp B\ (sH, H')$$

for all H, H' in $\mathfrak{h}$. Here c is a constant which is easily determined by the condition $\int_{G_0} dx = 1$.

Let A_0 be the Cartan subgroup of G_0 corresponding to $\mathfrak{h}_0$, and let dx^* denote the invariant measure on the factor space $G^* = G_0/A_0$. Define $x^*H = xH$ $(H \in \mathfrak{h})$, where x is any element in the coset $x^* \in G^*$. Let $\mathfrak{h}_0'$ be the set of those elements $H \in \mathfrak{h}_0$, where $\pi(H) \neq 0$. Then, if $f \in \mathcal{C}(\mathfrak{g}_0)$, the integral

$$F_f(H) = \pi(H) \int_{G^*} f(x^*H)\, dx^*$$

is convergent for $H \in \mathfrak{h}_0'$, and F_f is of class C^∞ on $\mathfrak{h}_0'$. Moreover, it follows from Theorem 1 that $F_{\partial(p)f} = \partial(\bar{p})\, F_f$ $(p \in I(\mathfrak{g}))$. This relation, in its turn, implies the following result.

THEOREM 2. *The mapping $f \to F_f$ is a continuous mapping of $\mathcal{C}(\mathfrak{g}_0)$ into $\mathcal{C}(\mathfrak{h}_0')$.*

For any $f \in \mathcal{C}(\mathfrak{g}_0)$, let $\tilde{f}$ denote the Fourier transform of f, so that

$$\tilde{f}(X) = \int_{\mathfrak{g}_0} \exp\,(iB(X, Y))\, f(Y)\, dY \qquad (X \in \mathfrak{g}_0),$$

where dY is the (suitably normalized) Euclidean measure on $\mathfrak{g}_0$. Then it is possible to obtain interesting relations between F_f and $F_{\tilde{f}}$. For example, if G_0 is either compact or complex,

$$F_{\tilde{f}}(H) = \int_{\mathfrak{h}_0} \exp\,(iB(H, H'))\, F_f(H')\, dH' \qquad (H \in \mathfrak{h}_0')$$

where dH' is the (suitably normalized) Euclidean measure on $\mathfrak{h}_0$. Similar but more complicated relations hold in other cases as well, under suitable restrictions on f.

If G_0 is a complex semisimple group, the above relation between F_f and $F_{\tilde{f}}$ can be obtained directly without much difficulty. This, in fact, is the starting point of the proof of Theorem 1.

[1] See *Trans. Am. Math. Soc.*, **75**, 194, 1953.

[2] This theorem should be compared with Lemma 2 of *Bull. Am. Math. Soc.*, **61**, 394, 1955.

[3] $\varepsilon(s) = 1$ or -1, according as s takes an even or odd number of positive roots into negative roots.