INTEGRABLE AND SQUARE-INTEGRABLE REPRESENTATIONS OF A SEMISIMPLE LIE GROUP

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Let G be a connected semisimple Lie group. We shall suppose for simplicity that the center of G is finite. Let π be an irreducible unitary representation of G on a Hilbert space $\mathfrak{H}$. We say that π is integrable (square-integrable) if there exists an element $\psi \neq 0$ in $\mathfrak{H}$ such that the function $(\psi, \pi(x)\psi)$ $(x \in G)$ is integrable (square-integrable) on G, with respect to the Haar measure. Assuming that the Haar measure dx has been normalized in some way once for all and that π is square-integrable, we denote by d_π the positive constant given by the relation[1]

$$\int_G |(\psi, \pi(x)\psi)|^2 \, dx = \frac{1}{d_\pi},$$

where ψ is any unit vector in $\mathfrak{H}$. Let $C_c^\infty(G)$ denote the set of all complex-valued functions on G which are everywhere indefinitely differentiable and which vanish outside a compact set. Then the following result is an easy consequence of the Schur orthogonality[1] relations.

THEOREM 1. *Let π be an irreducible unitary representation of G on $\mathfrak{H}$ which is square-integrable, and let T_π denote the character*[2] *of π. Then, if $f \in C_c^\infty(G)$,*

$$T_\pi(f) = d_\pi \int_G dx \left(\int_G f(xyx^{-1}) \, (\phi, \pi(y)\phi) \, dy\right),$$

where ϕ is any unit vector in $\mathfrak{H}$.

Now suppose G has a Cartan subgroup A which is compact. We extend A to a maximal compact subgroup K. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ its complexification. We shall assume for simplicity that G has a finite-dimensional faithful representation and therefore there exists a complex analytic group G_c with the Lie algebra $\mathfrak{g}$ such that G is the (real) analytic subgroup of G_c corresponding to $\mathfrak{g}_0$. Let $\mathfrak{h}_0$ and $\mathfrak{k}_0$ be the subalgebras of $\mathfrak{g}_0$ which correspond to A and K, respectively. We define $\mathfrak{p}_0$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{p}$ as in a previous note[3] and introduce a lexicographic order among roots (of $\mathfrak{g}$ with respect to $\mathfrak{h}$). For any root α we define X_α, $X_{-\alpha}$ and $H_\alpha = [X_\alpha, X_{-\alpha}]$ as before,[3] so that $\alpha(H_\alpha) = 2$. Put $\mathfrak{n}_+ = \sum_{\alpha > 0} CX_\alpha$, $\mathfrak{n}^- = \sum_{\alpha > 0} CX_{-\alpha}$, where C is the field of complex numbers and α runs over all positive roots. Then $\mathfrak{n}_+$, $\mathfrak{n}_-$ are subalgebras of $\mathfrak{g}$. Let A_c, N_c^+, N_c^- be the complex analytic subgroups of G_c corresponding to $\mathfrak{h}$, $\mathfrak{n}_+$, $\mathfrak{n}_-$, respectively. Then $G_c^0 = N_c^- A_c G$ is an open submanifold of G_c. If ξ is a holomorphic character of A_c, we can choose a (unique) linear function Λ on $\mathfrak{h}$ such that $\xi(\exp H) = e^{\Lambda(H)}$ ($H \in \mathfrak{h}$). We denote by $\mathfrak{H}_\Lambda$ the space of all holomorphic functions ϕ on G_c^0 such that

(i) $\phi(naw) = \xi(a)\phi(w) \qquad (n \in N_c^-,\ a \in A_c,\ w \in G_c^0)$,

(ii) $\|\phi\|^2 = \int_G |\phi(x)|^2 \, dx < \infty$.

Then $\mathfrak{H}_\Lambda$ is a Hilbert space under the norm $\|\cdot\|$, and we get a representation π_Λ of G on $\mathfrak{H}_\Lambda$ if we put $(\pi_\Lambda(x)\phi)(y) = \phi(yx)$ ($\phi \in \mathfrak{H}_\Lambda$; $x, y \in G$). If $\mathfrak{H}_\Lambda \neq \{0\}$, π_Λ is an irreducible unitary representation[4] which is square-integrable. Let 2ρ denote the sum of all positive roots. In case G is simple and not compact, the following four conditions are both necessary and sufficient in order that $\mathfrak{H}_\Lambda \neq \{0\}$:

1. The first Betti number of G is 1.
2. Every noncompact[3] positive root is totally positive.[3]
3. $\Lambda(H_\alpha)$ is a nonnegative integer for every positive compact[3] root α.
4. $\Lambda(H_\beta) + \rho(H_\beta)$ is a negative integer for every positive noncompact[3] root β.

Let $2\rho_+$ denote the sum of all positive noncompact roots. Then, in the presence of the first three conditions, the following condition is sufficient to insure the integrability of π:

4'. $\Lambda(H_\beta) + \rho(H_\beta) < -2\rho_+(H_\beta) + 1$ for every positive noncompact root β.

From now on we shall suppose that G is simple and not compact and that conditions 1 and 2 are fulfilled. We shall also assume that G_c is simply connected. Then it is possible[3] to choose a fundamental system $(\alpha_0, \alpha_1, \ldots, \alpha_l)$ of positive roots such that α_0 is noncompact, while $\alpha_1, \ldots, \alpha_l$ are all compact. Let Λ_i denote the linear function on $\mathfrak{h}$ given by $\Lambda_i(H_{\alpha_j}) = \delta_{ij}$ $(0 \leq i, j \leq l)$. Since G_c is simply connected, there exists a holomorphic character ξ_i of A_c such that $\xi_i(\exp H) = e^{\Lambda_i(H)}$ $(H \in \mathfrak{h}, 0 \leq i \leq l)$. Moreover, from the theory of finite-dimensional representations of G_c one can deduce the existence of a unique holomorphic function g_i on G_c

such that $g_i(nan') = \xi_i(a)$ $(n \in N_c^-, a \in A_c, n' \in N_c^+)$. Put $g = g_0^{m_0}g_1^{m_1}\ldots g_l^{m_l}$, where $m_0, \ldots, m_l$ are nonnegative integers, and let $g_y(x) = g(xy)$ $(x, y \in G_c)$. Then, if V is the vector space spanned over C by the functions g_y $(y \in G_c)$ and π is the representation of G_c on V given by $(\pi(y)f)(x) = f(xy)$ $(x, y \in G_c, f \in V)$, π is a finite-dimensional irreducible representation of G_c. The corresponding representation of $\mathfrak{g}$ is complex-linear, and its highest weight is $m_0\Lambda_0 + \ldots + m_l\Lambda_l$. On the other hand, suppose that Λ is a linear function on $\mathfrak{h}$ satisfying conditions 3 and 4. Put $\lambda_i = \Lambda(H_{\alpha_i})$ $(0 \leq i \leq l)$. Then $\lambda_0, \ldots, \lambda_l$ are all integers, and $\Lambda = \lambda_0\Lambda_1 + \ldots + \lambda_l\Lambda_l$. Moreover, $\lambda_0 < 0$, while $\lambda_1, \ldots, \lambda_l \geq 0$. Put $g_\Lambda = g_0^{\lambda_0}g_1^{\lambda_1}\ldots g_l^{\lambda_l}$. Then g_Λ is a meromorphic function on G_c. However, it can be shown that g_0 is never zero on G_c^0, and so g_Λ is holomorphic on G_c^0. Also, one can prove that $g_\Lambda \in \mathfrak{H}_\Lambda$, and therefore $\mathfrak{H}_\Lambda$ is the closure of the space spanned by the right translates of g_Λ under G. Thus the analogy with the finite-dimensional case mentioned above is rather close.

Let F be the set of all linear functions Λ on $\mathfrak{h}$ which satisfy conditions 3 and 4. For every $\Lambda \in F$ we have defined a square-integrable representation π_Λ above. Put $d_\Lambda = d_{\pi_\Lambda}$. Then, if m is the number of totally positive roots, we have the following result:

THEOREM 2. *It is possible to normalize the Haar measure of G in such a way that*

$$d_\Lambda = (-1)^m \prod_{\alpha > 0} \left\{\frac{\Lambda(H_\alpha) + \rho(H_\alpha)}{\rho(H_\alpha)}\right\}$$

for every $\Lambda \in F$.

The analogy with Weyl's formula[5] for the degree of an irreducible finite-dimensional representation in terms of its highest weight is obvious.

Let $\mathfrak{E}$ be the set of all equivalence classes of irreducible unitary representations of G. Let $\mathfrak{E}_0$ denote the subset of $\mathfrak{E}$ consisting of those classes ω which correspond to square-integrable representations. If $\omega \in \mathfrak{E}_0$ and $\pi \in \omega$, we put $d_\omega = d_\pi$. Also, define

$$N_\omega(f) = \left\|\int f(x)\pi(x)\,dx\right\|^2 \qquad (f \in C_c^\infty(G), \omega \in \mathfrak{E}),$$

where π is any representation in ω and $\|A\|$ denotes the Hilbert-Schmidt norm[2] of an operator A. It follows from the work of von Neumann[6] and Mautner[7] and a previous result of mine[8] that there exists a *unique* positive measure μ on $\mathfrak{E}$ such that

$$\int_G |f(x)|^2\,dx = \int_{\mathfrak{E}} N\omega(f)\,d\mu$$

for all $f \in C_c^\infty(G)$. On the other hand, one can prove the following result:

THEOREM 3. *The μ-measure of a single point ω in $\mathfrak{E}_0$ is exactly d_ω.*

For any $\Lambda \in F$ let ω_Λ denote the equivalence class of π_Λ. Then $\Lambda \rightarrow \omega_\Lambda$ is a one-one mapping of F into $\mathfrak{E}_0$. We denote by $\mathfrak{E}_F$ the image of F under this mapping. It is obvious from Theorems 2 and 3 that we now have an explicit formula for the restriction of the measure μ on $\mathfrak{E}_F$.

[1] See these PROCEEDINGS, **40,** 1076, 1954, Theorem 2; and R. Godement, *Compt. rend. Acad. sci.* (Paris), **225,** 657–659, 1947.

[2] See these PROCEEDINGS, **37,** 366–369, 1951.

[3] See *ibid.*, **40**, 1078, 1954, hereafter cited as "RVI."

[4] See RVI. The last statement of RVI is not correct. In order to rectify it, we have to replace π_ξ there by the definition of π_Λ given above in the present note.

[5] H. Weyl, *Math. Z.*, **24**, 328–395, 1925.

[6] J. von Neumann, *Ann. Math.*, **50**, 401–485, 1949.

[7] F. I. Mautner, *Ann. Math.*, **52**, 528–555, 1950.

[8] *Trans. Am. Math. Soc.*, **75**, 230, 1953, Theorem 7.