INTEGRABLE AND SQUARE-INTEGRABLE REPRESENTATIONS OF A SEMISIMPLE LIE GROUP

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Let $G$ be a connected semisimple Lie group. We shall suppose for simplicity that the center of $G$ is finite. Let $\pi$ be an irreducible unitary representation of $G$ on a Hilbert space $\mathcal{H}$. We say that $\pi$ is integrable (square-integrable) if there exists an element $\psi \neq 0$ in $\mathcal{H}$ such that the function $\langle \psi, \pi(x)\psi \rangle$ is integrable (square-integrable) on $G$, with respect to the Haar measure. Assuming that the Haar measure $dx$ has been normalized in some way once for all and that $\pi$ is square-integrable, we denote by $d_\pi^*$ the positive constant given by the relation

$$\int_G |\langle \psi, \pi(x)\psi \rangle|^2 \, dx = \frac{1}{d_\pi^*},$$

where $\psi$ is any unit vector in $\mathcal{H}$. Let $C_c^\infty(G)$ denote the set of all complex-valued functions on $G$ which are everywhere indefinitely differentiable and which vanish outside a compact set. Then the following result is an easy consequence of the Schur orthogonality relations.
THEOREM 1. Let \( \pi \) be an irreducible unitary representation of \( G \) on \( \mathcal{S} \) which is square-integrable, and let \( T_\pi \) denote the character of \( \pi \). Then, if \( f \in C_c^\infty(G) \),

\[
T_\pi(f) = \int_G f(x) \, d\phi,
\]

where \( \phi \) is any unit vector in \( \mathcal{S} \).

Now suppose \( G \) has a Cartan subgroup \( A \) which is compact. We extend \( A \) to a maximal compact subgroup \( K \). Let \( g_0 \) be the Lie algebra of \( G \) and \( g \) its complexification. We shall assume for simplicity that \( G \) has a finite-dimensional faithful representation and therefore there exists a complex analytic group \( G_c \) with the Lie algebra \( g \) such that \( G \) is the (real) analytic subgroup of \( G_c \) corresponding to \( g_0 \).

Let \( h_0 \) and \( f_0 \) be the subalgebras of \( g_0 \) which correspond to \( A \) and \( K \), respectively. We define \( p_0, \overline{p}, \overline{f} \) as in a previous note and introduce a lexicographic order among roots of \( g \) with respect to \( h_0 \). For any root \( \alpha \) define \( X_\alpha, X_{-\alpha} \) and \( H_\alpha = [X_\alpha, X_{-\alpha}] \) as before, so that \( \alpha(H_\alpha) = 2 \). Put

\[
u_+ = \sum_{\alpha > 0} CX_\alpha, \quad \nu_- = \sum_{\alpha > 0} CX_{-\alpha},
\]

where \( C \) is the field of complex numbers and \( \alpha \) runs over all positive roots. Then \( \nu_+, \nu_- \) are subalgebras of \( g \). Let \( A_\alpha, N^+, N^- \) be the complex analytic subgroups of \( G_c \) corresponding to \( h_0, \nu_+, \nu_- \), respectively. Then \( G_c^0 = N^- A_\alpha G \) is an open submanifold of \( G_c \).

If \( \xi \) is a holomorphic character of \( A_\alpha \), we can choose a (unique) linear function \( \Lambda \) on \( h_0 \) such that \( \xi(\exp H) = e^{\Lambda(H)} \) \((H \in h_0)\). We denote by \( \mathcal{S}_\Lambda \) the space of all holomorphic functions \( \phi \) on \( G_c^0 \) such that

\[
\begin{align*}
(\phi(\text{naw}) &= \xi(a)\phi(w) & (n \in N^-, a \in A_\alpha, w \in G_c^0), \\
\|\phi\|^2 &= \int_G |\phi(x)|^2 \, dx < \infty.
\end{align*}
\]

Then \( \mathcal{S}_\Lambda \) is a Hilbert space under the norm \( || \cdot || \), and we get a representation \( \pi_\Lambda \) of \( G \) on \( \mathcal{S}_\Lambda \) if we put \( (\pi_\Lambda(x)\phi)(y) = \phi(xy) \) \((\phi \in \mathcal{S}_\Lambda; x, y \in G)\). If \( \mathcal{S}_\Lambda \not\cong \{0\} \), \( \pi_\Lambda \) is an irreducible unitary representation which is square-integrable. Let \( 2\rho \) denote the sum of all positive roots. In case \( G \) is simple and not compact, the following four conditions are both necessary and sufficient in order that \( \mathcal{S}_\Lambda \not\cong \{0\} \):

1. The first Betti number of \( G \) is 1.
2. Every noncompact positive root is totally positive.
3. \( \Lambda(H_\alpha) \) is a nonnegative integer for every positive compact root \( \alpha \).
4. \( \Lambda(H_\beta) + \rho(H_\beta) \) is a negative integer for every positive noncompact root \( \beta \).

Let \( 2\rho_+ \) denote the sum of all positive noncompact roots. Then, in the presence of the first three conditions, the following condition is sufficient to insure the integrability of \( \pi \):

\[
\Lambda(H_\beta) + \rho(H_\beta) < -2\rho_+ \Lambda(H_\beta) + 1 \quad \text{for every positive noncompact root } \beta.
\]

From now on we shall suppose that \( G \) is simple and not compact and that conditions 1 and 2 are fulfilled. We shall also assume that \( G_c \) is simply connected. Then it is possible to choose a fundamental system \((\alpha_0, \alpha_1, \ldots, \alpha_l)\) of positive roots such that \( \alpha_0 \) is noncompact, while \( \alpha_1, \ldots, \alpha_l \) are all compact. Let \( A_i \) denote the linear function on \( h \) given by \( A_i(H_\alpha) = \delta_{ij} (0 \leq i, j \leq l) \). Since \( G_c \) is simply connected, there exists a holomorphic character \( \xi_i \) of \( A_i \) such that \( \xi_i(\exp H) = e^{\Lambda_i(H)} \) \((H \in h, 0 \leq i \leq l)\). Moreover, from the theory of finite-dimensional representations of \( G_c \) one can deduce the existence of a unique holomorphic function \( g_i \) on \( G_c \)
such that $g_i(n\alpha') = \xi_i(a)$ $(n \in N_c, a \in A_c, \alpha' \in N_c^+)$. Put $g = g_{\delta_0}^{m_0} g_{\delta_1}^{m_1} \ldots g_{\delta_l}^{m_l}$, where $m_0, \ldots, m_l$ are nonnegative integers, and let $g_\delta(x) = g(xy) (x, y \in G_c)$. Then, if $V$ is the vector space spanned over $C$ by the functions $g_\delta (y \in G_c)$ and $\pi$ is the representation of $G_c$ on $V$ given by $(\pi(y)f)(x) = f(xy) (x, y \in G, f \in V)$, $\pi$ is a finite-dimensional irreducible representation of $G_c$. The corresponding representation of $g$ is complex-linear, and its highest weight is $m_0 \Lambda_0 + \ldots + m_l \Lambda_l$. On the other hand, suppose that $\Lambda$ is a linear function on $\mathfrak{h}$ satisfying conditions 3 and 4. Put $\lambda_i = \Lambda(H_{\alpha_i}) (0 \leq i \leq l)$. Then $\lambda_0, \ldots, \lambda_l$ are all integers, and $\Lambda = \lambda_0 \Lambda_1 + \ldots + \lambda_l \Lambda_l$. Moreover, $\lambda_0 < 0$, while $\lambda_1, \ldots, \lambda_l \geq 0$. Put $g_\Lambda = g_0^{\lambda_0} g_1^{\lambda_1} \ldots g_l^{\lambda_l}$. Then $g_\Lambda$ is a meromorphic function on $G_c$. However, it can be shown that $g_\Lambda$ is never zero on $G_c^0$, and so $g_\Lambda$ is holomorphic on $G_c^0$. Also, one can prove that $g_\Lambda \in \mathcal{S}_\Lambda$, and therefore $\mathcal{S}_\Lambda$ is the closure of the space spanned by the right translates of $g_\Lambda$ under $G$. Thus the analogy with the finite-dimensional case mentioned above is rather close.

Let $F$ be the set of all linear functions $\Lambda$ on $\mathfrak{h}$ which satisfy conditions 3 and 4. For every $\Lambda \in F$ we have defined a square-integrable representation $\pi_\Lambda$ above. Put $d_\Lambda = d_{\pi_\Lambda}$. Then, if $m$ is the number of totally positive roots, we have the following result:

**Theorem 2.** It is possible to normalize the Haar measure of $G$ in such a way that

$$d_\Lambda = (-1)^m \prod_{\alpha > 0} \left( \frac{\Lambda(H_{\alpha}) + \rho(H_{\alpha})}{\rho(H_{\alpha})} \right)$$

for every $\Lambda \in F$.

The analogy with Weyl's formula for the degree of an irreducible finite-dimensional representation in terms of its highest weight is obvious.

Let $\mathcal{E}$ be the set of all equivalence classes of irreducible unitary representations of $G$. Let $\mathcal{E}_0$ denote the subset of $\mathcal{E}$ consisting of those classes $\omega$ which correspond to square-integrable representations. If $\omega \in \mathcal{E}_0$ and $\pi \in \omega$, we put $d_\omega = d_\pi$. Also, define

$$N_\omega(f) = \| \int f(x) \pi(x) \, dx \|^2$$

for $f \in C_c^\infty(G)$, where $\pi$ is any representation in $\omega$ and $\|A\|$ denotes the Hilbert-Schmidt norm of an operator $A$. It follows from the work of von Neumann and Mautner that there exists a unique positive measure $\mu$ on $\mathcal{E}$ such that

$$\int_{\mathcal{E}} |f(x)|^2 \, dx = \int_{\mathcal{E}} N_\omega(f) \, d \mu$$

for all $f \in C_c^\infty(G)$. On the other hand, one can prove the following result:

**Theorem 3.** The $\mu$-measure of a single point $\omega$ in $\mathcal{E}_0$ is exactly $d_\omega$.

For any $\Lambda \in F$ let $\omega_\Lambda$ denote the equivalence class of $\pi_\Lambda$. Then $\Lambda \rightarrow \omega_\Lambda$ is a one-one mapping of $F$ into $\mathcal{E}_0$. We denote by $\mathcal{E}_F$ the image of $F$ under this mapping. It is obvious from Theorems 2 and 3 that we now have an explicit formula for the restriction of the measure $\mu$ on $\mathcal{E}_F$.

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2 See these Proceedings, 37, 366--369, 1951.
5 See *ibid.*, 40, 1078, 1954, hereafter cited as "RVI."
6 See RVI. The last statement of RVI is not correct. In order to rectify it, we have to replace $\pi_1$ there by the definition of $\pi_4$ given above in the present note.