REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. VI

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We keep to the notation of the preceding note.[1] Since $\mathfrak{m}_0$ is reductive, there will be no essential loss of generality from the point of view of irreducible unitary representations of M_0 if we assume that $\mathfrak{m}_0$ is semisimple. Then $\mathfrak{m}_0 \cap \mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{m}_0$, and $\mathfrak{h}_{\mathfrak{k}_0}$ is a Cartan subalgebra of $\mathfrak{m}_0$. Moreover, $\mathfrak{m}_0 = \mathfrak{m}_0 \cap \mathfrak{k}_0 + \mathfrak{m}_0 \cap \mathfrak{p}_0$. Our problem of constructing irreducible unitary representations of M_0 is therefore the same as that for G, under the additional assumption that a maximal abelian subalgebra of $\mathfrak{k}_0$ is also maximal abelian in $\mathfrak{g}_0$. Hence we shall now assume that $\mathfrak{h}_0 \subset \mathfrak{k}_0$. Let C be the field of complex numbers and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. We denote by $\mathfrak{h}$, $\mathfrak{k}$, and $\mathfrak{p}$ the subspaces of $\mathfrak{g}$ spanned over C by $\mathfrak{h}_0$, $\mathfrak{k}_0$, and $\mathfrak{p}_0$, respectively. For any root α of $\mathfrak{g}$ (with respect to $\mathfrak{h}$), choose an element $X_\alpha \neq 0$ in $\mathfrak{g}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha$ ($H \in \mathfrak{h}$). Then X_α is unique apart from a factor in C, and it lies either in $\mathfrak{k}$ or in $\mathfrak{p}$. We call α a compact root if $X_\alpha \in \mathfrak{k}$. Either α and $-\alpha$ are both compact, or they are both noncompact.

Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{X}$ the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{k})$. Let π be a representation of $\mathfrak{B}$ on a vector space V. We shall say that π has an extreme vector if there exists an element $\psi \neq 0$ in V and a linear function Λ on $\mathfrak{h}$ such that the following conditions hold:

1. For every root α at least one of the two vectors $\pi(X_\alpha)\psi$, $\pi(X_{-\alpha})\psi$ is zero.
2. $\pi(H)\psi = \Lambda(H)\psi$ for all $H \in \mathfrak{h}$, and dim $\pi(\mathfrak{X})\psi$ is finite.

ψ is then called an extreme vector belonging to the weight Λ. Now, if ψ is an extreme vector, it is possible to define a lexicographic order (with respect to a suitable base on $(-1)^{1/2}\mathfrak{h}_0$ over R) in the space of all linear functions on $\mathfrak{h}$ which take real values on $(-1)^{1/2}\mathfrak{h}_0$, such that[2] $\pi(X_\alpha)\psi = 0$ for every root $\alpha > 0$. From now on we shall keep this order fixed and say that an extreme vector ψ is positive if $\pi(X_\alpha)\psi = 0$ for all $\alpha > 0$. Put $H_\alpha = [X_\alpha, X_{-\alpha}]$ and $\epsilon_\alpha = 1$ or -1 according as α is compact or not. Then it is possible to select X_α in such a way that $\alpha(H_\alpha) = 2$ and $(-1)^{1/2}(X_\alpha + \epsilon_\alpha X_{-\alpha})$, $(X_\alpha - \epsilon_\alpha X_{-\alpha})$ are both in $\mathfrak{g}_0$ for every root α. Let $\alpha_1, \ldots, \alpha_r$ be all the positive compact roots. We say that a root β is totally positive if every root of the form $\beta + m_1\alpha_1 + \ldots + m_r\alpha_r$ (where $m_1, \ldots, m_r$ are integers) is positive. Clearly, if β is totally positive, it is positive and noncompact. Let Q be the set of all totally positive roots and Q' the remaining set of positive roots.

Theorem 1. *Let π be an irreducible representation of $\mathfrak{B}$ on V with a positive extreme vector ψ belonging to the weight Λ. Then, if $\alpha \in Q'$, $\lambda_\alpha = \Lambda(H_\alpha)$ is a nonnegative integer and $\pi(X_{-\alpha}^{\lambda_\alpha + 1})\psi = 0$. Moreover, if Q is empty, dim V is finite.*

Conversely, suppose that we are given a linear function Λ on $\mathfrak{h}$ such that $\Lambda(H_\alpha)$ is a nonnegative integer for every $\alpha \in Q'$. Then it can be shown that there exists an irreducible representation π of $\mathfrak{B}$ on a vector space V with a positive extreme vector ψ belonging to the weight Λ, and π is unique up to equivalence. Furthermore, assuming that G is simply connected, we can find an irreducible quasisimple[3] representation σ of G on a Hilbert space $\mathfrak{H}$ and an element $\phi \in \mathfrak{H}$ which is well behaved[3] under σ such that the representation of $\mathfrak{B}$ defined on $\mathfrak{H}_0 = \sigma(\mathfrak{B})\phi$ is equivalent to π under the mapping $\sigma(b)\phi \leftrightarrow \pi(b)\psi$ $(b \in \mathfrak{B})$. In order that π be infinitesimally unitary,[4] it is necessary that $\Lambda(H_\beta)$ be real and ≤ 0 for all positive noncompact roots β.

If $\mathfrak{g}_0$ is simple, only the two following cases are possible:

1. There are no totally positive roots.
2. Every positive noncompact root is totally positive.

The second case can occur only if the center of $\mathfrak{k}_0$ is $\neq \{0\}$. Moreover, if $\mathfrak{g}_0$ is not compact, the representation π of Theorem 1 can never be infinitesimally unitary unless we are in case 2. So now let us assume that $\mathfrak{g}_0$ is not compact and we are in case 2. Then we can select a fundamental system of positive roots $\alpha_0, \alpha_1, \ldots, \alpha_l$ such that α_0 is noncompact, while α_i $(1 \leq i \leq l)$ are compact. A linear function Λ on $\mathfrak{h}$ fulfils the condition of Theorem 1 if and only if $\Lambda(H_{\alpha_i})$ $(1 \leq i \leq l)$ are nonnegative integers. We denote by π_Λ the corresponding representation of $\mathfrak{B}$. We have seen that π_Λ cannot be infinitesimally unitary unless $\Lambda(H_{\alpha_0})$ is real and ≤ 0. On the other hand, it can be shown that, if λ_i $(1 \leq i \leq l)$ are given nonnegative integers, there exists a real number $\lambda_0 \leq 0$ such that π_Λ is infinitesimally unitary whenever $\Lambda(H_{\alpha_i}) = \lambda_i$ $(1 \leq i \leq l)$ and $\Lambda(H_{\alpha_0}) \leq \lambda_0$.

Let us now suppose that G (which is again semisimple) has a faithful finite-dimensional representation.[5] Then there exists a complex analytic group G_c with the Lie algebra $\mathfrak{g}$, which contains G as the (real) analytic subgroup corresponding to $\mathfrak{g}_0$. Let $\mathfrak{n} = \sum_{\alpha > 0} CX_\alpha$, and let A_c and N_c be the complex analytic subgroups of G_c corresponding to $\mathfrak{h}$ and $\mathfrak{n}$, respectively. Then $G_c^0 = GA_cN_c$ is an open connected subset, and therefore a complex submanifold, of G_c. Let ξ be a complex analytic character of A_c. We consider the space $\mathfrak{H}_\xi$ of all holomorphic functions ϕ on G_c^0 such that $\phi(xan) = \phi(x)\xi(a)$ $(x \in G_c^0, a \in A_c, n \in N_c)$, and

$$||\phi||^2 = \int_G |\phi(x)|^2\, dx < \infty.$$

(Here dx is the Haar measure of G.) It is not difficult to see[6] that $\mathfrak{H}_\xi$ is complete with respect to the above norm, and so it is a Hilbert space. If $\mathfrak{H}_\xi \neq \{0\}$, we can define a unitary representation π_ξ of G on it by the rule $(\pi_\xi(x)\ \phi)(y) = \phi\ (x^{-1}y)$ $(x \in G, y \in G_c^0)$. It can be proved that this representation is irreducible and square-integrable.[7] Now assume again that G is simple and not compact. Then $\mathfrak{H}_\xi = \{0\}$ unless we are in case 2. So now let us suppose that we are in case 2. Given two complex analytic characters ξ and ξ' of A_c, we shall say that $\xi' \leq \xi$ if $\xi'(\exp H_{\alpha_i}) = \xi(\exp H_{\alpha_i})$ $(1 \leq i \leq l)$ and $|\xi'\ (\exp H_{\alpha_0})| \leq |\xi(\exp H_{\alpha_0})|$. Again $\mathfrak{H}_\xi = \{0\}$, unless $|\xi\ (\exp H_{\alpha_i})| \geq 1$ $(1 \leq i \leq l)$. On the other hand, if ξ_0 is any (complex analytic) character of A_c such that $|\xi_0\ (\exp H_{\alpha_i})| \geq 1$ $(1 \leq i \leq l)$, we can find a character $\xi_1 \leq \xi_0$ such that $\mathfrak{H}_\xi \neq \{0\}$ if $\xi \leq \xi_1$. If Λ is a linear function on $\mathfrak{h}$ such that $\xi\ (\exp H) = e^{\Lambda(H)}$ $(H \in \mathfrak{h})$ and $\mathfrak{H}_\xi \neq \{0\}$, the corresponding representation of $\mathfrak{B}$ defined under π_ξ is equivalent to π_Λ.

[1] These PROCEEDINGS, **40,** 1076–1077, 1954.

[2] This result has also been obtained independently by A. Borel.

[3] See these PROCEEDINGS, **37,** 170–173, 1951.

[4] See *Trans. Am. Math. Soc.*, **75,** 233, 1953.

[5] It is possible to avoid this assumption, which is made here only for simplicity.

[6] See S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton, N.J.: Princeton University Press, 1948), p. 117.

[7] Our procedure for constructing π_ξ is a generalization of the method used by Bargmann (*Ann. Math.*, **48,** 620, 1947) and Gelfand and Graev (*Izvest. Akad. Nauk, S.S.S.R.* **17,** 189–248, 1953) in certain special cases.