REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. V

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Let G be a connected semisimple Lie group and $\mathfrak{g}_0$ its Lie algebra over the field R of real numbers. Let $x \to Ad(x)$ denote the adjoint representation of G. A maximal connected abelian subgroup A of G is called a Cartan subgroup if $\mathfrak{g}_0$ is fully reducible under $Ad(A)$. The corresponding subalgebra of $\mathfrak{g}_0$ is called a Cartan subalgebra. Two Cartan subgroups A_1, A_2 are called conjugate (in G) if $A_2 = xA_1x^{-1}$ for some $x \in G$. Experience shows[1] that there is a close connection between the classes of conjugate Cartan subgroups and the various series of irreducible unitary representations of G which appear in the reduction of its regular representation. The object of this note is to give a general method of constructing irreducible unitary representations of G from each such class.

Let $X \to adX$ denote the adjoint representation $\mathfrak{g}_0$. Put $B(X,Y) = \mathrm{sp}\,(adXadY)\cdot(X, Y \in \mathfrak{g}_0)$. A subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}_0$ is called compact if the quadratic form $B(X, X)$ is negative definite on $\mathfrak{l}_0$. Let $\mathfrak{k}_0$ be a maximal compact subalgebra of $\mathfrak{g}_0$. We denote by $\mathfrak{p}_0$ the subspace of $\mathfrak{g}_0$ orthogonal to $\mathfrak{k}_0$ under the bilinear form $B(X, Y)$. Then $\mathfrak{g}_0$ is the direct sum of $\mathfrak{k}_0$ and $\mathfrak{p}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. It is possible to choose $\mathfrak{k}_0$ in such a way that $\mathfrak{h}_0 = \mathfrak{h}_{\mathfrak{p}_0} + \mathfrak{h}_{\mathfrak{k}_0}$, where $\mathfrak{h}_{\mathfrak{p}_0} = \mathfrak{h}_0 \cap \mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{h}_{\mathfrak{k}_0} = \mathfrak{h}_0 \cap \mathfrak{k}_0$. Let $H_1, \ldots, H_p$ be a base for $\mathfrak{h}_{\mathfrak{p}_0}$ over R. We order real linear functions on $\mathfrak{h}_{\mathfrak{p}_0}$ lexicographically with respect to this base. For any such function λ let $\mathfrak{g}_{0,\lambda}$ denote the set of all $X \in \mathfrak{g}_0$ such that $[H, X] = \lambda(H)X$ $(H \in \mathfrak{h}_{\mathfrak{p}_0})$. Then $\mathfrak{n}_0 = \sum_{\lambda > 0} \mathfrak{g}_{0,\lambda}$ is a nilpotent subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{m}_0$ denote the set of all $X \in \mathfrak{g}_0$ such that $[X, H] = 0$ and $B(X, H) = 0$ for all $H \in \mathfrak{h}_{\mathfrak{p}_0}$. Then $\mathfrak{m}_0$ is a reductive[2] subalgebra of $\mathfrak{g}_0$, and $\mathfrak{h}_{\mathfrak{k}_0}$ is a maximal abelian subalgebra of $\mathfrak{m}_0$.

Let K, A_+, M_0, N be the analytic subgroups of G corresponding to $\mathfrak{k}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{m}_0$, and $\mathfrak{n}_0$. Let M_k be the centralizer and M'_k the normalizer of A_+ in K. Then $W = M'_k/M_k$ is a finite group. Put $M = M_kM_0$ and $S = MA_+N$. Then M and S are closed subgroups of G, and we have the following generalization of a lemma of F. Bruhat:[3]

Lemma 1. *There exist only a finite number of distinct double cosets SxS $(x \in G)$. In case $\mathfrak{h}_{\mathfrak{p}_0}$ is a maximal abelian subspace of $\mathfrak{p}_0$, these double cosets are in a natural 1-1 correspondence with the elements of W.*

Let λ_0 be an irreducible unitary representation of M on a Hilbert space U and ξ a (unitary) character of A_+. If we put $\lambda(man) = \lambda_0(m)\xi(a)$ $(m \in M, a \in A_+, n \in N)$, λ defines a representation of S. Let π be the induced[4] representation of G corresponding to λ, and let $\mathfrak{H}$ denote the representation space of π. If Z is the center of G, $Z \subset M$, and therefore both λ and π map Z into scalars. Let Ω denote the set of all equivalence classes of finite-dimensional irreducible representations of K, and let $\mathfrak{H}_{\mathfrak{D}}$ $(\mathfrak{D} \in \Omega)$ be the set of all elements of $\mathfrak{H}$ which transform under $\pi(K)$ according to $\mathfrak{D}$.

Theorem 1. *π is a unitary quasi-simple[5] representation of G, and $\dim \mathfrak{H}_{\mathfrak{D}} < \infty$ for every $\mathfrak{D} \in \Omega$.*

It follows from this theorem that π decomposes into a (discrete) direct sum of at most a countable number of irreducible unitary representations. However, in view of Lemma 1 and Bruhat's results,[6] it seems likely that π is irreducible at least "in general." Since M/M_0Z is finite, the above procedure will be applicable as soon as we have a method of constructing irreducible unitary representations of M_0. Such a method is given in the following note.[7]

Let $x \to x^*$ denote the natural mapping of G onto $G^* = G/Z$, and let dx^* denote the Haar measure on G^*. Let π be a unitary irreducible representation of G on a Hilbert space $\mathfrak{H}$. Then for any fixed ψ in $\mathfrak{H}$, $|(\psi, \pi(x)\psi)|$ depends only on x^*. We say that π is square-integrable if $\int_{G^*}|(\psi, \pi(x)\psi)|^2 \, dx^* < \infty$ for every $\psi \in \mathfrak{H}$. One can prove that this is indeed the case if there exist a finite number of elements ϕ_i, ψ_i $(1 \leq i \leq r)$ in $\mathfrak{H}$ such that the function $f(x) = \sum_{i=1}^{r} (\phi_i, \pi(x)\psi_i)$ is not identically zero and $\int_{G^*}|f(x)|^2 dx^* < \infty$. The following theorem gives the analogue of the Schur orthogonality relations for square-integrable representations.[8]

THEOREM 2. *Let π and π' be two irreducible unitary representations of G on the Hilbert space $\mathfrak{H}$ and $\mathfrak{H}'$, respectively. Suppose that they are both square-integrable and that they both define the same character of Z. Then*

$$\int_{G^*}\overline{(\phi, \pi(x)\psi)}(\phi', \pi'(x)\psi') \, dx^* = 0 \qquad (\phi, \psi \in \mathfrak{H}, \qquad \phi', \psi' \in \mathfrak{H}'),$$

unless π and π' are equivalent. On the other hand, if U is a unitary mapping of $\mathfrak{H}$ on $\mathfrak{H}'$ under which π and π' are equivalent, then

$$\int_{G^*}\overline{(\phi, \pi(x)\psi)} \, (\phi', \pi'(x)\psi') \, dx^* = c(\phi', U\phi) \, (U\psi, \psi'),$$

where c is a positive real number independent of ϕ, ψ, ϕ', ψ'.

We shall give a general method of constructing square-integrable representations in the following note.[7]

[1] See V. Bargmann, *Ann. Math.*, **48**, 568–640, 1947; Harish-Chandra, these PROCEEDINGS, **38**, 337–342, 1952; and I. M. Gelfand and M. I. Graev, *Doklady Akad. Nauk S.S.S.R.*, N.S., **92**, 221–224, 1953.

[2] See L. Koszul, *Bull. Soc. Math. France*, **78**, 22, 1950.

[3] F. Bruhat, *Compt. rend. Acad. sci.* (Paris), **238**, 550–553, 1954.

[4] G. W. Mackey, *Ann. Math.*, **55**, 106, 1952.

[5] See Harish-Chandra, these PROCEEDINGS, **37**, 170–173, 1951.

[6] F. Bruhat, *Compt. rend. Acad. sci.* (Paris), **238**, 38–40, 1954.

[7] These PROCEEDINGS, **40**, 1078–1080, 1954.

[8] See Bargmann, *op. cit.*, p. 634. R. Godement (*Compt. rend. Acad. sci.* (Paris), **225**, 657–659, 1947) has proved a similar result for any locally compact unimodular group. However, in case the center of G is infinite, his theorem does not seem to include ours.