# ON THE PLANCHEREL FORMULA FOR THE RIGHT-INVARIANT FUNCTIONS ON A SEMISIMPLE LIE GROUP

BY HARISH-CHANDRA

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

*Communicated by Paul A. Smith, January 21, 1954*

Let $G$ be a connected semisimple Lie group and $K$ the complete inverse image in $G$ of a maximal compact subgroup of its adjoint group. Let $\mu$ denote the invariant measure on the space $G/K$ of all cosets $xK(x \in G)$. We construct the space $L_2(G/K)$ with respect to $\mu$ and consider the problem of the complete reduction of the natural representation $\sigma$ of $G$ on $L_2(G/K)$. Let $D$ be any subgroup contained in the center of $G$. Then $D \subset K$ and so if we replace $G$ by $G/D$ our problem is not affected in any way. Hence we may assume that the center of $G$ is finite and $K$ is compact. We can now regard $L_2(G/K)$ as a closed subspace of $L_2(G)$ consisting of those functions $f$ for which $f(x) = f(xu)$ $(x \in G,\ u \in K)$. Then $\sigma$ is the restriction of the left-regular representation of $G$ to $L_2(G/K)$. Let $dx$ denote the Haar measure on $G$. Put

$$(f, g) = \int_G \overline{f(x)} g(x) \cdot dx \qquad (f, g \in L_2(G))$$

where the bar denotes complex conjugate.

Let $\mathfrak{g}$ and $\mathfrak{k}_0$ be the Lie algebras of $G$ and $K$ respectively. In order to save space we shall adhere strictly to the notation of an earlier paper.[1] Thus any symbol should automatically be given the same meaning as there unless it is explicitly defined anew. Let $\eta$ be the anti-automorphism of $\mathfrak{B}$ over $R$ such that $\eta(X + (-1)^{1/2}Y) = -X + (-1)^{1/2}Y$ $(X, Y \in \mathfrak{g}_0)$. We denote by $\mathfrak{B}_H$ the set of those elements in $\mathfrak{B}$ which are left fixed by $\eta$. Let $\mathfrak{Q}$ be the centralizer of $\mathfrak{k}$ in $\mathfrak{B}$. Put $\mathfrak{Q}_0 = \mathfrak{Q} \cap \mathfrak{B}_0$, $\mathfrak{Q}_H = \mathfrak{Q} \cap \mathfrak{B}_H$, and $\mathfrak{Q}_{0, H} = \mathfrak{Q}_0 \cap \mathfrak{Q}_H$. (Here $\mathfrak{B}_0$ is the algebra generated over $R$ by $(1, \mathfrak{g}_0)$.) Let $\mathfrak{P}_d$ $(d \geqslant 0)$ denote the subspace of $\mathfrak{B}$ spanned over $C$ by the basic canonical elements[2] of degree $d$ formed out of a base for $\mathfrak{p}$. Let $\mathfrak{P} = \sum_{d \geqq 0} \mathfrak{P}_d$. An element in $\mathfrak{P}$ is called homogeneous of degree $d$ if it lies in $\mathfrak{P}_d$. Let $l_+ = \dim_R \mathfrak{h}_{\mathfrak{p}_0} = \dim A_+$.

LEMMA 1. *The algebra $\mathfrak{Q}^* = (\mathfrak{Q} + \mathfrak{B}\mathfrak{k})/\mathfrak{B}\mathfrak{k}$ is abelian and there exist $l_+$ elements $z_i$ $1 \leq i \leq l_+$ in $\mathfrak{Q}_{0, H}$ such that their residue-classes mod $\mathfrak{B}\mathfrak{k}$ generate $\mathfrak{Q}^*$. $z_i$ may be chosen to be homogeneous elements in $\mathfrak{P} \cap \mathfrak{Q}_{0, H}$.*

Let $C_c(G)$ denote the space of all complex-valued continuous functions $f$ on $G$ which vanish outside a compact set and $C_c^\infty(G)$ the subspace consisting of those $f$ which are indefinitely differentiable everywhere. Moreover let $C_c(G/K)$ and $I_c(G)$, respectively, be the subsets of those $f \in C_c(G)$ for which $f(xu) = f(x)$ and $f(uxu^{-1}) = f(x) (x \in G,\ u \in K)$. Put $I_c(G/K) = I_c(G) \cap C_c(G/K)$ and denote the closures of $I_c(G)$ and $I_c(G/K)$ in $L_1(G)$ and $L_2(G/K)$ by $I_1$ $(G)$and $I_2(G/K)$ respectively. For any $f$ and $g$ in $L_1(G)$ we denote by $f{*}g$ their convolution and by $\sigma(f)$ the operator $\int f(x)\sigma(x)\, dx$. Then $f \rightarrow \sigma(f)$ is a continuous representation of $L_1(G)$ and it is clear that if $f \in I_1(G)$, $I_2(G/K)$ is invariant under $\sigma(f)$. Let $\tau(f)$ denote its restriction on $I_2(G/K)$. Then $f \rightarrow \tau(f)$ is a representation of the subalgebra $I_1(G)$ on $I_2(G/K)$.

LEMMA 2. *$\tau$ is an abelian representation of $I_1(G)$.*

Let $V$ denote the Garding subspace[3] of $L_2(G/K)$ with respect to $\sigma$ and $b \rightarrow \sigma(b)(b \in \mathfrak{B})$ the corresponding representation of $\mathfrak{B}$ on $V$. It is obvious that for any $z \in \mathfrak{Q}$, $\sigma(z)$ leaves $V_I = V \cap I_2(G/K)$ invariant. We denote by $\tau(z)$ the restriction of $\sigma(z)$ on $V_I(z \in \mathfrak{Q})$.

LEMMA 3. *If $z \in \mathfrak{Q}_H$, $\tau(z)$ is an essentially hypermaximal operator on $I_2(G/K)$.*

For any $z \in \mathfrak{Q}_H$ let $U(z)$ denote the Cayley transform of $\tau(z)$. Then $U(z)$ is a unitary operator on $I_2(G/K)$ and

$$U(z)(\sigma(z) + (-1)^{1/2}I)g = (\sigma(z) - (-1)^{1/2}I)g \qquad (g \in V_I)$$

where $I$ is the unit operator on $I_2(G/K)$. Put $U_i = U(z_i)(1 \leq i \leq l_+)$ where $z_i$ are the elements defined in Lemma 1. Let $\mathfrak{A}$ be the smallest $C^*$-algebra[4] of operators on $I_2(G/K)$ containing $\tau(f)(f \in I_1(G))$ and $U_i$ $1 \leq i \leq l_+$. One proves without difficulty that $\mathfrak{A}$ is abelian. Let $\Gamma$ be the spectrum of $\mathfrak{A}$. Then $\Gamma$ is a compact Hausdorff space. For any $\omega \in \Gamma$ we denote by $\xi_\omega$ the corresponding continuous character of $\mathfrak{A}$. Let $\lambda$ and $\rho$, respectively, denote the left- and the right-regular representations of $G$ on $L_1(G)$. Put

$$E = \int_K \lambda(u)\rho(u)\, du$$

where $du$ is the normalized Haar measure on $K$ so that $\int_K du = 1$. Then $E$ is a projection from $L_1(G)$ to $I_1(G)$ and $||Ef||_1 \leq ||f||_1\ (f \in L_1(G))$ where $||\ \ ||_1$ denotes the norm in $L_1(G)$. Now set

$$\xi_\omega(f) = \xi_\omega(\tau(Ef)) \qquad (f \in L_1(G)).$$

Then if $\tilde{f}(x) = \overline{f(x^{-1})}(f \in L_1(G),\ x \in G)$ it is easy to verify that $|\xi_\omega(f)| \leq ||f||_1$ and $\xi_\omega(\tilde{f}*f)$ is real and non-negative $(f \in L_1(G))$. Let $\Gamma_0$ denote the set of all $\omega \in \Gamma$ such that $\xi_\omega$ vanishes identically on $L_1(G)$. Then it follows from well-known results[6] that if $\omega \notin \Gamma_0$ there exists a unitary representation $\pi$ of $G$ on a Hilbert space $\mathfrak{H}$ and a vector $\psi \in \mathfrak{H}$ such that

(i) $|\psi| = 1$ and $\xi_\omega(f) = \int_G f(x)(\psi,\ \pi(x)\psi)\, dx\ (f \in L_1(G))$.

(ii) The space spanned by $\pi(x)\psi(x \in G)$ is dense in $\mathfrak{H}$.

Here $(\psi_1,\ \psi_2)$ denotes the scalar product and $|\psi|$ the norm in $\mathfrak{H}$. It is easy to verify that $\xi_\omega(\lambda(u)f) = \xi_\omega(\rho(u)f) = \xi_\omega(f)(f \in L_1(G),\ u \in K)$. From this one can prove that $\pi(u)\psi = \psi(u \in K)$ and $\pi$ is irreducible. Moreover the choice of $\psi$ is unique, apart from a unimodular factor, and $\psi$ is well-behaved[6] under $\pi$. Put $\xi_\omega(b) = (\psi,\ \pi(b)\psi)(b \in \mathfrak{B})$. Then it is not difficult to show that

$$\xi_\omega(U_i) = (\xi_\omega(z_i) - (-1)^{1/2})(\xi_\omega(z_i) + (-1)^{1/2})^{-1} \qquad 1 \leq i \leq l_+.$$

Now let $\omega_1,\ \omega_2$ be two points in $\Gamma - \Gamma_0$ and $\pi_1,\ \pi_2$ the corresponding irreducible representations. Then if $\xi_{\omega_1}(U_i) = \xi_{\omega_2}(U_i)$, $1 \leq i \leq l_+$, it follows from Lemma 1 that $\xi_{\omega_1}(z) = \xi_{\omega_2}(z)(z \in \mathfrak{Q})$ and therefore[7] $\pi_1$ and $\pi_2$ are equivalent. Hence $\xi_{\omega_1}(f) = \xi_{\omega_2}(f)$ for all $f \in L_1(G)$ and so $\omega_1 = \omega_2$. Let $\mathfrak{E}$ denote the set of all equivalence classes of irreducible unitary representations $\pi$ of $G$ such that the trivial representation of $K$ occurs in the reduction of $\pi$. Then if $\omega \in \Gamma - \Gamma_0$ and $\alpha(\omega)$ is the class of the representation $\pi$ defined above for $\omega$, $\omega \rightarrow \alpha(\omega)$ is a 1-1 mapping of $\Gamma - \Gamma_0$ onto a subset $\mathfrak{E}_0$ of $\mathfrak{E}$. We identify $\Gamma - \Gamma_0$ with $\mathfrak{E}_0$ under this mapping.

On the other hand it is well known that there exists a unique projection-valued (regular) measure $dP_\omega$ on the class of all Borel subsets of $\Gamma$ such that

$$A = \int_\Gamma \xi_\omega(A) dP_\omega$$

for every operator $A$ in $\mathfrak{A}$. Since $\Gamma_0$ is clearly compact, it is Borel measurable and if $P_{\Gamma_0}$ is the corresponding projection it is obvious that $\tau(f)P_{\Gamma_0} = 0$ for all $f \in I_1(G)$. Moreover since it is possible to choose a sequence $f_n \in I_1(G)$ such that $\tau(f_n)g \rightarrow g$ for every $g \in I_2(G/K)$, it follows that $P_{\Gamma_0} = 0$. From this we can conclude that $\Gamma_0$ contains no interior point and therefore if $\omega_0 \in \Gamma_0$ there exists a sequence $\omega_n \in \Gamma - \Gamma_0$ such that $\omega_n \rightarrow \omega_0$ in $\Gamma$. Furthermore

$$A = \int_{\mathfrak{E}_0} \xi_\omega(A) dP_\omega \qquad (A \in \mathfrak{A}).$$

For any $\omega \in \mathfrak{E}_0$ let $Q(\omega)$ denote the point $(\xi_\omega(z_1), \ldots, \xi_\omega(z_{l_+}))$ in the real Euclidean space of dimension $l_+$.

LEMMA 4. *Let $\omega_n$ be a sequence in $\mathfrak{E}_0$ which converges in $\Gamma$ to a point $\omega_0 \in \Gamma_0$. Then the sequence $Q(\omega_n)$ diverges to infinity.*

In view of the above-mentioned relationship between $\xi_\omega(U_i)$ and $\xi_\omega(z_i)$ for $\omega$ in $\mathfrak{E}_0$, it follows that $\xi_{\omega_n}(U_i) \rightarrow 1$ for at least one value of $i$ $(1 \leq i \leq l_+)$. This proves that $\xi_{\omega_0}(U_i) = 1$ for some $i$ if and only if $\omega_0 \in \Gamma_0$.

Now let $\omega_1$, $\omega_2$ be two points in $\Gamma$ such that $\xi_{\omega_1}(U_i) = \xi_{\omega_2}(U_i)$, $i = 1, \ldots, l_+$. Suppose $\omega_1 \in \Gamma_0$. Then $\xi_{\omega_1}(U_i) = 1$ for some $i$. Therefore since $\xi_{\omega_2}(U_i) = \xi_{\omega_1}(U_i) = 1$ it follows that $\omega_2 \in \Gamma_0$. Hence $\omega_1$, $\omega_2$ are both in $\Gamma_0$ and so $\xi_{\omega_1}$, $\xi_{\omega_2}$ are identically zero on $L_1(G)$. It is then clear that $\omega_1 = \omega_2$. On the other hand if $\omega_1 \notin \Gamma_0$, we conclude in the same way that $\omega_2 \notin \Gamma_0$ and therefore again in view of our earlier remarks $\omega_1 = \omega_2$. This shows that $\xi_\omega(U_i)$ $1 \leq i \leq l_+$ is a separating set of continuous functions on $\Gamma$. Therefore from the Stone-Wierstrass theorem we can conclude that $\mathfrak{A}$ is the smallest $C^*$-algebra containing the operators $U_i$, $1 \leq i \leq l_+$. Then it follows from Lemma 4 that $\omega \rightarrow Q(\omega)$ is a topological mapping of $\mathfrak{E}_0$ onto a subset $Q(\mathfrak{E}_0)$ of the Euclidean space. Let $\tau_i$ denote the unique hypermaximal extension of $\tau(z_i)$, $1 \leq i \leq l_+$, and let $dP_\lambda$ denote the projection-valued Borel measure on $Q(\mathfrak{E}_0)$ corresponding to $dP_\omega$ on $\mathfrak{E}_0$. Then it is clear from the above remarks that $Q(\mathfrak{E}_0)$ is the spectrum of the abelian set $\tau_i$, $1 \leq i \leq l_+$, of hypermaximal operators on $I_2(G/K)$ and

$$\tau_i = \int_{Q(\mathfrak{E}_0)} \lambda_i \, dP_\lambda \qquad 1 \leq i \leq l_+$$

where $\lambda_i(Q(\omega)) = \xi_\omega(z_i)$ $(\omega \in \mathfrak{E}_0)$.

On the other hand it is not difficult to show that there exists a uniquely determined (regular) Borel measure $d\omega$ on $\mathfrak{E}_0 = \Gamma - \Gamma_0$ such that

$$d(f, P_\omega g) = \overline{\xi_\omega(f)}\xi_\omega(g) \, d\omega \qquad (f, g \in I_c(G/K), \omega \in \mathfrak{E}_0).$$

From this one can prove that

$$\int |f(x)|^2 dx = \int_{\mathfrak{E}_0} \xi_\omega(\tilde{f}*f) \, d\omega \qquad (f \in C_c(G/K).$$

Let $\pi$ be any unitary representation in the class $\omega \in \mathfrak{E}_0$. Then if $N_\omega(f)$ denotes the square of the Hilbert-Schmidt norm of the operator $\int f(x)\pi(x) \, dx$ $(f \in C_c(G))$ it is easy to verify that $\xi_\omega(\tilde{f}*f) = N_\omega(f)$ $(f \in C_c(G/K))$. Hence we have the Plancherel formula

$$\int |f(x)|^2 \, dx = \int_{\mathfrak{E}_0} N_\omega(f) \, d\omega \qquad (f \in C_c(G/K)).$$

Our results may therefore be summarized in the following theorem.

THEOREM. *Let $F$ be the spectrum of the abelian family of hypermaximal operators $\tau_i$, $1 \leq i \leq l_+$, regarded as a subset of the real Euclidean space $E$ of dimension $l_+$. Let $P_\lambda$ denote the corresponding resolution of the identity so that*

$$\tau_i = \int_F \lambda_i \, dP_\lambda \qquad (1 \leq i \leq l_+)$$

*where $\lambda_i$ is the* ith *coordinate of any point $\lambda \in E$. Then there exists a* 1-1 *mapping $\lambda \rightarrow \omega(\lambda)$ of $F$ on a subset $\mathfrak{E}_0$ of $\mathfrak{E}$ with the following property.*

*If $\pi$ is any representation in $\omega(\lambda)$ and $\psi$ is the (essentially unique) vector in its representation space such that $|\psi| = 1$ and $\pi(u)\psi = \psi (u \in K)$, then*

$$\lambda_i = (\psi, \pi(z_i)\psi) \qquad 1 \leq i \leq l_+.$$

*Put* $\varphi_\lambda(x) = (\psi, \pi(x)\psi)(x \in G)$. *Then there exists a uniquely determined (regular) Borel measure* $d\lambda$ *on* $F$ *such that*

$$d(f, P_\lambda f) = |\textstyle\int f(x)\varphi_\lambda(x)\, dx|^2\, d\lambda \qquad (f \in I_c(G/K)).$$

*Moreover we have the following Plancherel formula*

$$\int |f(x)|^2\, dx = \int_F N_{\omega(\lambda)}(f)\, d\lambda \qquad (f \in C_c(G/K)).$$

Let $W$ denote the group of all those linear mappings $s$ of $\mathfrak{h}_{\mathfrak{p}_0}$ into itself for which we can find an element $u \in K$ such that $sH = Ad(u)H(H \in \mathfrak{h}_{\mathfrak{p}_0})$. Then $W$ is a finite group. Let $I_c(\mathfrak{h}_{\mathfrak{p}_0})$ denote the set of all continuous functions $f$ on $\mathfrak{h}_{\mathfrak{p}_0}$ vanishing outside a compact set such that $f(H) = f(sH)$ $(s \in W,\ H \in \mathfrak{h}_{\mathfrak{p}_0})$. For any $f \in I_c(G/K)$ put $f_0(H) = f(\exp H)$ $(H \in \mathfrak{h}_{\mathfrak{p}_0})$. Since $G = KA_+K$ it is easy to show that $f \rightarrow f_0$ is a 1-1 mapping of $I_c(G/K)$ onto $I_c(\mathfrak{h}_{\mathfrak{p}_0})$ and

$$\int |f(x)|^2\, dx = \int_{\mathfrak{h}_{\mathfrak{p}_0}} |f_0(H)|^2 \Delta_+(H)\, dH$$

where $\Delta_+(H) = \prod_{\alpha \in P_\lambda} |e^{\alpha(H)} - e^{-\alpha(H)}|$ and $dH$ is the Euclidean measure on $\mathfrak{h}_{\mathfrak{p}_0}$. Let $L_2(\mathfrak{h}_{\mathfrak{p}_0})$ denote the Hilbert space of all functions on $\mathfrak{h}_{\mathfrak{p}_0}$ which are square-integrable with respect to the measure $\Delta_+(H)\, dH$. Let $I_2(\mathfrak{h}_{\mathfrak{p}_0})$ denote the closure of $I_c(\mathfrak{h}_{\mathfrak{p}_0})$ in $L_2(\mathfrak{h}_{\mathfrak{p}_0})$. Then the above mapping $f \rightarrow f_0$ can be extended uniquely to a unitary isomorphism of $I_2(G/K)$ with $I_2(\mathfrak{h}_{\mathfrak{p}_0})$. Under this isomorphism the operators $\tau(z_i)$ $1 \leq i \leq l_+$ go over into certain differential operators $D_i$ whose coefficients are meromorphic functions on $\mathfrak{h}_{\mathfrak{p}_0}$. $F$ and $dP_\lambda$ of our theorem may now, respectively, be interpreted as the spectrum and the spectral measure of the family $D_i$ $1 \leq i \leq l_+$ and the above Plancherel formula then becomes the "Parseval formula" corresponding to the problem of expanding a given function in $I_2(\mathfrak{h}_{\mathfrak{p}_0})$, in terms of eigen-functions of the family $D_i$ $1 \leq i \leq l_+$ of differential operators.

[1] *Trans. Am. Math. Soc.*, **75**, 185–243 (1953). See especially §2 and §12 for notation. We shall refer to this paper as *RI*.

[2] See *RI*, p. 193.

[3] See §4 of *RI* for the definition of the Garding subspace.

[4] A C*-algebra is an algebra $\mathfrak{A}$ of bounded operators on a Hilbert space such that (i) the unit operator is in $\mathfrak{A}$ (ii) the adjoint of any operator in $\mathfrak{A}$ is also in $\mathfrak{A}$ (iii) $\mathfrak{A}$ is closed under uniform topology.

[5] I. Gelfand and D. Raikov, *Recueil Math. (Moscow)*, **13**, 316 (1943).

[6] See Theorem 6, §11 of *RI*.

[7] See PROC. NATL. ACAD. SCI., **37**, 170–173 (1951); Theorem 7.