ON THE PLANCHEREL FORMULA FOR THE RIGHT-ININVARIANT
FUNCTIONS ON A SEMISIMPLE LIE GROUP

BY HARISH-CHANDRA

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

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Let $G$ be a connected semisimple Lie group and $K$ the complete inverse image in $G$ of a maximal compact subgroup of its adjoint group. Let $\mu$ denote the invariant measure on the space $G/K$ of all cosets $xK(x \in G)$. We construct the space $L_2(G/K)$ with respect to $\mu$ and consider the problem of the complete reduction of the natural representation $\sigma$ of $G$ on $L_2(G/K)$. Let $D$ be any subgroup contained in the center
of $G$. Then $D \subset K$ and so if we replace $G$ by $G/D$ our problem is not affected in any way. Hence we may assume that the center of $G$ is finite and $K$ is compact. We can now regard $L_2(G/K)$ as a closed subspace of $L_2(G)$ consisting of those functions $f$ for which $f(x) = f(xu)$ $(x \in G, u \in K)$. Then $\sigma$ is the restriction of the left-regular representation of $G$ to $L_2(G/K)$. Let $dx$ denote the Haar measure on $G$. Put

$$
(f, g) = \int_G f(x)g(x) \, dx 
$$

where the bar denotes complex conjugate.

Let $\mathfrak{g}$ and $\mathfrak{h}_0$ be the Lie algebras of $G$ and $K$ respectively. In order to save space we shall adhere strictly to the notation of an earlier paper. Thus any symbol should automatically be given the same meaning as there unless it is explicitly defined anew. Let $\eta$ be the anti-automorphism of $\mathfrak{g}$ over $R$ such that $\eta(X + (-1)^{1/2}Y) = X + (-1)^{1/2}Y$ $(X, Y \in \mathfrak{g}_0)$. We denote by $\mathfrak{g}_H$ the set of those elements in $\mathfrak{g}$ which are left fixed by $\eta$. Let $\mathfrak{O}$ be the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Put $\mathfrak{O}_0 = \mathfrak{O} \cap \mathfrak{g}_0$, $\mathfrak{O}_H = \mathfrak{O} \cap \mathfrak{g}_H$, and $\mathfrak{O}_{0, H} = \mathfrak{O}_0 \cap \mathfrak{O}_H$. (Here $\mathfrak{g}_0$ is the algebra generated over $R$ by $(1, \mathfrak{g}_0)$.) Let $\mathfrak{g}_d$ $(d \geq 0)$ denote the subspace of $\mathfrak{g}$ spanned over $C$ by the basic canonical elements of degree $d$ formed out of a base for $\mathfrak{g}$. Let $\mathfrak{g} = \sum_d \mathfrak{g}_d$. An element in $\mathfrak{g}$ is called homogeneous of degree $d$ if it lies in $\mathfrak{g}_d$.

Let $l_+ = \dim_{K} \mathfrak{g}_{\mathfrak{h}_0} = \dim A_{+}$.

**Lemma 1.** The algebra $\mathfrak{O}^* = (\mathfrak{O} + \mathfrak{h}_0)/\mathfrak{h}_0$ is abelian and there exist $l_+$ elements $z_1, 1 \leq i \leq l_+ \in \mathfrak{O}_{0, H}$ such that their residue-classes mod $\mathfrak{h}_0$ generate $\mathfrak{O}^*$. $z_i$ may be chosen to be homogeneous elements in $\mathfrak{g} \cap \mathfrak{O}_{0, H}$.

Let $C_c(G)$ denote the space of all complex-valued continuous functions $f$ on $G$ which vanish outside a compact set and $C_c^\infty(G)$ the subspace consisting of those $f$ which are indefinitely differentiable everywhere. Moreover let $C_c(G/K)$ and $I_c(G)$, respectively, be the subsets of those $f \in C_c(G)$ for which $f(xu) = f(x)$ and $f(u_0u^{-1}) = f(x)(x \in G, u \in K)$. Put $I_c(G/K) = I_c(G) \cap C_c(G/K)$ and denote the closures of $I_c(G)$ and $I_c(G/K)$ in $L_0(G)$ and $L_0(G/K)$ by $I_1(G)$ and $I_1(G/K)$ respectively. For any $f$ and $g$ in $L_1(G)$ we denote by $f * g$ their convolution and by $\sigma(f)$ the operator $\int f(x) \sigma(x) \, dx$. Then $f \mapsto \sigma(f)$ is a continuous representation of $L_0(G)$ and it is clear that if $f \in I_1(G)$, $I_2(G/K)$ is invariant under $\sigma(f)$. Let $\tau(f)$ denote its restriction on $I_2(G/K)$. Then $f \mapsto \tau(f)$ is a representation of the sub-algebra $I_1(G)$ on $I_2(G/K)$.

**Lemma 2.** $\tau$ is an abelian representation of $I_1(G)$.

Let $V$ denote the Garding subspace of $L_2(G/K)$ with respect to $\sigma$ and $b \mapsto \sigma(b)(b \in \mathfrak{g})$ the corresponding representation of $\mathfrak{g}$ on $V$. It is obvious that for any $z \in \mathfrak{O}$, $\sigma(z)$ leaves $V_I = V \cap I_2(G/K)$ invariant. We denote by $\tau(z)$ the restriction of $\sigma(z)$ on $V_I(z \in \mathfrak{O})$.

**Lemma 3.** If $z \in \mathfrak{O}_H$, $\tau(z)$ is an essentially hypermaximal operator on $I_2(G/K)$.

For any $z \in \mathfrak{O}_H$ let $U(z)$ denote the Cayley transform of $\tau(z)$. Then $U(z)$ is a unitary operator on $I_2(G/K)$ and

$$
U(z)(\sigma(z) + (-1)^{1/2}I)g = (\sigma(z) - (-1)^{1/2}I)g \quad (g \in V_I)
$$

where $I$ is the unit operator on $I_2(G/K)$. Put $U_i = U(z_i)(1 \leq i \leq l_+)$ where $z_i$ are the elements defined in Lemma 1. Let $\mathfrak{A}$ be the smallest $C^*$-algebra of operators
on $L_2(G/K)$ containing $\tau(f)(f \in I_1(G))$ and $U_i \leq i \leq l_+$. One proves without difficulty that $\mathfrak{A}$ is abelian. Let $\Gamma$ be the spectrum of $\mathfrak{A}$. Then $\Gamma$ is a compact Hausdorff space. For any $\omega \in \Gamma$ we denote by $\xi_\omega$ the corresponding continuous character of $\mathfrak{A}$. Let $\lambda$ and $\rho$, respectively, denote the left- and the right-regular representations of $G$ on $L_1(G)$. Put

$$E = \int_\Gamma \lambda(u)\rho(u) \, du$$

where $du$ is the normalized Haar measure on $K$ so that $\int_K du = 1$. Then $E$ is a projection from $L_1(G)$ to $I_1(G)$ and $||E||_1 \leq ||f||_1$ (if $f \in L_1(G)$) where $|| \cdot ||_1$ denotes the norm in $L_1(G)$. Now set

$$\xi_\omega(f) = \xi_\omega(\tau(Ef)) \quad (f \in L_1(G)).$$

Then if $\tilde{f}(x) = \overline{f(x^{-1})}(f \in L_1(G), x \in G)$ it is easy to verify that $|\xi_\omega(f)| \leq ||f||_1$ and $\xi_\omega(\tilde{f}ef)$ is real and non-negative ($f \in L_1(G)$). Let $\Gamma_0$ denote the set of all $\omega \in \Gamma$ such that $\xi_\omega$ vanishes identically on $L_1(G)$. Then it follows from well-known results that if $\omega \in \Gamma_0$ there exists a unitary representation $\pi$ of $G$ on a Hilbert space $\mathcal{S}$ and a vector $\psi \in \mathcal{S}$ such that

(i) $|\psi| = 1$ and $\xi_\omega(f) = \int_\mathcal{G} f(x)(\psi, \pi(x)\psi) \, dx$ ($f \in L_1(G)$).

(ii) The space spanned by $\pi(x)\psi(x \in G)$ is dense in $\mathcal{S}$.

Here $(\psi_1, \psi_2)$ denotes the scalar product and $|\psi|$ the norm in $\mathcal{S}$. It is easy to verify that $\xi_\omega(\lambda(u)f) = \xi_\omega(\rho(u)f) = \xi_\omega(f)(f \in L_1(G), u \in K)$. From this one can prove that $\pi(u)\psi = \psi(u \in K)$ and $\pi$ is irreducible. Moreover the choice of $\psi$ is unique, apart from a unimodular factor, and $\psi$ is well-behaved under $\pi$. Put $\xi_\omega(b) = (\psi, \pi(b)\psi)(b \in \mathfrak{A})$. Then it is not difficult to show that

$$\xi_\omega(U_i) = (\xi_\omega(z_i) - (-1)^{1/2})(\xi_\omega(z_i) + (-1)^{1/2})^{-1} \quad 1 \leq i \leq l_+.$$

Now let $\omega_1, \omega_2$ be two points in $\Gamma - \Gamma_0$ and $\pi_1, \pi_2$ the corresponding irreducible representations. Then if $\xi_\omega(U_i) = \xi_\omega(U_i), 1 \leq i \leq l_+$, it follows from Lemma 1 that $\xi_\omega(z) = \xi_\omega(z)(z \in \mathbb{C})$ and therefore $\pi_1$ and $\pi_2$ are equivalent. Hence $\xi_\omega(f) = \xi_\omega(f)$ for all $f \in L_1(G)$ and so $\omega_1 = \omega_2$. Let $\mathcal{C}$ denote the set of all equivalence classes of irreducible unitary representations $\pi$ of $G$ such that the trivial representation of $K$ occurs in the reduction of $\pi$. Then if $\omega \in \Gamma - \Gamma_0$ and $\alpha(\omega)$ is the class of the representation $\pi$ defined above for $\omega$, $\omega \rightarrow \alpha(\omega)$ is a 1-1 mapping of $\Gamma - \Gamma_0$ onto a subset $\mathcal{C}_0$ of $\mathcal{C}$. We identify $\Gamma - \Gamma_0$ with $\mathcal{C}_0$ under this mapping.

On the other hand it is well known that there exists a unique projection-valued (regular) measure $dP_\omega$ on the class of all Borel subsets of $\Gamma$ such that

$$A = \int_\Gamma \xi_\omega(A) \, dP_\omega$$

for every operator $A$ in $\mathfrak{A}$. Since $\Gamma_0$ is clearly compact, it is Borel measurable and if $P_{\Gamma_0}$ is the corresponding projection it is obvious that $\tau(f)P_{\Gamma_0} = 0$ for all $f \in I_1(G)$. Moreover since it is possible to choose a sequence $f_\nu \in I_1(G)$ such that $\tau(f_\nu)g \rightarrow g$ for every $g \in L_2(G/K)$, it follows that $P_{\Gamma_0} = 0$. From this we can conclude that $\Gamma_0$ contains no interior point and therefore if $\omega_0 \in \Gamma_0$ there exists a sequence $\omega_\nu \in \Gamma - \Gamma_0$ such that $\omega_\nu \rightarrow \omega_0$ in $\Gamma$. Furthermore

$$A = \int_\mathfrak{A} \xi_\omega(A) \, dP_\omega \quad (A \in \mathfrak{A}).$$
For any \( \omega \in \mathbb{C}_0 \) let \( Q(\omega) \) denote the point \((\xi_\omega(z_1), \ldots, \xi_\omega(z_{l+}))\) in the real Euclidean space of dimension \( l_+ \).

**Lemma 4.** Let \( \omega_n \) be a sequence in \( \mathbb{C}_0 \) which converges in \( \Gamma \) to a point \( \omega_0 \in \mathbb{C}_0 \). Then the sequence \( Q(\omega_n) \) diverges to infinity.

In view of the above-mentioned relationship between \( \xi_\omega(U_i) \) and \( \xi_\omega(z_i) \) for \( \omega \in \mathbb{C}_0 \), it follows that \( \xi_\omega(U_i) \rightarrow 1 \) for at least one value of \( i \) (\( 1 \leq i \leq l_+ \)). This proves that \( \xi_\omega(U_i) = 1 \) for some \( i \) if and only if \( \omega_0 \in \Gamma_0 \).

Now let \( \omega_1, \omega_2 \) be two points in \( \Gamma \) such that \( \xi_\omega(U_i) = \xi_\omega(U_j), i = 1, \ldots, l_+ \). Suppose \( \omega_1 \in \Gamma_0 \). Then \( \xi_\omega(U_i) = 1 \) for some \( i \). Therefore since \( \xi_\omega(U_i) = \xi_\omega(U_j) = 1 \) it follows that \( \omega_2 \in \Gamma_0 \). Hence \( \omega_1, \omega_2 \) are both in \( \Gamma_0 \) and so \( \xi_\omega, \xi_\omega \) are identically zero on \( L_n(G) \). It is then clear that \( \omega_1 = \omega_2 \). On the other hand if \( \omega_1 \notin \Gamma_0 \), we conclude in the same way that \( \omega_2 \notin \Gamma_0 \) and therefore again in view of our earlier remarks \( \omega_1 = \omega_2 \). This shows that \( \xi_\omega(U_i) \leq i \leq l_+ \) is a separating set of continuous functions on \( \Gamma \). Therefore from the Stone-Wierstrass theorem we can conclude that \( \mathfrak{A} \) is the smallest \( \mathfrak{C}^* \)-algebra containing the operators \( U_i, 1 \leq i \leq l_+ \). Then it follows from Lemma 4 that \( \omega \rightarrow Q(\omega) \) is a topological mapping of \( \mathbb{C}_0 \) onto a subset \( Q(\mathbb{C}_0) \) of the Euclidean space. Let \( \tau_i \) denote the unique hypermaximal extension of \( \tau(z_i), 1 \leq i \leq l_+ \), and let \( dP_\lambda \) denote the projection-valued Borel measure on \( Q(\mathbb{C}_0) \) corresponding to \( dP_\omega \) on \( \mathbb{C}_0 \). Then it is clear from the above remarks that \( Q(\mathbb{C}_0) \) is the spectrum of the abelian set \( \tau_i, 1 \leq i \leq l_+, \) of hypermaximal operators on \( L_2(G/K) \) and

\[
\tau_i = \int Q(\omega) \lambda_i dP_\lambda \quad 1 \leq i \leq l_+
\]

where \( \lambda_i(Q(\omega)) = \xi_\omega(z_i) \) (\( \omega \in \mathbb{C}_0 \)).

On the other hand it is not difficult to show that there exists a uniquely determined (regular) Borel measure \( d\omega \) on \( \mathbb{C}_0 = \Gamma - \Gamma_0 \) such that

\[
d(f, P_\omega) = \overline{\xi_\omega(f)} \xi_\omega(g) d\omega \quad (f, g \in L_2(G/K), \omega \in \mathbb{C}_0).
\]

From this one can prove that

\[
\int |f(x)|^2 dx = \int_{\mathbb{C}_0} \xi_\omega(f*f) d\omega \quad (f \in C_c(G/K)).
\]

Let \( \pi \) be any unitary representation in the class \( \omega \in \mathbb{C}_0 \). Then if \( N_\omega(f) \) denotes the square of the Hilbert-Schmidt norm of the operator \( \int f(x) \pi(x) dx \) \( (f \in C_c(G)) \) it is easy to verify that \( \xi_\omega(f*f) = N_\omega(f) \) \( (f \in C_c(G/K)) \). Hence we have the Plancherel formula

\[
\int |f(x)|^2 dx = \int_{\mathbb{C}_0} N_\omega(f) d\omega \quad (f \in C_c(G/K)).
\]

Our results may therefore be summarized in the following theorem.

**Theorem.** Let \( F \) be the spectrum of the abelian family of hypermaximal operators \( \tau_i, 1 \leq i \leq l_+, \) regarded as a subset of the real Euclidean space \( E \) of dimension \( l_+ \). Let \( P_\lambda \) denote the corresponding resolution of the identity so that

\[
\tau_i = \int_{\mathbb{C}_0} \lambda_i dP_\lambda \quad (1 \leq i \leq l_+)
\]

where \( \lambda_i \) is the \( i \)th coordinate of any point \( \lambda \in E \). Then there exists a 1-1 mapping \( \lambda \rightarrow \omega(\lambda) \) of \( F \) on a subset \( \mathbb{C}_0 \) of \( \mathbb{C} \) with the following property.

If \( \pi \) is any representation in \( \omega(\lambda) \) and \( \psi \) is the (essentially unique) vector in its representation space such that \( \|\psi\| = 1 \) and \( \pi(u)\psi = \psi(u \in K) \), then
\[ \lambda_i = (\psi, \pi(z_i)\psi) \quad 1 \leq i \leq l. \]

Put \( \varphi_\lambda(x) = (\psi, \pi(x)\psi)(x \in G) \). Then there exists a uniquely determined (regular) Borel measure \( dx \) on \( F \) such that

\[
d(f, P_xf) = | \int f(x) \varphi_\lambda(x) \, dx |^2 \, d\lambda \quad (f \in I_c(G/K)).
\]

Moreover we have the following Plancherel formula

\[
\int |f(x)|^2 \, dx = \int_{I_c(h_\mathcal{O})} N_{\omega(\Omega)}(f) \, d\lambda \quad (f \in C_c(G/K)).
\]

Let \( W \) denote the group of all those linear mappings \( s \) of \( h_\mathcal{O} \) into itself for which we can find an element \( u \in K \) such that \( sH = Ad(u)H(H \in h_\mathcal{O}) \). Then \( W \) is a finite group. Let \( L_2(h_\mathcal{O}) \) denote the Hilbert space of all functions \( f \) on \( h_\mathcal{O} \) square-integrable with respect to the measure \( \Delta^+(H) \, dH \) and \( dH \) is the Euclidean measure on \( h_\mathcal{O} \). Let \( L_2(h_\mathcal{O}) \) denote the Hilbert space of all functions on \( h_\mathcal{O} \) which are square-integrable with respect to the measure \( \Delta^+(H) \, dH \). Let \( I_c(h_\mathcal{O}) \) denote the closure of \( I_c(G/K) \) in \( L_2(h_\mathcal{O}) \). Then the above mapping \( f \rightarrow f_0 \) can be extended uniquely to a unitary isomorphism of \( I_c(G/K) \) with \( I_c(h_\mathcal{O}) \). Under this isomorphism the operators \( \tau(z_i) \) \( 1 \leq i \leq l \) go over into certain differential operators \( D_i \) whose coefficients are meromorphic functions on \( h_\mathcal{O} \). \( F \) and \( dP_\lambda \) of our theorem may now, respectively, be interpreted as the spectrum and the spectral measure of the family \( D_i, 1 \leq i \leq l \) and the above Plancherel formula then becomes the "Parseval formula" corresponding to the problem of expanding a given function in \( I_c(h_\mathcal{O}) \), in terms of eigen-functions of the family \( D_i, 1 \leq i \leq l \) of differential operators.

\(^1\) Trans. Am. Math. Soc., 75, 185-243 (1953). See especially §2 and §12 for notation. We shall refer to this paper as \( RI \).

\(^2\) See \( RI \), p. 193.

\(^3\) See §4 of \( RI \) for the definition of the Garding subspace.

\(^4\) A \( C^*\)-algebra is an algebra \( \mathfrak{A} \) of bounded operators on a Hilbert space such that (i) the unit operator is in \( \mathfrak{A} \) (ii) the adjoint of any operator in \( \mathfrak{A} \) is also in \( \mathfrak{A} \) (iii) \( \mathfrak{A} \) is closed under uniform topology.

\(^5\) I. Gelfand and D. Raikov, Recueil Math. (Moscow), 13, 316 (1943).

\(^6\) See Theorem 6, §11 of \( RI \).