REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. IV

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The present note is meant to be a continuation of the three earlier ones of this series which have appeared in these PROCEEDINGS.[1] In order to save space we shall adhere closely to the notation of *RII* and thus agree that all symbols shall have the same meaning as there unless they are explicitly defined anew.[2]

In *RII* we have defined the notion of the infinitesimal equivalence of two quasi-simple irreducible representations of G on two Hilbert spaces. This definition may clearly be extended, without any change whatsoever, to the case when the two representation spaces are Banach spaces. Let π be a representation of G on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be two closed invariant subspaces of $\mathfrak{H}$ such that $\mathfrak{H}_1 > \mathfrak{H}_2$. We regard the factor space $\mathfrak{H}_1/\mathfrak{H}_2$ as a Hilbert space in the usual way and consider the representation π' of G induced on it under π. Any representation π' obtained in this fashion will be said to be *deducible* from π.

Let Z be the center of G. Put $K^* = K/Z \cap D$. Then K^* is compact. Let $v \rightarrow v^*$ be the natural mapping of K on K^* and dv^* the element of Haar measure on K^* such that $\int_{K^*} dv^* = 1$. Let $L_2(K^*)$ be the Hilbert space consisting of all measurable and square-integrable functions on K^*. For any $x \in G$ and $v \in K$ we have defined v_x, $H(x, v)$ aud $\Gamma(x, v)$ in *RII*. It is easily seen that for x fixed $(v_x)^*$, $H(x, v)$ and $\Gamma(x, v)$ depend only on v^*.

Hence we may write them as v_x^*, $H(x, v^*)$ and $\Gamma(x, v^*)$. Let μ and ν be two linear functions on $\mathfrak{C}$ and $\mathfrak{h}_{\mathfrak{P}}$ respectively. We define a representation $\pi_{\mu,\nu}$ of G on $L_2(K^*)$ as follows:

$$\pi_{\mu,\nu}(x)f(v^*) = e^{-\mu(\Gamma(x^{-1}, v^*))}e^{-\nu(H(x^{-1}, v^*))}f(v^*_{x^{-1}}).$$

Here $x \in G$, $f \in L_2(K^*)$ and $v^* \in K^*$ and $\pi_{\mu,\nu}(x)f(v^*)$ denotes the value of $\pi_{\mu,\nu}(x)f$ at v^*. Let $\mathfrak{E}$ denote the set of all irreducible representations of G each of which is deducible from some $\pi_{\mu,\nu}$. It is easy to verify that every representation in $\mathfrak{E}$ is quasi-simple.

Let $\mathfrak{X}$ be the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{k})$. Since there is a natural 1-1 correspondence between finite-dimensional representations of K and $\mathfrak{k}$ (and therefore also of $\mathfrak{X}$), any $\mathfrak{D} \in \Omega$ may also be regarded as an equivalence class of finite-dimensional irreducible representations of $\mathfrak{k}$ (or $\mathfrak{X}$). Our first theorem may now be stated as follows:

THEOREM 1. *Let $\mathfrak{R}$ be a maximal left ideal in $\mathfrak{B}$ with the following two properties:*

(1) There exists a homomorphism χ of $\mathfrak{Z}$ into C such that $z - \chi(z) \in \mathfrak{R}$ for all $z \in \mathfrak{Z}$.

(2) The natural representation of $\mathfrak{X}$ on $\mathfrak{X}/\mathfrak{R} \cap \mathfrak{X}$ is finite-dimensional and the equivalence class of at least one of its irreducible components lies in Ω_F.

Then there exists a quasi-simple irreducible representation π of G on a Hilbert space $\mathfrak{H}$ and a well-behaved element $\psi \in \mathfrak{H}$ such that $\pi(b)\psi = 0$ $(b \in \mathfrak{B})$ if and only if $b \in \mathfrak{R}$. Moreover π may be chosen in $\mathfrak{E}$.

COROLLARY. *Let π be an irreducible quasi-simple representation of G on a Banach space $\mathfrak{H}$ such that some $\mathfrak{D} \in \Omega_F$ occurs in π. Then π is infinitesimally equivalent to some representation in $\mathfrak{E}$.*

Now let π be an irreducible quasi-simple representation of G on a Hilbert $\mathfrak{H}$. For any $\mathfrak{D} \in \Omega$ let $\mathfrak{H}_{\mathfrak{D}}$ denote the set of all elements in $\mathfrak{H}$ which transform under $\pi(K)$ according to $\mathfrak{D}$. We know from Theorem 3 of *RI* that $\dim \mathfrak{H}_{\mathfrak{D}} < \infty$ for all $\mathfrak{D} \in \Omega$. Moreover we may suppose without loss of generality that the subspaces $\mathfrak{H}_{\mathfrak{D}}$ are mutually orthogonal for different $\mathfrak{D}$. Let $E_{\mathfrak{D}}$ denote the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{H}_{\mathfrak{D}}$ $(\mathfrak{D} \in \Omega)$. Let μ be a linear function on $\mathfrak{C}$ such that $\pi(\exp \Gamma) = e^{\mu(\Gamma)}\pi(1)$ $(\Gamma \in \mathfrak{C}_0)$ whenever $\exp \Gamma \in D \cap Z$. Define a representation π^* of K^* on $\mathfrak{H}$ as follows:

$$\pi^*((u \exp \Gamma)^*) = e^{-\mu(\Gamma)}\pi(u \exp \Gamma) \qquad (u \in K', \Gamma \in \mathfrak{C}_0).$$

It is clear that for every $\mathfrak{D} \in \Omega$, $\mathfrak{H}_{\mathfrak{D}}$ is invariant under $\pi^*(K^*)$. Let $\mathfrak{D}^*$ denote the equivalence class of any irreducible component of the representation of K^* induced on $\mathfrak{H}_{\mathfrak{D}}$ under π^*. We choose an orthonormal base for each $\mathfrak{H}_{\mathfrak{D}}$. These bases all taken together form an orthonormal base for $\mathfrak{H}$. Let M be the subgroup of K as defined in *RII* and let ω be the set of all equivalence classes of finite-dimensional irreducible representations of M. For any $\delta \in \omega$ we denote by λ_δ the highest weight of δ.

THEOREM 2. *Suppose $\mathfrak{H}_{\mathfrak{D}} \neq \{0\}$ for some $\mathfrak{D} \in \Omega_F$. Then there exists a $\delta \in \omega$ and a linear function Λ on $\mathfrak{h}$ such that the following conditions are fulfilled:*

(1) χ_Λ is the infinitesimal character of π.

(2) Λ coincides with λ_δ on $\mathfrak{h}_{\mathfrak{k}}$.

(3) For any $\mathfrak{D} \in \Omega$

$$\dim \mathfrak{H}_{\mathfrak{D}} \leqq d(\mathfrak{D})n(\mathfrak{D}, \delta)$$

where $d(\mathfrak{D})$ is the degree of any representation $\sigma \in \mathfrak{D}$ and $n(\mathfrak{D}, \delta)$ is the number of times δ occurs in the reduction of $\sigma(M)$.

(4) Let $\mathfrak{D}_1, \mathfrak{D}_2$ be two elements in Ω such that

$$\mathfrak{H}_{\mathfrak{D}_i} \neq \{0\} \quad i = 1, 2.$$

Then the matrix coefficients of $E_{\mathfrak{D}_1}\pi(x)E_{\mathfrak{D}_2}$ $(x \in G)$ with respect to the above base are finite linear combinations with constant coefficients of functions of the form

$$\int_{K^*} g_{\mathfrak{D}_1^*}(v_x^*)g_{\mathfrak{D}_2^*}(v^{*-1})e^{\Lambda(H(x, v^*))}e^{\mu(\Gamma(x, v^*))}\, dv^*$$

where $g_{\mathfrak{D}_1^*}$ and $g_{\mathfrak{D}_2^*}$ are some matrix coefficients of representations in $\mathfrak{D}_1^*$ and $\mathfrak{D}_2^*$ respectively.

In case $\mathfrak{D}_1^*, \mathfrak{D}_2^*$ both correspond to the trivial representation of K^* the above result gives theorem 3 of *RII*.

THEOREM 3. *Let π be a quasi-simple irreducible representation of G on a Hilbert space $\mathfrak{H}$. Put $V = \sum_{\mathfrak{D} \in \Omega} \mathfrak{H}_{\mathfrak{D}}$. Suppose it is possible to define a new scalar product $(\varphi, \psi)'$ $(\varphi, \psi \in V)$ in V such that*

$$(\pi(X)\varphi, \psi)' = -(\varphi, \pi(X)\psi)' \qquad (X \in \mathfrak{g}_0).$$

Let $\mathfrak{H}'$ be the Hilbert space obtained by completing V with respect to the corresponding metric. Then there exists an irreducible unitary representation π' of G on $\mathfrak{H}'$ such that

$$\pi(X)\psi = \lim_{t \to 0} \frac{1}{t}\{\pi'(\exp tX)\psi - \psi\} \qquad (t \in R, \text{ limit in } \mathfrak{H}')$$

or all $X \in \mathfrak{g}_0$ and $\psi \in V$. Moreover π' is uniquely determined.

In view of theorems 2 and 3 it is clear that the problem of constructing all irreducible unitary representations of G is now largely reduced to that of determining the irreducible representations of $\mathfrak{B}$ which are "formally unitary." In fact it seems likely that all the above theorems actually remain true if we replace Ω_F by Ω everywhere even though our proofs are then no longer applicable. In case G has a finite-dimensional representation which is faithful on K' it is easily seen that $\Omega = \Omega_F$ and so the above theorems then hold in all generality. This is so in particular if G is a complex semisimple Lie group.

[1] PROC. NATL. ACAD. SCI., **37**, 170–173, 362–365, 366–369 (1951), quoted hereafter as *RI*, *RII* and *RIII* respectively.

[2] I take this opportunity of acknowledging the fact that one or two minor errors have crept into the latter portion of *RIII*. Since they are of a rather computational nature and do not, in any way, affect the general line of argument, it seems best to wait until the publication of the full details of the proof. However, I should like to correct some misprints in *RII*. On p. 363 in line 22 the last P should be $P-$. Also in lines 23 and 25 P should be replaced by $P-$ everywhere, so that we now have

$$\mathfrak{M} = \mathfrak{h}_{\mathfrak{g}} + \sum_{\alpha \in P_-} C X_\alpha + \sum_{\alpha \in P_-} CX_{-\alpha}$$