We shall adhere strictly to the notation of the preceding note. Making use of an unpublished result of Chevalley one can prove the following theorem.

**Theorem 1.** Let \( \pi \) be a quasisimple irreducible representation of \( G \) on a Hilbert space \( \mathcal{H} \). Then there exists an integer \( N \) such that

\[
\dim \mathcal{H}_D \leq N(d(D))^2
\]

for any \( D \in \Omega \).

Moreover if \( \mathcal{H}_D \neq \{0\} \) for some \( D \in \Omega \) then it can be shown that we may take \( N \) equal to the order of the Weyl group \( W \).

Let \( \pi \) be as above and let \( C^\infty_c(G) \) denote the class of all complex-valued functions on \( G \) which are indefinitely differentiable everywhere and which vanish outside a compact set. Let \( A \) be a bounded operator on \( \mathcal{H} \). We say that \( A \) has a trace if for every complete orthonormal set \( \{\psi_1, \psi_2, \ldots, \psi_n, \ldots\} \) in \( \mathcal{H} \) the series \( \sum_{i \geq 1} (\psi_i, A\psi_i) \) converges to a finite number independent of the choice of this orthonormal set. We denote this number by \( spA \). Let \( A^* \) be the adjoint of \( A \). We say that \( A \) is of the Hilbert-Schmidt class if \( AA^* \) has a trace.

**Theorem 2.** Let \( \pi \) be a quasisimple irreducible representation of \( G \) on a Hilbert space \( \mathcal{H} \). Then for any \( f \in C^\infty_c(G) \) the operator \( \int_G f(x)\pi(x) \, dx \) has a trace. Put

\[
T_\pi(f) = sp(\int_G f(x)\pi(x) \, dx)
\]

and for any \( a \in G \) let \( af_a \) denote the function \( af_a(x) = f(a^{-1}x) \) \( (x \in G) \). Then \( T_\pi \) is a distribution in the sense of L. Schwartz and

\[
T_\pi(af_a) = T_\pi(f) \quad (f \in C^\infty_c(G), a \in G).
\]

We shall call the distribution \( T_\pi \) the character of the representation \( \pi \).

**Theorem 3.** Let \( \pi_1 \) and \( \pi_2 \) be quasisimple irreducible representations of \( G \) on two Hilbert spaces. If \( T_\pi_1 = cT_\pi_2 (c \in C) \) then \( \pi_1 \) and \( \pi_2 \) are infinitesimally equivalent. Conversely if \( \pi_1 \) and \( \pi_2 \) are infinitesimally equivalent \( T_{\pi_1} = T_{\pi_2} \).

Since for irreducible unitary representations infinitesimal equivalence is the same as ordinary equivalence, such a representation is completely determined within equivalence by its character.
**Theorem 4.** Let \( \pi \) be a quasisimple irreducible representation of \( G \) and let \( f \) be any measurable function on \( G \) such that \( f \) vanishes outside a compact set and \( \int_{G} |f(x)|^{4} \, dx < \infty \). Then the operator \( \int_{G} f(x) \pi(x) \, dx \) is of the Hilbert-Schmidt class.

For any \( X \in \mathfrak{g}_0 \) and \( f \in C_c^{\infty}(G) \) put
\[
(*Xf)(x) = \left\{ \frac{d}{dt} f(\exp(-tX)x) \right\}_{t=0}.
\]

Then the mapping \( X \mapsto *X \) is a representation of \( \mathfrak{g}_0 \) on \( C_c^{\infty}(G) \) which can be extended to a representation \( b \mapsto *b \) (\( b \in \mathfrak{B} \)) of \( \mathfrak{B} \). Let \( \varphi \) be the anti-automorphism of \( \mathfrak{B} \) such that \( \varphi(X) = -X \) (\( X \in \mathfrak{g} \)). If \( T \) is any distribution on \( G \) we define \( bT \) (\( b \in \mathfrak{B} \)) as follows:
\[
bT(f) = T(\varphi(b)f) \quad (f \in C_c^{\infty}(G)).
\]

Moreover for any function \( f \) on \( G \) we denote by \( yf, f_1, f_2 \) (\( y, z, e G \)) the functions
\[
yf(z) = f(y^{-1}x), \quad f_1(x) = f(xz), \quad f_2(x) = f(y^{-1}xz) \quad (z \in G).
\]

Let \( Z \) denote the center of \( G \) and let \( \pi \) be a quasisimple irreducible representation of \( G \) on a Hilbert space. Let \( x \) be the infinitesimal character of \( \pi \) and let \( \eta \) be the homomorphism of \( Z \) into \( \mathbb{C} \) such that \( \pi(a) = \eta(a) \pi(1) \) (\( a \in Z \)). Then if \( T_\pi \) is the character of \( \pi \) it is easily seen that
\[
zT_\pi = \chi(z)T_\pi \quad (z \in Z).
\]

Now put \( M_1 = MZ \) and let \( x \mapsto x^* \) denote the adjoint representation of \( G \). Put \( (x)^{y*} = yxy^{-1}(x) \) (\( y \in G \)) and let \( C_c(G) \) denote the class of all continuous functions on \( G \) which vanish outside a compact set. Let \( \alpha \) and \( g \) be continuous functions on \( A_+ \) and \( M_1 \), respectively. Put
\[
T(f) = \int f((n^{-1}h^{-1}m^{-1})^*) \alpha(h)g(m) \, dm \, dh \, dn \, dm^* \quad (f \in C_c(G)).
\]

Here \( dm, dh, dn, dm^* \) are the left invariant Haar measures on \( M_1, A_+, N, K^* \), respectively, and the integral extends over \( K^* \times M_1 \times A_+ \times N \). It can be shown that \( T(f) = T((yf_\sigma)(y \in G) \). Let \( \sigma \) be an irreducible representation of \( M_1 \) on a finite-dimensional Hilbert space \( U \). Let \( \delta \) be the equivalence class of the representation \( \sigma_0 \) of \( M \) defined by \( \sigma \). Let \( \psi_0 \neq 0 \) be an element in \( U \) belonging to the highest weight \( \lambda_\delta \) of \( \sigma_0 \) and let \( \eta \) be the homomorphism of \( Z \) into \( \mathbb{C} \) such that \( \sigma(a) = \eta(a) \sigma(1) \) (\( a \in Z \)). Put \( g(m) = \langle \psi_0, \sigma(m^{-1})\psi_0 \rangle (m \in M_1) \) and \( \alpha(h) = e^{-1(1+2\rho) \log h} \) where \( \nu \) is a linear function on \( \hbar \) and \( \log h \) is the unique element in \( \hbar \) such that \( \exp(\log h) = h \). If we regard \( T \) as a distribution it is easily seen that
\[
zT = \chi_A(z)T \quad (z \in Z).
where \( \Lambda(H_1 + H_2) = \nu(H_1) + \lambda_\delta(H_2) \) \((H_1 \in \mathfrak{h}_1, H_2 \in \mathfrak{h}_2)\). Moreover

\[
T(f_a) = \eta(a)T(f) \quad (a \in \mathbb{Z}, f \in C_c(G))
\]

and it is easy to check that

\[
T(f) = \int f((n^{-1} - h^{-1}m^{-1})^*)e^{-(n+2\rho)(\log h)} \xi(m) \, mh \, dn \, du^* \quad (f \in C_c(G))
\]

where

\[
\xi(m) = \frac{1}{d(\delta)} |\psi_0|^2 \, sp\sigma(m)
\]

and \(d(\delta)\) is the degree of \(\sigma\). Let \(A_-\) be the analytic subgroup of \(M\) corresponding to \(\mathfrak{h}_0\). Put \(A = A_1A_-\) and \(A_1 = AZ\). Let \(V\) be the set of all elements in \(G\) which can be written in the form \(xxyx^{-1}\) with \(x \in G\) and \(y \in A_1N\). We shall say that an element \(y \in V\) is regular if \(y = xhx^{-1}\) for some \(x \in G\) and \(h \in \mathfrak{h}_1\) and \(y^*\) has exactly 5 eigen-values equal to 1. Let \(V_0\) be the set of all elements in \(V\) which are regular. Then \(V_0\) is open in \(G\) and \(V\) is the closure of \(V_0\). Let \(W_0\) be the subgroup of the Weyl group \(W\) consisting of those elements \(s \in W\) for which there exists an \(x \in G\) such that \(sH = x^*H\) for all \(H \in \mathfrak{h}\). It is easily seen that every \(s \in W_0\) leaves both \(\mathfrak{h}_0\) and \(\mathfrak{h}_1\) invariant. Put

\[
\Delta^-(H) = \prod_{\alpha \in P_-} (e^{4\alpha(H)} - e^{-4\alpha(H)}), \quad \Delta^+(H) = \prod_{\alpha \in P_+} (e^{4\alpha(H)} - e^{-4\alpha(H)})
\]

and define \(\epsilon(s) = \pm 1 \quad (s \in W_0)\) in such a way that

\[
\Delta^-(sH) = \epsilon(s)\Delta^-(H)
\]

for all \(H \in \mathfrak{h}_2\). In particular if \(P_-\) is empty \(\epsilon(s) = 1 \quad \forall s \in W_0\). Consider the function \(\Theta_{\Lambda, \eta}\) on \(V_0\) defined as follows:

\[
\Theta_{\Lambda, \eta}(y) = \eta(\gamma) \frac{\sum_{s \in W_0} \epsilon(s)e^{s(H_1 + H_2)}}{\Delta^-(H_1)\Delta^+(H_1 + H_2)} \quad (y \in V_0)
\]

where \(y = x(\gamma \exp(H_1 + H_2))x^{-1}\) for some \(x \in G, \gamma \in Z, H_1 \in \mathfrak{h}_2\) and \(H_2 \in \mathfrak{h}_0\). It can be shown that in spite of the ambiguity in the choice of \(\gamma, H_1\) and \(H_2, \Theta_{\Lambda, \eta}\) is well defined on \(V_0\). We extend \(\Theta_{\Lambda, \eta}\) on \(G\) by defining it to be zero outside \(V_0\). Then it can be proved that

\[
T(f) = \int_G f(x) \Theta_{\Lambda, \eta}(x) \, dx \quad (f \in C_c(G))
\]

provided the Haar measure \(dx\) on \(G\) is suitably normalized. Let \(s_1, s_2, \ldots, s_r\) be a maximal set of distinct elements in the Weyl group \(W\) with the following properties:

(I) Let \(\Lambda_i = s_i(\Lambda + \rho) - \rho \quad 1 \leq i \leq r\). Then \(\Lambda_i + \rho \neq s(\Lambda_j + \rho)\) if \(i \neq j\) \((1 \leq i, j \leq r)\) and \(s \in W_0\).

(II) For each \(i\) \((1 \leq i \leq r)\) there exists a \(\delta_i \in \omega\) such that \(\Lambda_i\) coincides on
with the highest weight \( \lambda_i \) of \( \Delta_i \). Moreover if \( \sigma \in \Delta_i \) and \( \gamma \in M \cap Z \)

\[
\sigma(\gamma) = \eta(\gamma)\sigma(1).
\]

Put \( \Theta_i = \Theta_{\lambda_i, \eta} \) \( 1 \leq i \leq r \) and \( T_i(f) = \int_G f(x)\Theta_i(x) \, dx \) \( f \in C_\alpha(G) \).

Then we see that the distributions \( T_i \) \( 1 \leq i \leq r \) are solutions of the equations,

\[
sT = \chi_\lambda(\varepsilon)T \quad (s \in S)
\]

\[
T(yf) = T(f), \quad T(fd) = \eta(a)T(f) \quad (f \in C_\alpha(G), y \in G, a \in Z).
\]

We have seen above that if \( \pi \) is any quasisimple irreducible representation of \( G \) such that \( \chi_\lambda \) is the infinitesimal character of \( \pi \) and \( \pi(a) = \eta(a)\pi(1) \) \( a \in Z \), then its character \( T_{\pi} \) is also a solution of the above equations. Hence one might hope that in most cases \( T_{\pi} \) would be a linear combination of \( T_i \) \( 1 \leq i \leq r \).

In conclusion I should like to thank Professor C. Chevalley for his help and advice on several questions connected with the results of this note.

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2 Since \( \pi \) is irreducible it follows easily that \( \mathcal{H} \) is separable.

3 Here \( dx \) denotes the left invariant Haar measure on \( G \).


5 See RII for the meaning of the various symbols.