REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. II

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The object of this note is to announce some further results on representations of a connected semisimple Lie group on a Hilbert space. We shall omit all proofs and assume that the reader is familiar with the contents of an earlier note[1] (quoted henceforward as *RI*).

Let R and C denote the fields of real and complex numbers, respectively, and let G be a connected, simply connected, semisimple Lie group and $\mathfrak{g}_0$ its Lie algebra over R. Let $x \rightarrow Ad(x)(x \in G)$ denote the adjoint representation of G and let K be the complete inverse image in G of some maximal compact subgroup of $Ad(G)$. Then K is a closed connected subgroup of G. Let $\mathfrak{K}_0$ be the Lie algebra of K and let $X \to adX (X \in \mathfrak{g}_0)$ denote the adjoint representation of $\mathfrak{g}_0$. Put $B(X, Y) = sp(adXadY)(X, Y \in \mathfrak{g}_0)$. Let $\mathfrak{P}_0$ be the set of all elements $Y \in \mathfrak{g}_0$ such that $B(X, Y) = 0$ for all $X \in \mathfrak{K}_0$. Then $\mathfrak{g}_0 = \mathfrak{K}_0 + \mathfrak{P}_0$, $\mathfrak{K}_0 \cap \mathfrak{P}_0 = \{0\}$ and there exists an automorphism θ of $\mathfrak{g}_0$ over R such that $\theta(X + Y) = X - Y$ for $X \in \mathfrak{K}_0$, $Y \in \mathfrak{P}_0$. Let $\mathfrak{h}_{\mathfrak{P}_0}$ be a maximal abelian subspace of $\mathfrak{P}_0$. We extend $\mathfrak{h}_{\mathfrak{P}_0}$ to a maximal abelian subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Then $\mathfrak{h}_0 = \mathfrak{h}_{\mathfrak{K}_0} + \mathfrak{h}_{\mathfrak{P}_0}$ where $\mathfrak{h}_{\mathfrak{K}_0} = \mathfrak{h}_0 \cap \mathfrak{K}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and let $\mathfrak{P}$, $\mathfrak{K}$, $\mathfrak{h}$, $\mathfrak{h}_{\mathfrak{P}}$, $\mathfrak{h}_{\mathfrak{K}}$ be the subspaces of $\mathfrak{g}$ spanned by $\mathfrak{P}_0$, $\mathfrak{K}_0$, $\mathfrak{h}_0$, $\mathfrak{h}_{\mathfrak{P}_0}$, $\mathfrak{h}_{\mathfrak{K}_0}$ respectively, over C. We extend the bilinear form B and the automorphism θ on $\mathfrak{g}$ by linearity over C. Choose bases $(H_1, \ldots, H_p)$ and $(H_{p+1}, \ldots, H_l)$, respectively, for $\mathfrak{h}_{\mathfrak{P}_0}$ and $\sqrt{-1}\ \mathfrak{h}_{\mathfrak{K}_0}$ over R. Let $\mathfrak{F}$ be the space of all linear functions on $\mathfrak{h}$. Given $\lambda \in \mathfrak{F}$ we can find a unique element $H_\lambda \in \mathfrak{h}$ such that $\lambda(H) = B(H, H_\lambda)$ for all $H \in \mathfrak{h}$. We shall say that λ is real if $H_\lambda \in \mathfrak{h}_{\mathfrak{P}_0} + \sqrt{-1}\ \mathfrak{h}_{\mathfrak{K}_0}$. Moreover if λ is real and $H_\lambda = \sum_{1 \leqslant i \leqslant l} c_i H_i (c_i \in R)$ we say that $\lambda > 0$ if $\lambda \neq 0$ and $c_j > 0$ where j is the least index $(1 \leqslant j \leqslant l)$ such that $c_j \neq 0$. We know that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\theta\mathfrak{h} = \mathfrak{h}$. For every root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$ let $X_\alpha \neq 0$ denote an element in $\mathfrak{g}$ such that $[H, X_\alpha] = \alpha(H)X_\alpha (H \in \mathfrak{h})$. Let P be the set of all roots $\alpha > 0$. For any $\lambda \in \mathfrak{F}$ let $\theta\lambda$ denote the linear function given by $\theta\lambda(H) = \lambda(\theta H)(H \in \mathfrak{h})$. Then if α is α root $\theta\alpha$ is also a root. Let P_+ be the set of all $\alpha \in P$ such that $\theta\alpha < 0$ and P the remaining set of positive roots. Iwasawa[2] has shown that X_α, $X_{-\alpha} \in \mathfrak{K}$ for $\alpha \in P$. Put

$$\mathfrak{N} = \sum_{\alpha \in P_+} CX_\alpha, \ \mathfrak{M} = \mathfrak{h}_{\mathfrak{K}} + \sum_{\alpha \in P} CX_\alpha + \sum_{\alpha \in P} CX_{-\alpha}.$$

Then $\mathfrak{N}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{M}$ is a subalgebra of $\mathfrak{K}$. Put $\mathfrak{N}_0 = \mathfrak{N} \cap \mathfrak{g}_0$, $\mathfrak{M}_0 = \mathfrak{M} \cap \mathfrak{K}_0$. Then $\mathfrak{g}_0 = \mathfrak{K}_0 + \mathfrak{h}_{\mathfrak{P}_0} + \mathfrak{N}_0$ where the sum is direct[2] and $[\mathfrak{M}, \mathfrak{N}] \subset \mathfrak{N}$, $[\mathfrak{M}, \mathfrak{h}_{\mathfrak{P}}] = \{0\}$. Moreover $\mathfrak{K}$ and $\mathfrak{M}$ are reductive algebras, i.e., they are the direct sums of their centers and their derived algebras which are semisimple. Let $\mathfrak{C}$ be the center of $\mathfrak{K}$. Put $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{g}_0$ and $\mathfrak{K}_0' = [\mathfrak{K}_0, \mathfrak{K}_0]$. Let D, K', A_+, M and N, respectively, be the analytic subgroups of G corresponding to $\mathfrak{C}_0$, $\mathfrak{K}_0'$, $\mathfrak{h}_{\mathfrak{P}_0}$, $\mathfrak{M}_0$ and $\mathfrak{N}_0$. Then M is closed and the mappings $(\gamma, u) \to \gamma u (\gamma \in D, u \in K')$ and $(v, h, n) \to vhn (v \in K, h \in A_+, n \in N)$ are analytic isomorphisms of DXK' with K and $K \times A_+ \times N$ with G, respectively.

Let Ω, Ω' and ω, respectively, be the set of all equivalence classes of finite-dimensional irreducible representations of K, K' and M. Then if $\mathfrak{D} \in \Omega$ and $\sigma \in \mathfrak{D}$ it is easily seen that $\sigma(K')$ is irreducible and $\sigma(M)$ is fully reducible. We denote by $\mathfrak{D}'$ the equivalence class of the irreducible representation of K' defined by σ. Also if $\delta \in \omega$ we say that $\delta < \mathfrak{D}$ if δ occurs in the reduction of $\sigma(M)$. Let $\alpha \in \delta \in \omega$. Then α defines α representation β of $\mathfrak{M}_0$ and therefore of $\mathfrak{M}$. Let λ and μ be any two weights of β with respect to $\mathfrak{h}_{\mathfrak{K}}$. We extend λ and μ on $\mathfrak{h}$ by defining them to be zero on $\mathfrak{h}_{\mathfrak{P}}$. Then it is easily seen that $\lambda - \mu$ is a real linear function on $\mathfrak{h}$. Hence β has a weight λ such that $\lambda - \mu > 0$ for every other weight μ. Clearly λ depends on δ alone and we shall call it the highest weight of δ.

Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}$ and let $\mathfrak{Z}$ be the center of $\mathfrak{B}$. Let $C[x]$ denote the ring of all commutative polynomials in l independent variables $x_1, \ldots, x_l$ with coefficients in C. We denote by β the isomorphic mapping of $C[x]$ into $\mathfrak{B}$ given by $\beta(x_1^{m_1}x_2^{m_2}\ldots x_l^{m_l}) = H_1^{m_1}\ldots H_l^{m_l}$. For any $z \in \mathfrak{Z}$ there exists a unique element[3] $\chi_x(z) \in C[x]$ such that $z - \beta(\chi_x(z)) \in \sum_{\alpha \in P} \mathfrak{B}X_\alpha$. Λ being any linear function on $\mathfrak{h}$ we denote by $\chi_\Lambda(z)$ the value of the polynomial $\chi_x(z)$ at $x_i = \Lambda(H_i)| \leqslant i \leqslant l$. Then the mapping $\chi_\Lambda : z \rightarrow \chi_\Lambda(z)(z \in \mathfrak{Z})$ is a homomorphism of $\mathfrak{Z}$ into C and conversely every homomorphism of $\mathfrak{Z}$ into C is of the form[3] χ_Λ for some $\Lambda \in \mathfrak{F}$. Let $\rho = {}^1/_2 \sum_{\alpha \in P} \alpha$ and let W be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then if $\Lambda_1, \Lambda_2 \in \mathfrak{F}$, $\chi_{\Lambda_1} = \chi_{\Lambda_2}$ if and only if[3] $s(\Lambda_1 + \rho) = \Lambda_2 + \rho$ for some $s \in W$.

We denote by Ω_F the set of all $\mathfrak{D} \in \Omega$ for which there exists a finite-dimensional representation π of G such that $\mathfrak{D}'$ occurs in the reduction of $\pi(K')$.

THEOREM 1. *Let π be a quasisimple*[1] *representation of G on a Banach space with the infinitesimal character*[4] *χ. Suppose $\mathfrak{D}$ is an element in Ω_F such that $\mathfrak{D}$ occurs*[5] *in π. Then there exists a linear function Λ on $\mathfrak{h}$ and a $\delta \in \omega$ such that $\delta < \mathfrak{D}$ and $\chi = \chi_\Lambda$ and $\Lambda(H) = \lambda_\delta(H)(H \in \mathfrak{h}_{\mathfrak{K}})$, where λ_δ is the highest weight of δ. Conversely suppose we are given a linear function Λ on $\mathfrak{h}$ and $\mathfrak{D} \in \Omega$ such that Λ coincides on $\mathfrak{h}_{\mathfrak{K}}$ with the highest weight λ_δ of some $\delta < \mathfrak{D}$ $(\delta \in \omega)$. Then there exists a quasisimple irreducible representation π of G on a Hilbert space with the infinitesimal character χ_Λ such that $\mathfrak{D}$ occurs in π.*

Remarks.—It seems likely that the first part of the above theorem is actually true for all $\mathfrak{D} \in \Omega$ and not merely for $\mathfrak{D} \in \Omega_F$, but so far it has not been possible to prove this. Notice that if G is a complex semisimple group $\Omega = \Omega_F$ and so in this case the theorem holds without any restriction on $\mathfrak{D}$.

Let π_1 and π_2 be two quasisimple representations of G on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. For any $\mathfrak{D} \in \Omega$ let $\mathfrak{H}_{i,\mathfrak{D}}$ denote the set of all elements in $\mathfrak{H}_i$ which transform[1] under $\pi_i(K)$ according to $\mathfrak{D}(i = 1, 2)$. Put $\mathfrak{H}_i^0 = \sum_{\mathfrak{D} \in \Omega} \mathfrak{H}_{i,\mathfrak{D}}$. Then we get a representation π_i^0 of $\mathfrak{B}$ on $\mathfrak{H}_i^0$ such that

$$\pi_i^0(X)\psi = \lim_{t \rightarrow 0} \frac{1}{t} \{\pi_i(\operatorname{expt} X)\psi - \psi\} \qquad (\psi \in \mathfrak{H}_i^0, X \in \mathfrak{g}_0, t \in R).$$

We say that π_1 and π_2 are infinitesimally equivalent if the representations π_1^0 and π_2^0 are algebraically equivalent, i.e., if there exists an isomorphism α of $\mathfrak{H}_1^0$ onto $\mathfrak{H}_2^0$ such that $\pi_2^0(b)\alpha\psi = \alpha\pi_1^0(b)\psi(b \in \mathfrak{B}, \psi \in \mathfrak{H}_1^0)$. Clearly if π_1 and π_2 are equivalent they are also infinitesimally equivalent. Conversely it can be shown that if π_1 and π_2 are both unitary, their infinitesimal equivalence implies their equivalence in the usual sense.

Let π be a quasisimple irreducible representation of G on a Hilbert space. We know (see Theorem 3 of *RI*) that dim $\mathfrak{H}_{\mathfrak{D}} < \infty$ for all $\mathfrak{D} \in \Omega$. Moreover, we may assume without loss of generality that the subspaces $\mathfrak{H}_{\mathfrak{D}}$ are all mutually orthogonal for distinct $\mathfrak{D}$. Let $E_{\mathfrak{D}}$ denote the orthogonal projection on $\mathfrak{H}$ on $\mathfrak{H}_{\mathfrak{D}}$. Put

$$\varphi_{\mathfrak{D}}^{\pi}(x) = sp(E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}}) \qquad (x \in G).$$

Then we can restate Theorem 7 of *RI* in a slightly improved form as follows.

THEOREM 2. *Let π_1, π_2 be irreducible quasisimple representations of G on two Hilbert spaces. Suppose that for some $\mathfrak{D} \in \Omega$ and $c \in C$, $\varphi_{\mathfrak{D}}^{\pi_1} = c\varphi_{\mathfrak{D}}^{\pi_2} \neq 0$. Then π_1 and π_2 are infinitesimally equivalent. Conversely if π_1 and π_2 are infinitesimally equivalent $\varphi_{\mathfrak{D}}^{\pi_1} = \varphi_{\mathfrak{D}}^{\pi_2}$ for all $\mathfrak{D} \in \Omega$.*

We have seen above that every element $x \in G$ can be written uniquely in the form $x = vhn(v \in K, h \in A_+, n \in N)$. For any $v \in K$ and $x \in G$ let v_x and $H(x, v)$ denote the unique elements in K and $\mathfrak{h}_{\mathfrak{P}_0}$, respectively, such that $xv = v_x(\exp H(x, v))n$ for some $n \in N$. Moreover let $\Gamma(v)$ denote the element in $\mathfrak{C}_0$ such that $v = (\exp \Gamma(v))u$ for some $u \in K'$. Put $\Gamma(x, v) = \Gamma(v_x) - \Gamma(v)$. Let Z be the center of G and let $z \rightarrow z^*$ denote the adjoint representation of G. Then $K \supset Z$ and K^* is compact. It is easily seen that $(va)_x = v_xa$, $H(x, va) = H(x, v)$, $\Gamma(x, va) = \Gamma(x, v)(a \in Z)$. Hence we may write $H(x, v) = H(x, v^*)$, $\Gamma(x, v) = \Gamma(x, v^*)$. Let dv^* denote the Haar measure on K^* such that $\int_{K^*} dv^* = 1$. For any $\mathfrak{D} \in \Omega$ let $\mu_{\mathfrak{D}}$ denote the linear function on $\mathfrak{C}$ such that

$$\sigma(\exp \Gamma) = e^{\mu_{\mathfrak{D}}(\Gamma)}\sigma(1) \qquad (\Gamma \in \mathfrak{C}_0)$$

for $\sigma \in \mathfrak{D}$. Also let $d(\mathfrak{D})$ denote the degree of σ.

THEOREM 3. *Let π be a quasisimple irreducible representation of G on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{D}$ be an element in Ω such that $d(\mathfrak{D}) = 1$ and $\mathfrak{D}$ occurs[5] in π. Then dim $\mathfrak{H}_{\mathfrak{D}} = 1$ and there exists a linear function Λ on $\mathfrak{h}$ such that χ_Λ is the infinitesimal[4] character of π and*

$$\varphi_{\mathfrak{D}}^{\pi}(x) = \int_{K^*} e^{\mu_{\mathfrak{D}}(\Gamma(x, v^*))}e^{\Lambda(H(x, v^*))}\, dv^* \qquad (x \in G).$$

[1] Harish-Chandra, PROC. NATL. ACAD. SCI., **37**, 170–173 (1951).

[2] Iwasawa, K., *Ann. Math.*, **50**, 507–558 (1949).

[3] Harish-Chandra, *Trans. Am. Math. Soc.*, **70**, 28–96 (1951).

[4] The infinitesimal character of π was called simply the character of π in *RI*.

[5] This means that $\mathfrak{D}$ occurs in the reduction of $\pi(K)$.