REPRESENTATIONS OF SEMISIMPLE LIE GROUPS ON A
BANACH SPACE

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Let $G$ be a connected semisimple Lie group and $\mathfrak{g}$ a Banach space. By a representation of $G$ on $\mathfrak{g}$ we mean a mapping $\pi$ which assigns to every $x \in G$ a bounded linear operator $\pi(x)$ on $\mathfrak{g}$ such that the following two conditions are fulfilled:

1. $\pi(xy) = \pi(x)\pi(y)$ and $\pi(1) = I$ (here 1 is the unit element of $G$ and $I$ the identity operator on $\mathfrak{g}$).
2. The mapping $(x, \psi) \rightarrow \pi(x)\psi(x \in G, \psi \in \mathfrak{g})$ is a continuous mapping of $G \times \mathfrak{g}$ into $\mathfrak{g}$.

The object of this note is to announce a few theorems on these representations. No attempt is made to give proofs here. A detailed account with complete proofs will appear elsewhere in another paper.

Let $\mathbb{R}$ and $\mathbb{C}$, respectively, be the fields of real and complex numbers. Let $\mathfrak{g}_0$ be the Lie algebra of $G$ and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. We denote the universal enveloping algebra of $\mathfrak{g}$ by $\mathcal{B}$. Let $C_c^\infty(G)$ be the set of all complex-valued functions on $G$ which are indefinitely differentiable everywhere and which vanish outside a compact set. Let $\pi$ be a representation of $G$ on $\mathfrak{g}$ and $V$ the set of all elements in $\mathfrak{g}$ which can be written as finite linear combinations of elements of the form

$$\int_G f(x)\pi(x)\psi \, dx \quad (\psi \in \mathfrak{g}, f \in C_c^\infty(G))$$

where $dx$ is the left invariant Haar measure on $G$. $V$ is called the Gårding subspace of $\mathfrak{g}$ (with respect to $\pi$). For every $X \in \mathfrak{g}_0$ we can define a linear transformation $\pi_v(X)$ of $V$ into itself such that

$$\pi_v(x)\psi = \lim_{t \to 0} \frac{1}{t} \left\{ \pi(\exp tX) - I \right\} \psi \quad (\psi \in V, X \in \mathfrak{g}_0).$$
The mapping $X \rightarrow \pi(X)$ is a representation of $\mathfrak{g}_0$ and therefore it can be extended uniquely to a representation $\pi$ of $\mathfrak{g}$ on $V$. Let $\psi \in V$. Consider $\pi(\mathfrak{g})\psi$ where the bar denotes closure in $\hat{\mathfrak{g}}$. It turns out that in general $\pi(\mathfrak{g})\psi$ is not invariant under $\pi(G)$. In order to avoid this unpleasant state of affairs we replace the Gårding subspace $V$ by the space of all well-behaved elements. This is defined as follows. Let $U$ be a subspace of $\hat{\mathfrak{g}}$ (not necessarily closed). We say that $U$ is well-behaved under $\pi$ if the following conditions hold.

(1) There exists a representation $\pi_U$ of $\mathfrak{g}_0$ on $U$ such that

$$\pi_U(X)\psi = \lim_{t \to 0} \frac{1}{t} \left\{ \pi(\exp tX) - I \right\} \psi \quad (X \in \mathfrak{g}_0, \psi \in U).$$

(2) For any continuous linear function $f$ on $\hat{\mathfrak{g}}$ and $\psi \in U$ the function

$$f(\pi(x)\psi) \quad (x \in G)$$

is an analytic function on $G$. It is clear that if $U_1$, $U_2$ are two well-behaved subspaces of $\hat{\mathfrak{g}}$ then $U_1 + U_2$ is also well-behaved. From this it follows that the union $W$ of all well-behaved subspaces in $\hat{\mathfrak{g}}$ is itself a well-behaved subspace. An element $\psi \in \hat{\mathfrak{g}}$ will be called well-behaved if $\psi \in W$ and $W$ is called the space of all well-behaved elements. It is clear that the mapping $X \rightarrow \pi_W(X) \quad (X \in \mathfrak{g}_0)$ can be extended uniquely to a representation $\pi_W$ of $\mathfrak{g}$ on $W$. The following theorem justifies the notion of well-behaved elements.

**Theorem 1.** Let $\psi$ be a well-behaved element of $\hat{\mathfrak{g}}$. Then $\pi_W(\mathfrak{g})\psi$ is invariant under $\pi(G)$.

Let $G^*$ be the adjoint group of $G$ and let $K^*$ be a maximal compact subgroup of $G^*$. Let $K$ be the complete inverse image of $K^*$ in $G$. Then $K$ is connected though it is not necessarily compact. Let $P$ be the set of all equivalence classes of finite-dimensional irreducible representations of $K$. Let $D \in P$ and $\psi \in \hat{\mathfrak{g}}$. We say that $\psi$ transforms under $\pi(K)$ according to $D$ if the space $U$ spanned by $\pi(u)\psi$ for all $u \in K$ is finite-dimensional and the representation of $K$ induced on $U$ is fully reducible into a direct sum of irreducible components each of which lies in $D$. Let $\hat{\mathfrak{g}}_D$ be the set of all elements in $\hat{\mathfrak{g}}$ which transform under $\pi(K)$ according to $D$. Put $W_D = W \cap \hat{\mathfrak{g}}_D$. Let $Z$ be the center of $G$ and $\mathfrak{z}$ the center of $\mathfrak{g}$. Let $V$ be the Gårding subspace of $\hat{\mathfrak{g}}$ and $\pi_V$ the representation of $\mathfrak{g}$ on $V$ as defined above. We shall say that $\pi$ is a quasi-simple representation of $G$ if there exist homomorphisms $c$ and $\chi$ of $Z$ and $\mathfrak{z}$ respectively, into $C$ such that the following two conditions hold:

1. $\pi(d) = c(d)I \quad (d \in Z)$.
2. $\pi_V(z)\psi = \chi(z)\psi \quad (z \in \mathfrak{z}, \psi \in V)$. 

In case condition (2) is satisfied we say that \( \pi \) has the character \( \chi \).

**Theorem 2.** Let \( \pi \) be a quasi-simple representation of \( G \) on \( \mathcal{H} \). Then 
\[
\sum_{\mathcal{D} \in \mathcal{P}} W_{\mathcal{D}} \text{ is dense in } \mathcal{H} \text{ and } \mathcal{H}_{\mathcal{D}} = W_{\mathcal{D}} \langle \mathcal{B} \in \mathcal{P} \rangle.
\]

Here \( \sum_{\mathcal{D} \in \mathcal{P}} W_{\mathcal{D}} \) is the space consisting of all finite linear combinations of elements in \( \bigcup_{\mathcal{D} \in \mathcal{P}} W_{\mathcal{D}} \).

**Theorem 3.** Let \( \pi \) be quasi-simple and let \( \psi \in \sum_{\mathcal{D} \in \mathcal{P}} W_{\mathcal{D}} \). Put 
\[
U = \pi_w(\mathcal{B})\psi \quad \text{and} \quad U_{\mathcal{D}} = U \cap \mathcal{H}_{\mathcal{D}}(\mathcal{D} \in \mathcal{P}).
\]
Then \( \pi_w(\mathcal{B})\psi = \sum_{\mathcal{D} \in \mathcal{P}} U_{\mathcal{D}} \) and 
\[
dim U_{\mathcal{D}} < \infty \text{ for every } \mathcal{D} \in \mathcal{P}.
\]

Let \( U, V \) be two subspaces of \( \mathcal{H} \). We write \( U > V \) or \( V < U \) if \( U \supset V \) and \( U \neq V \). Let \( U \) and \( V \) be two closed subspaces of \( \mathcal{H} \) invariant under \( \pi(G) \). We shall say that \( V \) is maximal in \( U \) if \( U > V \) and there exists no closed invariant subspace \( U_1 \) such that \( U > U_1 > V \).

**Theorem 4.** Let \( \pi \) be a quasi-simple representation of \( G \) on a Banach space \( \mathcal{H} \neq \{0\} \). Then there exist two closed invariant subspaces \( U \) and \( V \) in \( \mathcal{H} \) such that \( V \) is maximal in \( U \).

Now we shall consider the special case when \( \mathcal{H} \) is a Hilbert space and \( \pi \) is a unitary representation. From Theorem 4 we deduce the following result.

**Theorem 5.** Let \( \pi \) be a quasi-simple unitary representation of \( G \) on a Hilbert space \( \mathcal{H} \neq \{0\} \). Then there exists a closed invariant subspace \( U \) of \( \mathcal{H} \) such that \( \{0\} \) is maximal in \( U \) (i.e., \( U > \{0\} \)) and \( U \) is irreducible under \( \pi(G) \).

The above theorem has the following significance in relation to the theory of factors of Murray and von Neumann.\(^3\) Let \( \pi \) be a unitary representation of \( G \) on a Hilbert space \( \mathcal{H} \). Let \( \mathfrak{A} \) be the smallest weakly closed algebra of bounded operators on \( \mathcal{H} \) containing \( \pi(G) \). It is known\(^4\) that if \( \mathfrak{A} \) is a factor, i.e., if the center of \( \mathfrak{A} \) consists of scalar multiples of \( I \), then \( \pi \) is quasi-simple. Therefore in this case Theorem 5 is applicable and so it follows from the results of Murray and von Neumann\(^5\) that \( \mathfrak{A} \) must be a factor of type \( I_\infty \) or \( I_\infty \). This shows that factors of Type II and Type III cannot arise from a unitary representation of a semisimple Lie group on a Hilbert space. For the consequences of this result we refer the reader to the above-quoted paper of Mautner.

Let \( \mathcal{D} \in \mathcal{P} \). We say that \( \mathcal{D} \) occurs in \( \pi \) if \( \mathcal{H}_{\mathcal{D}} \neq \{0\} \). It is known that all unitary irreducible representations of \( G \) are quasi-simple.

**Theorem 6.** Let \( \chi \) be a homomorphism of \( \mathfrak{B} \) into \( C \) such that \( \chi(1) = 1 \) and let \( \mathcal{D} \in \mathcal{P} \). Let \( \mathcal{C}(\chi, \mathcal{D}) \) denote the set of all representations \( \pi \) of \( G \) which have the following properties:

1. \( \pi \) is an irreducible unitary representation of \( G \) on some Hilbert space.
2. \( \mathcal{D} \) occurs in \( \pi \).
\( \pi \text{ has the character } \chi. \)

Then \( \mathcal{E}(\chi, D) \) contains only a finite number of inequivalent representations.

Let \( d(D) \) denote the degree of any representation in \( D \). Suppose \( d(D) = 1 \), \( \pi \in \mathcal{E}(\chi, D) \) and \( S \) is the representation space of \( \pi \). Then it can be shown that \( \dim S_D = 1 \) and the maximum number of inequivalent representations in \( \mathcal{E}(\chi, D) \) is not greater than the order of the Weyl group of \( g \) with respect to some Cartan subalgebra. Moreover if \( G \) is a complex semi-simple Lie group all representations in \( \mathcal{E}(\chi, D) \) are equivalent (provided \( d(D) = 1 \)).

Let \( \pi \) be an irreducible unitary representation of \( G \) on a Hilbert space \( \mathfrak{H} \). For any \( D \in P \) let \( E_D \) denote the orthogonal projection of \( \mathfrak{H} \) on \( D_D \).

Since \( \pi \) is irreducible and unitary it is quasi-simple and therefore it follows from Theorem 4 that \( \dim S_D < \infty \). Hence the function \( \phi_D^\pi(x) = sp(E_D \pi(x)E_D)(x \in G) \) is well defined. It is an analytic function on \( G \) and \( \phi_D^\pi(u x u^{-1}) = \phi_D^\pi(x) \) \((u \in K)\).

**Theorem 7.** Let \( \pi_1 \) and \( \pi_2 \) be irreducible unitary representations of \( G \) on two Hilbert spaces. Let \( P_{\pi_1} \) be the set of all elements in \( P \) which occur in \( \pi_1 \). Suppose that for some \( D \in P_{\pi_1} \) and \( c \in C \),

\[
\phi_D^{\pi_1} = c \phi_D^{\pi_2}.
\]

Then \( \pi_1 \) and \( \pi_2 \) are equivalent. Conversely if \( \pi_1 \) and \( \pi_2 \) are equivalent

\[
\phi_D^{\pi_1} = \phi_D^{\pi_2},
\]

for all \( D \in P \).

In case \( d(D) = 1 \) the function \( \phi_D^\pi \) is the same as the spherical function introduced by Gelfand and Naimark.\(^5\) Now suppose \( G \) is a complex semi-simple Lie group and \( d(D) = 1 \). Then we have seen above that the representation \( \pi \) is completely determined within equivalence by its character \( \chi \). Actually in this case an explicit formula for \( \phi_D^\pi \) in terms of \( \chi \) can be obtained quite easily. This formula is very similar to the one given by Gelfand and Naimark\(^6\) for representations of the principal series.