REPRESENTATIONS OF SEMISIMPLE LIE GROUPS ON A BANACH SPACE

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Let G be a connected semisimple Lie group and $\mathfrak{H}$ a Banach space. By a representation of G on $\mathfrak{H}$ we mean a mapping π which assigns to every $x \in G$ a bounded linear operator $\pi(x)$ on $\mathfrak{H}$ such that the following two conditions are fulfilled:

(1) $\pi(xy) = \pi(x)\pi(y)$ and $\pi(1) = I$ (here 1 is the unit element of G and I the identity operator on $\mathfrak{H}$).

(2) The mapping $(x, \psi) \rightarrow \pi(x)\psi(x \in G, \psi \in \mathfrak{H})$ is a continuous mapping of $G \times \mathfrak{H}$ into $\mathfrak{H}$.

The object of this note is to announce a few theorems on these representations. No attempt is made to give proofs here. A detailed account with complete proofs will appear elsewhere in another paper.

Let R and C, respectively, be the fields of real and complex numbers. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. We denote the universal enveloping[1] algebra of $\mathfrak{g}$ by $\mathfrak{B}$. Let $C_c^{\infty}(G)$ be the set of all complex-valued functions on G which are indefinitely differentiable everywhere and which vanish outside a compact set. Let π be a representation of G on $\mathfrak{H}$ and V the set of all elements in $\mathfrak{H}$ which can be written as finite linear combinations of elements of the form

$$\int_G f(x)\pi(x)\psi\, dx \qquad (\psi \in \mathfrak{H}, f \in C_c^{\infty}(G))$$

where dx is the left invariant Haar measure on G. V is called the Gårding subspace[2] of $\mathfrak{H}$ (with respect to π). For every $X \in \mathfrak{g}_0$ we can define a linear transformation $\pi_V(X)$ of V into itself such that

$$\pi_V(x)\psi = \lim_{t \rightarrow 0} \frac{1}{t}\left\{\pi(\exp tX) - I\right\}\psi \qquad (\psi \in V, t \in R).$$

The mapping $X \to \pi_V(X)$ is a representation of $\mathfrak{g}_0$ and therefore it can be extended uniquely to a representation π_V of $\mathfrak{B}$ on V. Let $\psi \in V$. Consider $\overline{\pi_V(\mathfrak{B})\psi}$ where the bar denotes closure in $\mathfrak{H}$. It turns out that in general $\overline{\pi_V(\mathfrak{B})\psi}$ is not invariant under $\pi(G)$. In order to avoid this unpleasant state of affairs we replace the Gårding subspace V by the space of all well-behaved elements. This is defined as follows. Let U be a subspace of $\mathfrak{H}$ (not necessarily closed). We say that U is well-behaved under π if the following conditions hold.

(1) There exists a representation π_U of $\mathfrak{g}_0$ on U such that

$$\pi_U(X)\psi = \lim_{t \to 0} \frac{1}{t} \left\{\pi(\exp tX) - I\right\} \psi \qquad (X \in \mathfrak{g}_0, \psi \in U).$$

(2) For any continuous linear function f on $\mathfrak{H}$ and $\psi \in U$ the function

$$f(\pi(x)\psi) \qquad (x \in G)$$

is an analytic function on G. It is clear that if U_1, U_2 are two well-behaved subspaces of $\mathfrak{H}$ then $U_1 + U_2$ is also well-behaved. From this it follows that the union W of all well-behaved subspaces in $\mathfrak{H}$ is itself a well-behaved subspace. An element $\psi \in \mathfrak{H}$ will be called well-behaved if $\psi \in W$ and W is called the space of all well-behaved elements. It is clear that the mapping $X \to \pi_W(X)$ $(X \in \mathfrak{g}_0)$ can be extended uniquely to a representation π_W of $\mathfrak{B}$ on W. The following theorem justifies the notion of well-behaved elements.

THEOREM 1. *Let ψ be a well-behaved element of $\mathfrak{H}$. Then $\overline{\pi_W(\mathfrak{B})\psi}$ is invariant under $\pi(G)$.*

Let G^* be the adjoint group of G and let K^* be a maximal compact subgroup of G^*. Let K be the complete inverse image of K^* in G. Then K is connected though it is not necessarily compact. Let P be the set of all equivalence classes of finite-dimensional irreducible representations of K. Let $\mathfrak{D} \in$ P and $\psi \in \mathfrak{H}$. We say that ψ transforms under $\pi(K)$ according to $\mathfrak{D}$ if the space U spanned by $\pi(u)\psi$ for all $u \in K$ is finite-dimensional and the representation of K induced on U is fully reducible into a direct sum of irreducible components each of which lies in $\mathfrak{D}$. Let $\mathfrak{H}_{\mathfrak{D}}$ be the set of all elements in $\mathfrak{H}$ which transform under $\pi(K)$ according to $\mathfrak{D}$. Put $W_{\mathfrak{D}} = W \cap \mathfrak{H}_{\mathfrak{D}}$. Let Z be the center of G and $\mathfrak{Z}$ the center of $\mathfrak{B}$. Let V be the Gårding subspace of $\mathfrak{H}$ and π_V the representation of $\mathfrak{B}$ on V as defined above. We shall say that π is a quasi-simple representation of G if there exist homomorphisms c and χ of Z and $\mathfrak{Z}$ respectively, into C such that the following two conditions hold:

$$(1) \quad \pi(d) = c(d)I \qquad (d \in Z).$$

$$(2) \quad \pi_V(z)\psi = \chi(z)\psi \qquad (z \in \mathfrak{Z}, \psi \in V).$$

In case condition (2) is satisfied we say that π has the character χ.

THEOREM 2. *Let π be a quasi-simple representation of G on $\mathfrak{H}$. Then* $\sum_{\mathfrak{D} \epsilon \mathrm{P}} W_{\mathfrak{D}}$ *is dense in $\mathfrak{H}$ and* $\mathfrak{H}_{\mathfrak{D}} = \overline{W_{\mathfrak{D}}}$ $(\mathfrak{B} \epsilon \mathrm{P})$.

Here $\sum_{\mathfrak{D} \epsilon \mathrm{P}} W_{\mathfrak{D}}$ is the space consisting of all finite linear combinations of elements in $\bigcup_{\mathfrak{D} \epsilon \mathrm{P}} W_{\mathfrak{D}}$.

THEOREM 3. *Let π be quasi-simple and let* $\psi \epsilon \sum_{\mathfrak{D} \epsilon \mathrm{P}} W_{\mathfrak{D}}$. *Put* $U = \overline{\pi_W(\mathfrak{B})\psi}$ *and* $U_{\mathfrak{D}} = U \cap \mathfrak{H}_{\mathfrak{D}}(\mathfrak{D} \epsilon \mathrm{P})$. *Then* $\pi_W(\mathfrak{B})\psi = \sum_{\mathfrak{D} \epsilon \mathrm{P}} U_{\mathfrak{D}}$ *and* $\dim U_{\mathfrak{D}} < \infty$ *for every* $\mathfrak{D} \epsilon \mathrm{P}$.

Let U, V be two subspaces of $\mathfrak{H}$. We write $U > V$ or $V < U$ if $U \supset V$ and $U \neq V$. Let U and V be two closed subspaces of $\mathfrak{H}$ invariant under $\pi(G)$. We shall say that V is maximal in U if $U > V$ and there exists no closed invariant subspace U_1 such that $U > U_1 > V$.

THEOREM 4. *Let π be a quasi-simple representation of G on a Banach space* $\mathfrak{H} \neq \{0\}$. *Then there exist two closed invariant subspaces U and V in $\mathfrak{H}$ such that V is maximal in U.*

Now we shall consider the special case when $\mathfrak{H}$ is a Hilbert space and π is a unitary representation. From Theorem 4 we deduce the following result.

THEOREM 5. *Let π be a quasi-simple unitary representation of G on a Hilbert space* $\mathfrak{H} \neq \{0\}$. *Then there exists a closed invariant subspace U of $\mathfrak{H}$ such that $\{0\}$ is maximal in U (i.e., $U > \{0\}$ and U is irreducible under $\pi(G)$).*

The above theorem has the following significance in relation to the theory of factors of Murray and von Neumann.[3] Let π be a unitary representation of G on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{A}$ be the smallest weakly closed algebra of bounded operators on $\mathfrak{H}$ containing $\pi(G)$. It is known[4] that if $\mathfrak{A}$ is a factor, i.e., if the center of $\mathfrak{A}$ consists of scalar multiples of I, then π is quasi-simple. Therefore in this case Theorem 5 is applicable and so it follows from the results of Murray and von Neumann[3] that $\mathfrak{A}$ must be a factor of type I_n or I_∞. This shows that factors of Type II and Type III cannot arise from a unitary representation of a semisimple Lie group on a Hilbert space. For the consequences of this result we refer the reader to the above-quoted paper of Mautner.

Let $\mathfrak{D} \epsilon \mathrm{P}$. We say that $\mathfrak{D}$ occurs in π if $\mathfrak{H}_{\mathfrak{D}} \neq \{0\}$. It is known that all unitary irreducible representations of G are quasi-simple.

THEOREM 6. *Let χ be a homomorphism of $\mathfrak{Z}$ into C such that $\chi(1) = 1$ and let $\mathfrak{D} \epsilon \mathrm{P}$. Let $\mathfrak{C}(\chi, \mathfrak{D})$ denote the set of all representations π of G which have the following properties:*

(1) *π is an irreducible unitary representation of G on some Hilbert space.*

(2) *$\mathfrak{D}$ occurs in π.*

(3) *π has the character χ.*

Then $\mathfrak{E}(\chi, \mathfrak{D})$ contains only a finite number of inequivalent representations.

Let $d(\mathfrak{D})$ denote the degree of any representation in $\mathfrak{D}$. Suppose $d(\mathfrak{D}) = 1$, $\pi \epsilon \mathfrak{E}(\chi, \mathfrak{D})$ and $\mathfrak{H}$ is the representation space of π. Then it can be shown that $\dim \mathfrak{H}_{\mathfrak{D}} = 1$ and the maximum number of inequivalent representations in $\mathfrak{E}(\chi, \mathfrak{D})$ is not greater than the order of the Weyl group of $\mathfrak{g}$ with respect to some Cartan subalgebra. Moreover if G is a complex semi-simple Lie group all representations in $\mathfrak{E}(\chi, \mathfrak{D})$ are equivalent (provided $d(\mathfrak{D}) = 1$).

Let π be an irreducible unitary representation of G on a Hilbert space $\mathfrak{H}$. For any $\mathfrak{D} \epsilon \mathrm{P}$ let $E_{\mathfrak{D}}$ denote the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{D}_{\mathfrak{D}}$. Since π is irreducible and unitary it is quasi-simple and therefore it follows from Theorem 4 that $\dim \mathfrak{H}_{\mathfrak{D}} < \infty$. Hence the function $\varphi_{\mathfrak{D}}^{\pi}(x) = sp(E_{\mathfrak{D}}\ \pi(x)E_{\mathfrak{D}})(x \epsilon G)$ is well defined. It is an analytic function on G and $\varphi_{\mathfrak{D}}^{\pi}(u\, x\, u^{-1}) = \varphi_{\mathfrak{D}}^{\pi}(x)$ $(u \epsilon K)$.

THEOREM 7. *Let π_1 and π_2 be irreducible unitary representations of G on two Hilbert spaces. Let P_{π_1} be the set of all elements in P which occur in π_1. Suppose that for some $\mathfrak{D} \epsilon \mathrm{P}_{\pi_1}$ and $c \epsilon C$,*

$$\varphi_{\mathfrak{D}}^{\pi_1} = c\varphi_{\mathfrak{D}}^{\pi_2}.$$

Then π_1 and π_2 are equivalent. Conversely if π_1 and π_2 are equivalent

$$\varphi_{\mathfrak{D}}^{\pi_1} = \varphi_{\mathfrak{D}}^{\pi_2}$$

for all $\mathfrak{D} \epsilon \mathrm{P}$.

In case $d(\mathfrak{D}) = 1$ the function $\varphi_{\mathfrak{D}}^{\pi}$ is the same as the spherical function introduced by Gelfand and Naimark.[5] Now suppose G is a complex semi-simple Lie group and $d(\mathfrak{D}) = 1$. Then we have seen above that the representation π is completely determined within equivalence by its character χ. Actually in this case an explicit formula for $\varphi_{\mathfrak{D}}^{\pi}$ in terms of χ can be obtained quite easily. This formula is very similar to the one given by Gelfand and Naimark[5] for representations of the principal series.

[1] See, for example, Harish-Chandra, *Ann. Math.*, **50,** 900–915 (1949).

[2] Garding, L., PROC. NATL. ACAD. SCI., **33,** 331–332 (1947).

[3] Murray, F. J., and von Neumann, J., *Ann. Math.*, **37,** 116–229 (1936).

[4] See Mautner, F. I., *Ann. Math.*, **52,** 528–556 (1950).

[5] Gelfand, I. M., and Naimark, M. A., *Doklady Akad. Nauk SSR* (*N. S.*), **63,** 225–228 (1948).