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Multicritical two-dimensional vertex models

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We study the multicritical behavior of a class of two-dimensional ice-type vertex models on different lattices using renormalization-group theory. The models are classified by an integer m, with $m=2$ corresponding to the known square lattice case. For $m>2$, the specific-heat exponent is $\alpha=(m-2)/(m-1)$ with an upper critical dimensional confluent $(\ln t)^{1/2}$ divergence for $m=3$. The nature of the transition is similar to the mth-order multicritical point, yet the exponents are *not* those known from $c<1$ conformal invariance. The models are anisotropic with $\nu_{\parallel}=1$ and $\nu_{\perp}=\frac{1}{2}$. A few special features of the models are discussed.

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The exact solutions of two-dimensional vertex models are well known with far-reaching consequences for our understanding of critical phenomena, and beyond [1]. These easy-to-define square-lattice (and a few triangular and Kagomé lattices [1(b),2]) models, with local geometric constraints, encompass a huge class of known critical behaviors, like the mth-order multicritical points [3,4], Z_N critical points [3,5], Kasteleyn-Pokrovsky-Talapov type [6], F or Kosterlitz-Thouless type [7], etc. In this paper we show that there is a generalization of the simple five-vertex model on different lattices that show distinct multicritical behavior. The first nontrivial one in this series is a version of the triangular-lattice vertex model (Fig. 1) which is really a model at its upper critical dimension. The specific heat diverges near the critical point T_c on the high-temperature side as $c\sim t^{-1/2}(\ln t)^{1/2}$, where $t\equiv(T-T_c)$, T being the temperature. The square-root logarithmic contribution is actually the signature of upper critical dimensionality. It is a coincidence that exponent $\alpha=\frac{1}{2}$ is the same as for the K model, or the square-lattice ferroelectric five- or six-vertex models [1(a),1(b)]—the underlying physics is completely different. The models are characterized by an integer m such that the specific heat for $m>3$ behaves like $c\sim t^{-(m-2)/(m-1)}$. ($m=3$ corresponds to the triangular lattice case; $m=2$, the square-lattice five-vertex model.) These represent multicritical points in an extended parameter space, and the specific heat or the free-energy singularity is identical to that of a conventional *mean-field* multicritical point of order m (say, ϕ^{2m} theory), but surprisingly *in two dimensions* and without any long-range interaction. Hence, these are not the conventional mth-order multicritical points as known from conformal invariance with the central charge $c<1$ [3,4]. Furthermore, we show that these models are anisotropic in space for all m. It seems that the multicriticality found here are the anisotropic ("nonrelativistic") counterpart of the conventional isotropic ("relativistic") multicritical points.

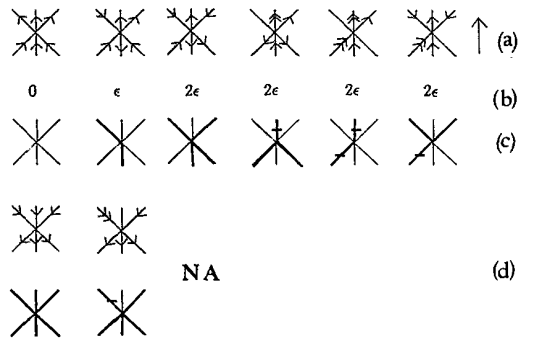

FIG. 1. (a) A few allowed vertices for $m=3$ on a triangular lattice, (b) energies, and (c) equivalent directed walks (thick lines), a horizontal bar indicating an additional walker on the edge. Vertex 0 with all arrows up constitutes a nondegenerate ground state. Excited states are created by flipping arrows and maintaining the "ice rule." The flipped arrows can, however, be on the same edge. Only one vertex of each type is shown; others can be obtained by permutations of the arrows. For a vertex of coordination number $2q$, the number of arrows in equals the number out which equals q ("ice rule"). (d) Three walker excitations are not allowed (NA). They are allowed for $m>3$.

The genesis of the models is in the recent formulation of the conventional ferroelectric five-vertex ($m=2$ in our classification) model in continuum for arbitrary dimensions through interacting directed walks [8–10]—a path-integral approach borrowed from the flux-lattice-melting (FLM) theory [11]. We showed that, using renormalization-group (RG) theory with dimensional regularization several quantities, like the fixed point, the second virial coefficient, and even the exponents (both thermodynamic and length scales), for the vertex model can be determined exactly [9,10]. Generalizations of the two-body Hamiltonian can be studied as well [12]. These were partly motivated by the recent attempts to handle the impurity-induced attraction in FLM, whence, for stability, higher-order repulsive interactions are required [13]. It was found, remarkably, that the generalized Hamiltonian with pure m-body interaction yielded, in an RG approach, an exact fixed point and an exact closed form mth-order virial coefficient [12]. These generalized many-body Hamiltonians, in turn, led to the models discussed in this paper through the mapping between vertex models and directed walks. After all, it is a truth universally acknowledged

that a simple model in possession of good (rich) physics must not be in dearth of applications [14].

Let us consider the triangular lattice with the diagonal direction as the z axis. (This is a preferred direction because, e.g., the vertex with all edges occupied is not allowed. This vertex would correspond to a complete polarization in the $-z$ direction.) We enforce the "ice rule" at each vertex but relax the "one-edge–one-arrow" criterion. There is a unique ground state with $+z$ polarization, and the excited states are obtained, as usual, by flipping arrows (see Fig. 1). The important points to note are that in the walk picture (i) the directed lines traverse the lattice in the z direction, (ii) each point visited by a walker costs an energy ε, (iii) two walkers can meet without paying any price, and continue their random walk, but (iv) three walkers are not allowed to be at the same site. It is this last restriction that makes it a three-body-interaction ($m=3$) system without any two-body interaction. (The square-lattice five-vertex model [1,8] forbids even two walkers at the same site and corresponds to the $m=2$ case. See below for further generalization.) It transpires that, unlike the $m=2$ case, a free-fermion picture is not suitable for this three-body interacting system. We might add here that the $m=2$ case has been studied in various forms in connection with the commensurate-incommensurate transition, and several types of multicritical features have been discussed (see, e.g., [6(b),15]), the models of this paper are not any of those.

Following the standard analysis for the five-vertex or the K model (see Refs. [6,9] and references therein), we conclude that the low-temperature phase of the model is just the ground state up to $T_c=\varepsilon/k\ln 3$. The transition temperature is obtained by studying the stability of the ground state with respect to the first excited state with free energy $F_1/N=\varepsilon-kT\ln 3$ for a lattice of size N in the z direction ($N\to\infty$). Our interest is, however, in the critical behavior and not in T_c. The thermodynamics is obtained by studying the excited states and noting that the number of chains is a layer-to-layer conserved quantity (i.e., a block-diagonal transfer matrix as for the conventional vertex models [1]). If λ_n is the number of configurations of n directed walks on the lattice, the free-energy density can be written as

$$F_n/NL=-a(T-T_c)n/L-kTL^{-1}\ln(\lambda_n^{1/N}/3^n)\,, \tag{1}$$

where $L(\to\infty)$ is the transverse size of the lattice. [Throughout the paper a ($=k\ln 3$, here) will stand for some unimportant constant, not necessarily the same everywhere.] In the thermodynamic limit ($N,L,n\to\infty$ with $\rho\equiv n/L$ a fixed value), free energy is to be a function of ρ and T and, therefore, we write, suppressing the unimportant T dependence,

$$f(\rho)=-at\rho+s(\rho)\,, \tag{2}$$

where $s(\rho)=\lim_{N,L\to\infty}kTL^{-1}\ln(3^n/\lambda_n^{1/N})$. This entropylike function is nonzero because of the interaction of the walkers and the problem is to evaluate the ρ dependence of $s(\rho)$. The thermodynamics is obtained by minimizing $f(\rho)$ with respect to ρ at a fixed t (i.e., finding the largest eigenvalue among the blocks of the transfer matrix), so that as $t\to 0$, the density ρ, and the specific heat c are given by

$$\partial s(\rho)/\partial\rho\sim t,\quad c\sim\partial\rho/\partial t\,. \tag{3}$$

In case there is a simple power-law dependence $s(\rho)\sim\rho^{\theta}$ as $\rho\to 0$, then $\rho\sim t^{\bar\beta}$ with

$$\bar\beta=(\theta-1)^{-1},\quad \alpha=1-\bar\beta\,. \tag{4}$$

For $m=2$, $\bar\beta$ is known as the incommensuration exponent [6,9]. These exponents are known for the five-vertex models for all dimensions, and in two dimensions $\theta=3$ producing $\bar\beta=\alpha=\frac{1}{2}$.

Once it is recognized that the entropy function $s(\rho)$ originates from the effective repulsion among the walkers, it is possible to obtain it through a path-integral approach in the continuum. Since the directed walks are really random walks in the transverse $d'=d-1$ directions, the walks are replaced by continuum Gaussian chains ("random walks") interacting with a short-range δ-function potential that contributes only if m (three here) chains have the same z coordinate. ($d'=1$ for the problem in hand, but general d' will be required soon.) Generalizing the two-body Hamiltonian [9,11] to m-chain interaction ($m=3$ here) we take [12]

$$H=\frac{1}{2}\sum_{\alpha}\int_0^N\left[\frac{\partial\mathbf{r}_\alpha}{\partial z}\right]^2 dz+\sum_{m\text{ chains}}v_m\int_0^N dz\,\delta^{d'}(\mathbf{r}_{\alpha_1}(z)-\mathbf{r}_{\alpha_2}(z))\cdots\delta^{d'}(\mathbf{r}_{\alpha_1}(z)-\mathbf{r}_{\alpha_m}(z))\,, \tag{5}$$

where $\mathbf{r}_\alpha(z)$ is the d' dimensional position of point z on chain α each of length N ($N\to\infty$). The v_m term is the repulsive m-body coupling constant (v_3 will be considered for the time being). We obtain $s(\rho)$ of Eq. (2) from this H by computing the v_m (v_3 now) dependent part of the free energy of n chains in the canonical ensemble, i.e., from the difference between the free energy of the interacting system and that of the corresponding noninteracting one.

A simple dimensional analysis shows that v_m is dimensionless at $d'=2/(m-1)$ so that v_3 is dimensionless at $d'=1$. This shows that the two-dimensional vertex model of Fig. 1 is at its upper critical dimension (UCD), demanding special care for the interaction term. We will study the problem through RG theory following the methods of Refs. [9,10]. This is facilitated by introducing the local-density variable $\rho(\mathbf{r},z)=\sum_\alpha\delta(\mathbf{r}(z)-\mathbf{r}_\alpha(z))$, in terms of which the interaction term in Eq. (5) is $\int_0^N dz\int d\mathbf{r}\,\rho^m(\mathbf{r},z)$.

In a mean-field (MF) approach, we ignore the fluctuation in ρ, to get

$$s(\rho)=av_m\rho^m \tag{6}$$

(a again a numerical constant). Note that the square-lattice five-vertex model also has a ρ^3 dependence [6], but, as shown in Ref. [9], this is a pure fluctuation effect of the two-body interaction (UCD is 3) in contrast to the MF three-body contribution here. On minimizing the free en-

ergy, Eq. (2), with this MF $s(\rho)$ for $m=3$, we get $c \sim t^{-1/2}$, similar to the five-vertex problem.

To go beyond the MF result, we calculate the effective coupling constant on any arbitrary length scale and then compute it on a correlation length scale $\sim\rho^{-1/d'}$ by integrating the β function or the RG recursion relation. This effective interaction (renormalized but not rescaled) can then be used for v_3 in Eq. (6) with $m=3$.

Let us define a dimensionless coupling constant with an arbitrary length scale L as

$$\zeta_m=(2\pi)^{(m-1)d'/2}m^{-d'/2}v_m L^{2-(m-1)d'}$$

($m=3$ for the present case). The β function for ζ_m is obtained, in the dimensional regularization scheme, from the mth (third for $m=3$) virial coefficient by absorbing the divergences in the perturbation series [12]. The resulting flow equation is

$$L\frac{\partial\zeta_m}{\partial L}=\epsilon_m\zeta_m\left[1-\frac{2\zeta_m}{\epsilon_m}\right], \tag{7}$$

where $\epsilon_m=2-(m-1)d'\geq 0$. It is shown in Ref. [12] that this equation is exact, i.e., valid to all orders of perturbation, just like the $m=2$ case of Ref. [10], and has no explicit m dependence—m is hidden in ζ_m and ϵ_m.

For $m=3$ and $d'=1$, as needed here, $\epsilon_3=0$, and we have $L\partial\zeta_m/\partial L=-2\zeta_m^2$. Integrating this to $L\sim\rho^{-1}$, we get the effective coupling as $v_3(\rho)\sim 1/\ln\rho$. On substitution in Eq. (6), this gives, from Eq. (2), $f(\rho)\sim -t\rho+v_3\rho^3/\ln\rho$, yielding $c\sim t^{-1/2}(\ln t)^{1/2}$. *The logarithm factor is a pure UCD effect.*

Let us now consider generalizations of the triangular-lattice model. We restrict ourselves to lattices with points of only even coordination numbers, maximum being $2m$. For example, one can take the union-jack lattice for $m=4$, or may construct lattices by adding the appropriate number of diagonals through neighboring squares of a square lattice. The energies of the vertices are chosen such that up to $m-1$ lines can meet at a point but not m lines. The models are, therefore, characterized by m which is related to the highest coordination number of the lattice and is the *minimum* number of lines that are *not* allowed to intersect. The continuum Hamiltonian is again given by Eq. (5) involving $m-1$ δ functions.

The upper critical dimension for the vertex model with m chain interaction is $d_m=1+2/(m-1)$ which is less than 2 if $m>3$. Hence, for the two-dimensional model, an MF analysis suffices yielding $s(\rho)\sim\rho^m$ with $\theta=m$. This gives [see Eq. (4)], for $m\geq 3$,

$$\bar{\beta}=(m-1)^{-1},\quad \alpha=(m-2)/(m-1). \tag{8}$$

We now determine the length scale exponents through a finite-size scaling (FSS) analysis. We just point out the crucial steps in the arguments, for details see Ref. [9]. Since FSS in its simplest form is applicable for dimensions below the UCD, we now take $d<d_m$.

Consider a finite lattice in the z direction. This implies that the z integrations in Eqs. (5) are for finite lengths N. Such interacting, large but finite polymers are described by scaling equations for, say, the osmotic pressure Π [16]

$$\Pi/kT\rho=g(\rho/\rho^*), \tag{9}$$

where ρ^* is the overlap concentration determining the concentration at which the mutual interaction is important and $g(x)$ is the scaling function. For polymers, this scaled variable ρ/ρ^* controls the crossover of any physical quantity from zero density (dilute limit or single-chain behavior) to nonzero density. The scale of concentration, ρ^*, is quantitatively determined by the appropriate virial coefficient as say the second virial coefficient for the two-chain interaction [16]. For the pure m-chain interaction, the first nonzero virial coefficient is the mth virial coefficient A_m, and $\rho^*=(A_m)^{-1/(m-1)}$. Since this variable is a function of N, and the osmotic pressure or the polymer free energy is related to the vertex-model free energy, Eq. (1), the N dependence should be what FSS dictates. Therefore, A_m is needed.

For the Hamiltonian of Eq. (5), A_m can be computed exactly at the RG fixed point $\zeta_m^*=\pi\epsilon_m$, and we find $a_m\sim N^{d'(m-1)/2}$ [12]. This result follows from the observation that the transverse size of a walk is $\langle R^2\rangle\sim N$ and A_m should have a dimension of $L^{d'(m-1)}$. This simplicity is a consequence of the absence of any anomalous dimensions mainly because of lack of any self-interaction [9,12].

Using the N dependence of A_m, and $\rho\sim t^{\bar{\beta}}$, Eq. (4), the scaled variable can be written as $tN^{d'/(2\bar{\beta})}$. According to FSS, near a critical point, the FSS variable is $tN^{1/\nu_\parallel}$ where $\nu_\parallel$ is the length-scale exponent in the z direction. A comparison of the two variables yields $\nu_\parallel$. The length-scale exponent in the transverse direction is given by the average distance between the lines, i.e., $\xi_\perp\sim\rho^{-1/d'}$. Hence, we obtain ($\xi_i\sim t^{-\nu_i}$ with i parallel or perpendicular)

$$\nu_\parallel=2\nu_\perp=2\bar{\beta}/d'. \tag{10}$$

The factor of 2 between $\nu_\parallel$ and $\nu_\perp$ can be traced to the differential treatment of $\mathbf{r}$ and z in Eq. (5).

We still have to calculate $\bar{\beta}$ for $d<d_m$. The procedure is similar to that of the three-chain case. Integrate the exact recursion relation for the m-chain interaction as given by Eq. (7) up to the transverse correlation length $\xi_\perp$, and obtain $v_m(\rho)\sim\rho^{[2-d'(m-1)]/d'}$. (Note again that there is no anomalous dimension.) Substituting this in the MF expression, Eq. (6), and minimizing Eq. (2), we obtain $\bar{\beta}=d'/2$ which is identical to the two-chain problem [9] but differs, as it should, from the MF result of Eq. (8). With this $\bar{\beta}$, we finally have

$$\nu_\parallel=2\nu_\perp=1. \tag{11}$$

These results are independent of m and d and agree with the exact result for the five-vertex model or the isomorphic K model [17]. From d independence it follows that they are also true for $d>d_m$.

For $d>d_m$, one expects the exponents to be the same as at d_m. This self-consistency is easily checked from the anisotropic hyperscaling, which gives $\alpha=2-(d-1)\nu_\perp-\nu_\parallel=(3-d)/2$. At $d=d_m=(m+1)/(m-1)$ the highest d for which hyperscaling is expected to be valid, we find $\alpha=(m-2)/(m-1)$ agreeing with the result of Eq. (8) (see Fig. 2).

The multicritical nature of these critical points follows from the simple fact that all the lower-order interactions are relevant at the transition. We do not elaborate on this, but just point out the MF analysis for the $m=3$ case.

Adding a two-body term in Eq. (1), we get $f(\rho) \sim -t\rho+v_2\rho^2+v_3\rho^3$. If $v_2>0$, then the two-body term dominates and the critical behavior corresponds to the $m=2$ case. If $v_2<0$ then there is a first-order transition [18]. The $m=3$ point is, therefore, analogous to a tricritical point. Generalizations follow suit.

We now point out a few special features of the models.

(1) The value of α is identical to that of a *classical* mth-order multicritical point described by, say, a ϕ^{2m} theory, and has a few formal similarities as mentioned earlier. However, a ϕ^{2m} theory has UCD of $2m/(m-2)$, and, therefore, possesses nontrivial exponents in two dimensions as given by conformal invariance with central charge $c<1$ or as found in the Andrews-Baxter-Forrester model [3,4]. In contrast, we are seeing this MF-like exponent in two dimensions. Moreover, the $m=2$ transition corresponds to a conformal field theory with central charge $c=1$ [19]. Also, the path-integral representation of Eq. (5) ensures a Hermitian "quantum" Hamiltonian, ensuring, in turn, unitarity of the models of this paper. The question of conformal invariance requires further studies.

(2) The models are anisotropic with two length-scale exponents $\nu_\parallel$ and $\nu_\perp$, Eq. (11), which are independent of m and d. As a consequence all of these models have identical exponents for $d<d_m$, even though the exponents like α and $\bar{\beta}$ are m dependent for $d>d_m$ (see Fig. 2). The fluctuation for $d<d_m$ conspires with the ρ^m term, Eq. (6), to wash out the m dependence.

(3) For $m\to\infty$, $d_m\to 1$, and α becomes 1 which indicates a first-order transition. It is straightforward to see that the transition is first order for all m in one dimension [20].

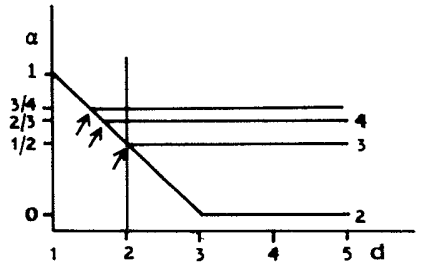

FIG. 2. Plot of α (specific-heat exponent) vs d. The numbers indicate the m values. An extra logarithmic factor for an m $[(\ln t)^{1/(m-1)}]$ appears at the corresponding UCD. These are indicated by the arrows. The vertical line is the $d=2$ line.

In conclusion, we have studied a class of vertex models classified by an integer m that is related to the highest coordination number of the lattice and/or the minimum number of lines that are not allowed to intersect. We used exact RG equations in a continuum path-integral formulation. The multicritical points found have similarities with the ordinary mth-order multicritical points, and the specific-heat exponent [Eq. (8)] is that of the MF ϕ^{2m} theory *but in two dimensions.* There is an additional logarithmic factor for $m=3$. The models are anisotropic with two length-scale exponents [see Eq. (11)], and are therefore not a ϕ^{2m} theory for which spatial anisotropy is irrelevant. The questions of conformal invariance and exact solvability need further attention.

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