# Short Papers

## Design of Multiunit Electronic Exchanges Through Decomposition

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*Abstract*—In this paper, we exploit the idea of decomposition to match buyers and sellers in an electronic exchange for trading large volumes of homogeneous goods, where the buyers and sellers specify marginal-decreasing piecewise constant price curves to capture volume discounts. Such exchanges are relevant for automated trading in many e-business applications. The problem of determining winners and Vickrey prices in such exchanges is known to have a worst-case complexity equal to that of as many as $(1 + m + n)$ NP-hard problems, where $m$ is the number of buyers and $n$ is the number of sellers. Our method proposes the overall exchange problem to be solved as two separate and simpler problems: 1) forward auction and 2) reverse auction, which turns out to be generalized knapsack problems. In the proposed approach, we first determine the quantity of units to be traded between the sellers and the buyers using fast heuristics developed by us. Next, we solve a forward auction and a reverse auction using fully polynomial time approximation schemes available in the literature. The proposed approach has worst-case polynomial time complexity and our experimentation shows that the approach produces good quality solutions to the problem.

*Note to Practitioners*— In recent times, electronic marketplaces have provided an efficient way for businesses and consumers to trade goods and services. The use of innovative mechanisms and algorithms has made it possible to improve the efficiency of electronic marketplaces by enabling optimization of revenues for the marketplace and of utilities for the buyers and sellers. In this paper, we look at single-item, multiunit electronic exchanges. These are electronic marketplaces where buyers submit bids and sellers ask for multiple units of a single item. We allow buyers and sellers to specify volume discounts using suitable functions. Such exchanges are relevant for high-volume business-to-business trading of standard products, such as silicon wafers, very large-scale integrated chips, desktops, telecommunications equipment, commoditized goods, etc. The problem of determining winners and prices in such exchanges is known to involve solving many NP-hard problems. Our paper exploits the familiar idea of decomposition, uses certain algorithms from the literature, and develops two fast heuristics to solve the problem in a near optimal way in worst-case polynomial time.

*Index Terms*—Forward auction, generalized knapsack problem, marginal-decreasing piecewise constant bids, multiunit exchanges, reverse auction, volume discounts.

## I. INTRODUCTION

In recent times, electronic marketplaces or electronic exchanges have provided an efficient mechanism for businesses and consumers to trade goods and services. In this paper, we consider single-item, multiunit electronic exchanges. These are e-marketplaces where buyers (or buying agents) and sellers (or selling agents) submit bids and ask for multiple units of a single item. Such exchanges are relevant for high-volume business-to-business (B2B) trading of standard products,

such as silicon wafers, very-large scale integrated (VLSI) chips, desktops, telecommunications equipment, commoditized goods, etc. We assume that the bids and asks submitted by the buyers and sellers are marginal-decreasing piecewise constant functions. Such functions help the buying agents and selling agents to specify volume discounts (often called quantity discounts). A buyer can specify a lower bound on the number of units he or she demands and a seller can specify an upper bound on the number of units he or she can supply. Here, we consider single-shot, sealed bid exchanges. There are two optimization problems involved with such exchanges:

1) *allocation problem* (also called trade determination problem or winner determination problem): determining how much will be bought by each buying agent and how much will be sold by each selling agent;
2) *pricing problem*: determining the net payment to be made by each (winning) buying agent and the net payment to be made to each (winning) selling agent.

Even with many restrictions on the structure of bids and asks submitted by the buyers and sellers, the allocation problem turns out to be intractable [1]. If, in addition, truth revelation properties are required from the buying and selling agents, the computation of payments will involve solving several such intractable problems [1]. In this paper, our approach to the problem is to solve it approximately, using a simple decomposition idea, and to come up with a computationally efficient solution that provides near optimal solutions.

### A. Relevant Work

There are excellent survey papers in the area of auctions and exchanges. For example, the reader is referred to the books [2]–[4] and the survey articles [1], [5]–[10]. Other popular references are [11]–[14]. Here, we provide a brief review of relevant literature in the areas of 1) single-item, multiunit auctions and 2) single-item, multiunit exchanges.

*1) Single-Item, Multiunit Auctions:* Our paper uses the results in the recent work of Kothari *et al.* [15] on approximately strategy-proof and tractable multiunit auctions. In [15], the authors consider single-item, multiunit auctions where the bidders use marginal-decreasing, piecewise constant functions to bid on homogeneous goods. Both forward auction (single seller and multiple buyers) and reverse auction (single buyer and multiple sellers) are considered. In the forward auction, the objective is to maximize the revenue for the seller and in the reverse auction, the objective is to minimize the cost for the buyer. It is shown that the allocation problems are generalizations of the classical $0/1$ knapsack problem, hence, NP hard. Computing Vickrey–Clarke–Groves (VCG) payments [1] is also addressed. The authors develop a fully polynomial time approximation scheme (FPTAS) for the generalized knapsack problem. This leads to an FPTAS algorithm for allocation in the auction which is approximately strategy proof and approximately efficient. It is also shown that VCG payments for the auctions can be computed in the worst-case $O(T \log n)$ time, where $T$ is the running time to compute a solution to the allocation problem.

Eso *et al.* [16] address the procurement problem faced by a buyer who wishes to buy large quantities of several heterogeneous products. Suppliers submit piecewise linear curves for each product indicating

the price as a function of the supplied quantity. The problem of minimizing the purchasing cost turns out to be intractable. The authors develop a flexible column generation-based heuristic that provides near-optimal solutions to the bid selection problem using branch and price methodology. Similar problems have been investigated in [17] and [18]. However, in all of these papers, the issue of pricing so as to induce truth revelation by the agents has not been discussed.

Dang and Jennings [19] consider multiunit auctions where the bids are piecewise linear curves. Maximizing the revenue of the auctioneer is the objective. Algorithms are provided for solving the allocation problem. In the case of multiunit, single-item auctions, the complexity of the allocation algorithm is $O(n(K + 1)^n)$, where $n$ is the number of bidders and $K$ is an upper bound on the number of segments of the piecewise linear pricing functions. The algorithm therefore has exponential complexity in the number of bids.

*2) Single-Item, Multiunit Exchanges:* Kalagnanam, Davenport, and Lee [20] consider continuous call double auctions which are also known in the literature as clearing houses or call markets. In such a market, the marketplace collects bids from buyers and asks from suppliers over a fixed time period and clears the market at the end of the time period. A bid specifies quantity and price. Similarly, an ask specifies a quantity and price. Three cases are considered.

1) Any part of a bid may be matched with any part of any ask. In this case, the allocation problem can be solved in log linear time.
2) When there are assignment constraints, that is, some demands can only be assigned to some supplies, then the allocation problem can be solved in polynomial time using network flow algorithms.
3) If the demand is indivisible, that is, a given demand is constrained to be satisfied by exactly one ask only, the allocation problem turns out to be NP-hard.

The above results are summarized in [1].

Dailianas *et al.* [21] consider marketplaces for bandwidth in a network services economy. The buyers specify bid curves which specify the unit price requested as a function of quantity. Similarly, the sellers specify offer curves that specify the unit price as a function of quantity offered. Three types of objectives are considered:

1) profit maximization: maximize profit on the price spread between the aggregated bids and aggregated offers;
2) buyer satisfaction: match the demand of all buyers and find the best combination of seller offers that will maximize the profit;
3) minimum liquidity: match the demands of at least a certain percentage of buyers while guaranteeing some minimum profit for the marketplace.

Both exact and heuristics-based solutions are explored for each of the three objectives and an analysis of the performance of the solutions is reported.

The authors in [22] discuss a variety of allocation algorithms. The authors consider markets where there are multiple indistinguishable units of an item for sale (or there are multiple units of multiple items for sale, but different items can be treated independently as belonging to different markets). The bids are in the form of supply curves (selling agents) and demand curves (buying agents) that specify price quantity relationships. These curves are assumed to be piecewise linear. The objective is to maximize the total surplus. Two different pricing schemes are considered: nondiscriminatory (all sellers share the same price and all buyers share the same price) and discriminatory (each seller and each buyer may be associated with a different price). The authors present a polynomial time algorithm for clearing nondiscriminatory markets and show that clearing discriminatory markets is NP complete. If the supply and demand curves are linear, then discriminatory markets can also be cleared in polynomial time.

### B. Contributions and Outline

We found the following research gaps in the literature:

- the allocation problem in the general case of multiunit, single item exchanges and auctions with marginal decreasing, piecewise constant bids is NP-hard; polynomial time algorithms have been proposed for the allocation problem only in very special cases;
- most of the papers do not consider the pricing problem; this is an important issue because appropriately computed prices can induce truthful bids by all of the agents.

Motivated by these research gaps, this paper explores the following directions in the context of single-item, multiunit exchanges where the bidders specify marginal decreasing piecewise constant price curves.

- We use the familiar idea of decomposition to solve the allocation and pricing problems by solving two separate simpler problems: a forward auction and a reverse auction.
- We propose two fast heuristics to compute the trading quantity to be used for the forward and reverse auctions. The heuristics have worst-case polynomial time complexity and produce nearly optimal values of trading quantity. The use of these heuristics in a decomposition-based approach has worst-case polynomial time complexity whereas the direct approach for solving the allocation and pricing problems has a computational complexity equal to that of as many as $(1 + m + n)$ NP-hard problems where $m$ is the number of buyers and $n$ is the number of sellers.
- Using appropriate and extensive numerical experiments, we show the efficacy of the proposed approach and the proposed heuristics, in terms of quality of solutions produced, computational efficiency, and ability to induce truth revelation by the bidders.

This paper is organized in the following way. Section II describes the notation and formulations that will be used in the rest of the paper. First, we present the formulation of optimization problems in a multiunit exchange where buying agents and selling agents submit marginal decreasing price functions. We show the formulation for the 1) allocation problem and 2) computation of Vickrey payments. We also show how the exchange problems can be solved using a simple decomposition approach involving a forward auction and a reverse auction. In Section III, we present two fast heuristics to determine the optimal quantity to be traded, which will be required in solving the forward auction and reverse auction. Section IV presents the results of a wide range of experiments carried out.

## II. ALLOCATION AND PRICING PROBLEMS IN SINGLE-ITEM MULTIUNIT EXCHANGES

The notation is described in Table I. The exchange we consider can be described as follows.

- There is a set of buying agents $M = \{1, \ldots, m\}$ and a set of selling agents $N = \{1, \ldots, n\}$.
- The buying agents submit bids $B = \{B_1, \ldots, B_m\}$, respectively. A bid is a list of pairs $B_i = [(u_i^1, b_i^1), \ldots, (u_i^{m_i-1}, b_i^{m_i-1})]$, with an upper bound of $u_i^{m_i}$ on the quantity, where $u_i^1 < u_i^2 < \cdots < u_i^{m_i}, b_i^1 > b_i^2 > \cdots > b_i^{m_i-1}$. Here, the bidder's valuation in the quantity range $[u_i^j, u_i^{j+1})$ is $b_i^j$ for each unit. Note that the bid structure here enables the buyers to specify quantity or volume discounts.
- The selling agents submit asks $A = \{A_1, \ldots, A_n\}$, respectively. An ask is a list of pairs $A_I = [(U_I^1, a_I^1), \ldots, (U_I^{M_I-1}, a_I^{M_I-1})]$ with an upper bound of $U_I^{M_I}$ on the quantity, where $U_I^1 < U_I^2 < \cdots < U_I^{M_I}, a_I^1 > a_I^2 > \cdots > a_I^{M_I-1}$. Note that the bid structure here enables the sellers to offer quantity or volume discounts.

TABLE I
NOTATION FOR THE ALLOCATION PROBLEM

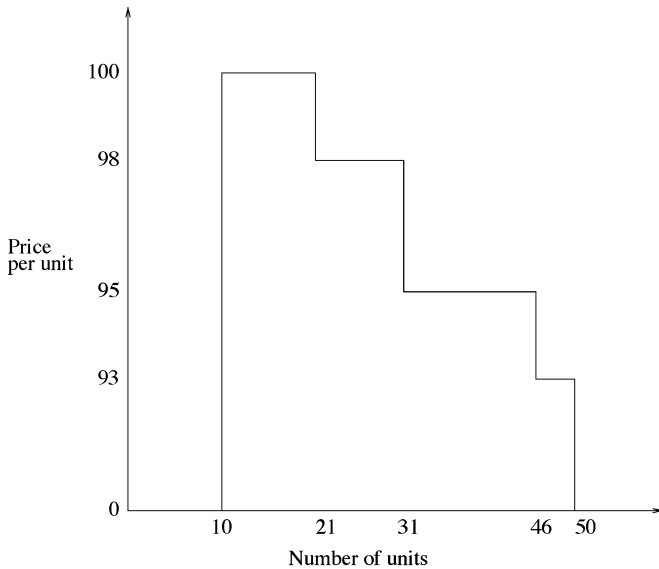| | |
|---|---|
| $M = \{1, \ldots, m\}$ | set of $m$ buyers |
| $N = \{1, \ldots, n\}$ | set of $n$ sellers |
| $i = 1, \ldots, m$ | index for buyers |
| $I = 1, \ldots, n$ | index for sellers |
| $m_i$ | number of steps in the *bid* of buyer $i$ |
| $M_I$ | number of steps in the *ask* of seller $I$ |
| $j = 1, \ldots, m_i - 1$ | index for step in the *bid* of buyer $i$ |
| $J = 1, \ldots, M_I - 1$ | index for step in the *ask* of seller $I$ |
| $Z$ | a large enough integer |
| $S(M, N)$ | surplus with $m$ buyers and $n$ sellers |
| $S(M \setminus \{k\}, N)$ | surplus when buyer $k$ does not participate |
| $S(M, N \setminus \{K\})$ | surplus when seller $K$ does not participate |
| $y_i^j$ | 1 if bid interval $j$ for buyer $i$ is selected |
| | 0 otherwise |
| $Y_I^J$ | 1 if bid interval $J$ for seller $I$ is selected |
| | 0 otherwise |
| $x_i^j$ | number of units in interval $j$ allocated to buyer $i$ |
| $X_I^J$ | number of units in interval $J$ allocated to seller $I$ |
| $\nu_{dis}^k$ | Vickrey discount to buyer $k$ |
| $\nu_{sur}^K$ | Vickrey surplus to seller $K$ |



Fig. 1. Bid submitted by a buying agent.

- We can interpret each list of tuples as a price function

$$P_{\text{bid},i}(q) = \begin{cases} qb_i^j, & \text{if } u_i^j \leq q < u_i^{j+1}, \\ & \text{where } j = 1, 2, \ldots, m_i - 1 \\ qb_i^{m_i - 1}, & \text{if } q = u_i^{m_i} \end{cases}$$

$$P_{\text{ask},I}(q) = \begin{cases} qa_I^J, & \text{if } U_I^J \leq q < U_I^{J+1}, \\ & \text{where } J = 1, 2, \ldots, M_I - 1 \\ qa_I^{M_I - 1}, & \text{if } q = U_I^{M_I}. \end{cases}$$

An example of a bid submitted by a buying agent is given in Fig. 1. Here, the buying agent bids a price of 100 per unit (p.u.) for a quantity in the range [10,21], 98 p.u. for the range [21,31], 95 p.u. for the range [31,46], and 93 p.u. for the range [46,50]. An example of an ask submitted by a selling agent is given in Fig. 2. Here, the selling agent offers a price of 40 p.u. for a quantity in the range [5,16], 38 p.u. for the range [16,36], and 37 p.u. for the range [36,50].

### A. Allocation Problem

In the exchange described above, the allocation problem is formulated as follows. We choose the surplus to the exchange (also called

revenue to the exchange) as the objective to maximize. The surplus is defined as the total payment received from all of the winning buyers minus the total payment made to all of the winning suppliers. The main constraint to be satisfied is that the total number of units sold to the buyers should be less than the total number of units procured from the sellers. The notation is described in Table I

Maximize
$$S(M, N) = \sum_{i \in M} \sum_{j = 1, \ldots, m_i - 1} x_i^j b_i^j$$
$$- \sum_{I \in N} \sum_{J = 1, \ldots, M_I - 1} X_I^J a_I^J$$

subject to

$$\sum_{i \in M} \sum_{j = 1, \ldots, m_i - 1} x_i^j \leq \sum_{I \in N} \sum_{J = 1, \ldots, M_I - 1} X_I^J \tag{1}$$

$$x_i^j \leq Z y_i^j$$
$$\forall i \in M, \ j = 1, \ldots, m_i - 1 \tag{2}$$

$$X_I^J \leq Z Y_I^J$$
$$\forall I \in N, \ J = 1, \ldots, M_I - 1 \tag{3}$$

$$\sum_{j = 1, \ldots, m_i - 1} y_i^j \leq 1 \, \forall i \in M \tag{4}$$

$$\sum_{J = 1, \ldots, M_I - 1} Y_I^J \leq 1 \, \forall I \in N \tag{5}$$

$$y_i^j u_i^j \leq x_i^j \quad \forall i \in M, \ j = 1, \ldots, m_i - 1 \tag{6}$$

$$Y_I^J U_I^J \leq X_I^J$$
$$\forall I \in N, \ J = 1, \ldots, M_I - 1 \tag{7}$$

$$x_i^j < u_i^{j+1}$$
$$\forall i \in M, \ j = 1, \ldots, m_i - 1 \tag{8}$$

$$X_I^J < U_I^{J+1}$$
$$\forall I \in N, \ J = 1, \ldots, M_I - 1 \tag{9}$$

$$y_i^j, Y_I^J \in 0, 1 \tag{10}$$

$$x_i^j, X_I^J \quad \text{integer.} \tag{11}$$

Constraint (1) guarantees that the number of units sold will not exceed the number of units procured. Constraint (2) assigns $x_i^j = 0$ if $y_i^j = 0$. Constraints (4) and (5) enforce the exchange mechanism to choose items from just one bid interval for each buyer and seller. Constraints [(6), (7), (8), (9)] ensure that $x_i^j$ and $X_i^j$ lie in the range of the $j$th interval for the $i$th buyer and the $i$th seller, respectively.

The classical 0/1 knapsack problem, which is a well-known NP-hard problem, is a special case of this problem, and this immediately implies that the above allocation problem is intractable.

### B. Pricing Problem

The pricing problem involves determining the actual payments to be made by the winning buyers to the exchange and the actual payments to be made to the winning sellers by the exchange. VCG payments are those that ensure that the bids and asks from the buyers and sellers reflect their true values [1]. Market mechanisms that follow VCG payments are often called strategy proof mechanisms. VCG payments for each winning agent can be determined as follows. First, solve the allocation problem by retaining the agent in the problem and determine the total surplus generated. Next, remove that agent from the scene, solve the allocation problem, and determine the total surplus (with the agent removed). The decrease in the surplus due to the absence of the agent is given as the Vickrey discount if the agent is a buying agent and is given as Vickrey surplus if the agent is a selling agent. It is easy to see that we need to solve up to $(m + n)$ intractable problems, one for each winning buyer and winning seller, to determine VCG payments.
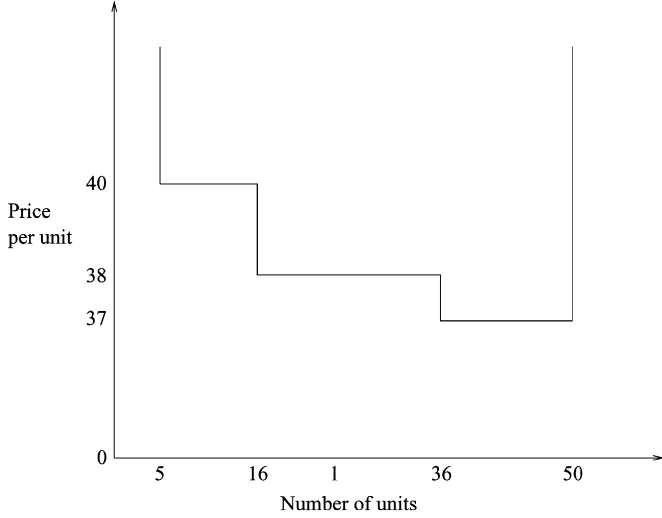
Fig. 2. Ask submitted by a selling agent.

### C. Decomposition Approach

We decompose the problem of a single-item, multiunit exchange into two natural, separate problems: forward auction and reverse auction. The approach involves the following steps.

1) *Determining the trading quantity, $Q \geq 0$, that is, the quantity of units that will be exchanged between the buyers and the sellers.*
2) *Solving the separate problems:*
   i) *Reverse auction*: Based on the bids submitted by the selling agents, procure a trading quantity $Q$ of the goods so as to maximize the total value of the selling agents.
   ii) *Forward auction*: Based on the bids submitted by the buying agents, sell the $Q$ goods to the buying agents, so as to maximize the total value for the buying agents.

We describe the reverse auction problem. The forward auction problem can be formulated on similar lines.

*1) Reverse Auction:* The formulation here is done on the lines of [15]

Minimize
$$\sum_{I \in N} \sum_{J=1,\ldots,M_I-1} X_I^J a_I^J$$

subject to
$$\sum_{I \in N} \sum_{J=1,\ldots,M_I-1} X_I^J \geq Q$$

$$X_I^J \leq Z Y_I^J$$
$$\forall I \in N, \forall J = 1,\ldots,M_I-1$$

$$\sum_{J=1,\ldots,M_I-1} Y_I^J \leq 1 \quad \forall I \in N$$

$$Y_I^J U_I^J \leq X_I^J$$
$$\forall I \in N, \forall J = 1,\ldots,M_I-1$$

$$X_I^J < U_I^{J+1}$$
$$\forall I \in N, \forall J = 1,\ldots,M_I-1$$

$$Y_I^J \in 0,1$$

$$X_I^J \quad \text{integer.}$$

Here, the first constraint ensures that the total number of units procured is greater than or equal to the trading quantity $Q$. This formulation is the same as that of a generalized knapsack problem [15]. Kothari, Parkes, and Suri [15] have proposed an $O(n^2)$ time 2-approximation algorithm for the generalized knapsack problem arising in reverse auction and also have presented a fully polynomial time approximation scheme based on this 2-approximation.

### III. HEURISTICS FOR DETERMINING TRADING QUANTITY

The decomposition approach produces a high-quality solution only if we use the optimal trading quantity. Determining the trading quantity to be used by the decomposition method is thus a critical problem. We address this problem in this section by proposing two heuristics to compute an almost optimal trading quantity.

#### A. Heuristic 1

Based on the bids and asks submitted, it is easy to determine a lower bound ($L$) and an upper bound ($U$) on the trading quantity between which the optimal quantity will lie. Once this range is determined, for different trading quantities in this range, our idea is to use a greedy method to determine the allocation to the sellers and buyers, and determine the surplus. We choose the quantity that maximizes this surplus.

First we determine

$$\text{minimum demand, } D_{\min} = \sum_{i \in M} u_i^1$$

$$\text{maximum demand, } D_{\max} = \sum_{i \in M} u_i^{m_i}$$

$$\text{minimum supply, } S_{\min} = \sum_{I \in N} U_I^1$$

$$\text{maximum supply, } S_{\max} = \sum_{I \in N} U_I^{M_I}.$$

Consider $S_{\min} < D_{\min}$. Here, three cases are possible:
1) $S_{\min} < S_{\max} < D_{\min} < D_{\max}$;
2) $S_{\min} < D_{\min} < S_{\max} < D_{\max}$;
3) $S_{\min} < D_{\min} < D_{\max} < S_{\max}$.

For Case 1) and Case 2), $L = S_{\min}$, $U = S_{\max}$. and for Case 3), $L = S_{\min}$, $U = D_{\max}$. Now, sort the tuples submitted by the buyers $t_i^j$ in descending order of unit price and sort the tuples submitted by the sellers $T_i^j$ in ascending order of the unit price. For different trade quantities $I$, we compute the total valuation of the buyers $V(I)$ and total valuation of the sellers $C(I)$ as discussed in the algorithm below. We scan through the sorted list and determine a feasible allocation. The trading quantity $Q$ is chosen as a value between $L$ and $U$ such that the difference between $V(I)$ and $C(I)$ is maximum.

The following describes our algorithm for determining the trading quantity $Q$.

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**Algorithm:** *Heuristic-1 for Determining Trading Quantity*

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1) Sort all pairs from the buyers in descending order of unit price and all pairs from the sellers in ascending order of unit price.
2) Vary the quantity to be traded, $I$ from $L$ to $U$.
3) Compute total valuation $V(I)$ of the buyers for quantity $I$ as follows:
   - set mark$(i) = 0$, for all bids $B_i$, $i = 1,\ldots,m$ Initialize the remaining quantity to be sold, $R_b = I$; the quantity allocated to buyer $i$, $Q_i = 0$; $V(I) = 0$;
   - scan the pairs in sorted order. Let the selected pair be $t_i^j$;
   - if mark$(i) = 1$ and $(Q_i + R_b) > (u_i^{j+1} - 1)$

   $$V(I) = V(I) + \delta$$

   where $\delta$ is the difference in the valuation of buyer $i$ for $(u_i^{j+1} - 1)$ and his or her valuation for $Q_i$ units

   $$R_b = R_b + Q_i - u_i^{j+1} - 1; \quad Q_i = u_i^{j+1}$$

go to scanning step;
- if mark$(i) = 1$ and $u_i^j \leq (Q_i + R_b) \leq (u_i^{j+1} - 1)$

$$V(I) = V(I) + \delta$$

where $\delta$ is the difference in the valuation of buyer $i$ for $(Q_i + R_b)$ units and his or her valuation for $Q_i$ units.
return $V(I)$,
else go to scanning step;
- if mark$(i) = 0$ and $R_b > (u_i^{j+1} - 1)$

$$V(I) = V(I) + \delta$$

where $\delta$ is the valuation of buyer $i$ for $(u_i^{j+1} - 1)$ units.

$$\text{mark}(i) = 1; \ Q_i = u_i^{j+1}; \ R_b = R_b - Q_i$$

go to scanning step;
- if mark$(i) = 0$ and $u_i^j \leq R_b \leq (u_i^{j+1} - 1)$

$$V(I) = V(I) + \delta$$

where $\delta$ is the valuation of buyer $i$ for $R_b$ units.
return $V(I)$.
4) Compute total valuation $C(I)$ of the sellers for quantity $I$ as follows:
- set mark$(i) = 0$, for all asks $A_i$, $i = 1, \ldots, n$
  Initialize the remaining quantity to be procured, $R_s = I$; the quantity allocated to buyer $i$, $Q_i = 0$; $C(I) = 0$;
- scan the pairs in sorted order. Let the selected pair be $T_i^j$;
- if mark$(i) = 1$,
  go to the scanning step;
- if mark$(i) = 0$ and $R_s > (U_i^{j+1} - 1)$

$$C(I) = C(I) + \delta$$

where $\delta$ is the valuation of seller $i$ for $(U_i^{j+1} - 1)$ units

$$\text{mark}(i) = 1; \ Q_i = U_i^{j+1}; \ R_s = R_s - Q_i$$

go to scanning step;
- if mark$(i) = 0$ and $U_i^j \leq R_S \leq (U_i^{j+1} - 1)$

$$C(I) = C(I) + \delta$$

where $\delta$ is the valuation of seller $i$ for $R_s$ units.
return $C(I)$.
5) if $V(I) - C(I) \geq \max_{\text{surplus}}$
   $Q = I$, update $\max_{\text{surplus}}$.
6) if $I \leq U$,
   go to step 2.
Once the initial sorting is done, the algorithm takes $O((U - L)N)$ running time, where $N = m + n$.

*B. Heuristic 2*

Here, we come up with a faster heuristic for determining the trading quantity based on the concept of determining the call market price-
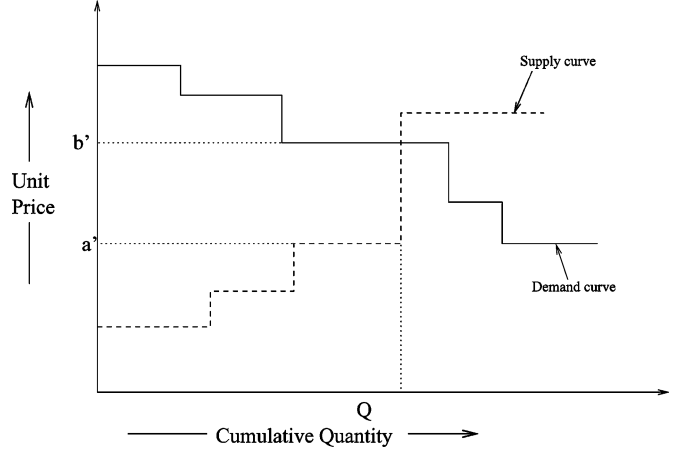


Fig. 3. Supply and demand curves for determining market price-quantity pairs.

quantity pair. First, we discuss an algorithm for clearing the call markets [23]. We will modify it for determining the trading quantity to be used. A call market is a sealed-bid, one-shot exchange which can be described as follows.
- There is a set of buying agents $M = \{1, \ldots, m\}$ and a set of selling agents $N = \{1, \ldots, n\}$.
- The buying agents submit bids $B = \{B_1, \ldots, B_m\}$, respectively. A bid $B_i$ is of form $B_i = (u_i, b_i)$ where buyer $i$ is willing to accept up to $u_i$ units at unit price $\leq b_i$.
- The selling agents submit asks $A = \{A_1, \ldots, A_n\}$, respectively. An ask $A_I$ is of form $A_I = (U_I, a_I)$, where seller $I$ is willing to sell up to $U_I$ unit at unit price $\geq a_I$.
In a call market, all trades clear at a market-clearing price. An algorithm for clearing a call market from [23] is described below.

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**Algorithm:** *Call Market Clearing Algorithm*

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- Sort the bids in decreasing order of unit price.
  Let the sorted order be $p_1 \geq p_2 \geq \ldots \geq p_m$.
- Sort the asks in ascending order of unit price.
  Let the sorted order be $q_1 \leq q_2 \leq \ldots \leq q_n$.
- At the buy side, the quantity of item available at price $p_r$ is

$$E_r = \sum_{i \in M} C_r^i$$

where

$$C_r^i = u_i, \text{ if } b_i > p_r$$
$$= 0, \text{ otherwise.}$$

- Similarly at the sell side, the quantity of item available at price $q_s$ is

$$F_s = \sum_{I \in N} D_s^I$$

where

$$D_s^I = U_I, \text{ if } a_I < q_s$$
$$= 0, \text{ otherwise.}$$

- Plot a graph between the price and cumulative quantity of the available item both for sellers and buyers (Fig. 3).
- The intersection point gives the optimal trading quantity $Q$.

Here, the quantity $Q$ will maximize the surplus, and the market-clearing price $\alpha$ will be $a' \leq \alpha \leq b'$. Notice that our exchange (single-item, multiunit exchange) is a variation of the above call market in the following ways.

1) For buyers, each bid corresponds to a range, that is

$$\left\{\left[u_i^j, u_i^{j+1}\right), b_i\right\}.$$

2) Each buyer submits XOR bids of the type

$$B_i = \left\{\left[u_i^1, u_i^2\right), b_i^1\right\} \oplus \ldots \oplus \left\{\left[u_i^{m_i-1}, u_i^{m_i}\right], b_i^{m_i-1}\right\}$$

where $b_i^1 > \ldots > b_i^{m_i-1}$ and $u_i^1 < \ldots < u_i^{m_i}$.

3) For sellers, each ask corresponds to a range, that is

$$\left\{\left[U_I^J, U_I^{J+1}\right), a_I\right\}.$$

4) Each seller submits XOR asks of the type

$$A_I = \left\{\left[U_I^1, U_I^2\right), a_I^1\right\} \oplus \ldots \oplus \left\{\left[U_I^{M_I-1}, U_I^{M_I}\right], a_I^{M_I-1}\right\}$$

where $a_I^1 > \ldots > a_I^{M_I-1}$ and $U_I^1 < \ldots < U_I^{M_I}$.

We now propose our heuristic for determining trading quantity for our exchange based on the call market-clearing algorithm.

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**Algorithm:** *Heuristic-2 for Determining Trading Quantity*

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1) Sort the bid prices of buyers $(b_1^1, \ldots, b_1^{m_1-1}), \ldots, (b_m^1, \ldots, b_1^{m_m-1})$ in descending order. Let the sorted order be $O_b = (p_1, \ldots, p_r, \ldots, p_R)$. $R$ is the number of terms in $O_b$.

2) At the buy side, the maximum quantity of item available at price $p_r$ is

$$E_r = \sum_{i \in M} C_r^i$$

where

$$C_r^i = \left[\left(u_i^{j'+1} - 1\right) \middle| j' = \max\left(j \middle| b_i^j > p_r\right)\right]$$
$$= 0 \text{ otherwise.}$$

3) Sort the ask prices of sellers in ascending order. Let the sorted order be $O_s = (q_1, \ldots, q_s, \ldots, q_S)$. $S$ is the number of terms in $O_s$.

4) At the sell side, the maximum quantity of the item available at price $q_s$ is

$$F_s = \sum_{I \in N} D_s^I$$

where

$$D_s^I = \left[\left(U_I^{J'+1} - 1\right) \middle| J' = \max\left(J \middle| b_I^J < q_s\right)\right]$$
$$= 0 \text{ otherwise.}$$

5) Observe that $E_r$ increases with a decrease in $p_r$. This is because each bid submitted by the buyers is marginally decreasing the piecewise constant valuation function. So, as the price decreases, the cumulative quantity of item available increases. Similarly, $F_s$ increases with an increase in $q_s$.

TABLE II
LOWER BOUND AND UPPER BOUND RANGES FOR BIDS AND ASKS

| Expt. No | Lower Bound Range | Upper Bound Range |
|---|---|---|
| 1 | (10,60) | (80,280) |
| 2 | (25,70) | (100,350) |
| 3 | (50,150) | (200,600) |
| 4 | (100,200) | (350,850) |

6) A graph of the total cumulative quantity of item and price both for sellers and buyers is similar to the one shown in Fig. 3.

7) Initialize $r = 1$; $s = 1$. Let $V(E_r)$ be the total valuation of the buyers for quantity $E_r$, and $C(F_s')$ be the total valuation of the sellers for quantity $F_s'$. $\max_{\text{surplus}}$ is used to store maximum surplus.
   Perform the following steps.
   while ($r \leq R$ and $s \leq S$)
   - if $p_r \geq q_s'$ and $E_r \leq F_s'$
     if $V(E_r) - C(F_s') \geq \max_{\text{surplus}}$
     $Q = E_r$; update $\max_{\text{surplus}}$;
   - if $E_r \geq F_s'$ $s = s + 1$;
   - if $E_r < F_s'$
     $r = r + 1$.

The above algorithm gives the trading quantity $Q$. But it may not be the optimal quantity because we do not consider the lower bound of each range of the bids and the asks. The running time of the algorithm can be easily seen to be $O(\max(m \log m, n \log n))$.

## IV. EXPERIMENTAL RESULTS

In this section, we present results of our numerical experiments to show the performance of the proposed decomposition approach and the proposed heuristics.

### A. Experimental Setup

We used an ILOG CPLEX solver package on a 3-GHz Xeon server with 2-GB random-access memory (RAM) to compute exact solutions. We refer to this as the direct solution approach. We used the same server for implementing our heuristics, our decomposition approach, and the FPTAS algorithms for forward auction and reverse auction.

The bids and asks required for the numerical experiments were generated to be as representative as possible. The bids and asks are marginal-decreasing piecewise constant valuation functions. We conducted experimentation with four sets of data. These sets of data differ with respect to the range of values for choosing the lower bound and upper bound on the number of units for each bid and ask. Table II gives these ranges for the four sets of data. In all of the experiments, we considered ten sellers and ten buyers. Also, we assumed the maximum number of steps in a bid or an ask to be 10. For each buyer, we generated the number of steps $(m_i - 1)$ in the bid through a discrete uniform random variate in the range 1 to 10. For each buyer, say buyer $i$ $(i = 1, \ldots, 10)$, we chose the minimum and maximum number of units $(u_i^1$ and $u_i^{m_i})$ in his or her bids using uniform random variates in the appropriate range. For each step $(j)$, we generated the price per unit $(b_i^j)$ randomly in the range $(0.5, 1)$ such that $b_i^1 > b_i^2 > \cdots > b_i^{m_i-1}$. We followed a similar method for each of the ten sellers.

The experiments on four different data sets were conducted 20 times using independent samples. We computed the average of the solution values for 2, 3, . . ., 20 replications and found that after 20 replications, the averages of the solution values remained invariant. The reported results are thus averaged over the 20 experiments conducted for each data set.

TABLE III
COMPARISON OF OPTIMAL SOLUTION WITH THE SOLUTION OBTAINED BY THE
DECOMPOSITION APPROACH USING OPTIMAL TRADING QUANTITY

| Expt No | $Q_o$ | $S_o$ | $S_d$ |
|---|---|---|---|
| 1 | 1326 | 305.52 | 305.15 |
| 2 | 1647 | 387.66 | 387.66 |
| 3 | 2794 | 634.52 | 634.52 |
| 4 | 4199 | 964.34 | 964.26 |

TABLE IV
COMPARISON OF THE HEURISTICS

| Expt. No. | $Q_1$ | $Q_2$ | $Q_o$ | $S_1$ | $S_2$ | $S_o$ |
|---|---|---|---|---|---|---|
| 1 | 1349 | 1314 | 1326 | 302 | 299 | 305 |
| 2 | 1646 | 1630 | 1647 | 387 | 386 | 387 |
| 3 | 2900 | 2840 | 2794 | 631 | 629 | 634 |
| 4 | 4057 | 4032 | 4199 | 958 | 957 | 964 |

### B. Performance of the Decomposition Approach With Optimal Trading Quantities

Our first experiment is to investigate how effectively the decomposition idea works. For this, we first solved the allocation problem to optimality use a direct solution approach (that is, without using decomposition) and obtained the optimal value of the total surplus (call it $S_o$) and the value of optimum quantity traded (call it $Q_o$). Using the value $Q_o$ in our decomposition approach and the FPTAS algorithms for forward auction and reverse auction problems, we then obtained the total surplus (call it $S_d$). Table III compares the values of the total surplus obtained using the direct solution approach and the decomposition approach. The table clearly shows that the allocation determined through the decomposition approach is very nearly optimal. Note that this experiment uses the optimal trading quantity in the decomposition approach and, hence, shows how well the FPTAS algorithms in the decomposition approach approximate the total surplus.

### C. Comparison of the Heuristics

Here, we use the heuristics presented in Section IV to determine the trading quantity and use this trading quantity for solving the forward auction and reverse auction problems using the FPTAS algorithms. Table IV first compares the trading quantities obtained using the two heuristics $Q_1$ and $Q_2$, with the optimal trading quantity $Q_o$ (computed using a direct solution approach). Then, it compares the total surplus values obtained using the decomposition approach with that computed using a direct solution approach. $S_1$ ($S_2$) is the surplus value obtained using the decomposition approach employing the trading quantity $Q_1$ ($Q_2$). $S_o$ is the surplus value obtained using a direct solution approach through an ILOG CPLEX solver. In Table IV, we have omitted the fractional component of the surplus values (by truncating the values to the nearest integer).

The table clearly shows that the values of quantity to be traded obtained using heuristic 1 and heuristic 2 are quite close to the optimal quantity and also the surplus generated is quite close to the optimal one. Heuristic 1 seems to provide better estimates compared to heuristic 2 as shown by the table. This is because heuristic 1 does an exhaustive search on a set of short listed candidate values whereas heuristic 2 may not always produce the optimal value (see Section III-B). However, our experimentation (not reported here) for large values of $n$ and $m$ has shown that the trading quantities estimated by the two heuristics are almost the same. In terms of running time, however, heuristic 2 is much faster than heuristic 1 (see Section IV-E for a discussion on this). The surplus values produced by the decomposition approach with the help of heuristics are quite close to the optimal surplus values, which shows the efficacy of the heuristics.

TABLE V
COMPARISON OF TOTAL VICKREY DISCOUNT AND TOTAL VICKREY
SURPLUS OBTAINED BY THE DECOMPOSITION APPROACH WITH
THOSE OF THE EXACT SOLUTION

| Expt. No. | $TVD_o$ | $TVS_o$ | $TVD_d$ | $TVS_d$ |
|---|---|---|---|---|
| 1 | 212 | 101 | 207 | 94 |
| 2 | 298 | 121 | 302 | 133 |
| 3 | 423 | 294 | 433 | 302 |
| 4 | 626 | 414 | 629 | 412 |

TABLE VI
COMPARISON OF INDIVIDUAL VICKREY DISCOUNTS
AND VICKREY SURPLUSES FOR EXPERIMENT 1

| Buyer | $VD_o$ | $VD_d$ | Seller | $VS_o$ | $VS_d$ |
|---|---|---|---|---|---|
| 1 | 15.0 | 16.2 | 1 | 1.7 | 0.8 |
| 2 | 49.2 | 49.6 | 2 | 10.4 | 12.3 |
| 3 | 22.0 | 22.8 | 3 | 12.6 | 11.6 |
| 4 | 24.4 | 23.4 | 4 | 11.0 | 9.0 |
| 5 | 28.8 | 27.5 | 5 | 0.86 | 0 |
| 6 | 9.2 | 7.4 | 6 | 14.9 | 14.7 |
| 7 | 7.7 | 5.6 | 7 | 34.0 | 33.2 |
| 8 | 17.9 | 17.8 | 8 | 15.2 | 15.2 |
| 9 | 21.2 | 19.8 | 9 | 12.3 | 13.4 |
| 10 | 16.9 | 16.6 | 10 | 7.6 | 6.9 |

### D. Degree of Strategy Proofness of the Decomposition Approach

Table V compares the total Vickrey discount ($TVD_o$) and total Vickrey surplus ($TVS_o$), respectively, for winning buyers and sellers computed by solving the problem to optimality using a direct solution approach with the values ($TVD_d$ and $TVS_d$) when the problem is solved using our decomposition approach. In the decomposition approach, we used heuristic 1 to determine the trading quantity. The table clearly shows that the total Vickrey discount and total Vickrey surplus values obtained by the decomposition approach are quite close to those obtained when the problem is solved optimally. To investigate this at a more detailed level, we computed the individual Vickrey discounts ($VD_o$ and $VD_d$) and Vickrey surpluses ($VS_o$ and $VS_d$) for the ten buyers and ten sellers. Table VI shows these results for data set 1. In the tables, we have omitted the fractional component of the surplus values (by truncating the values to the nearest integer). The results in this table also suggest that in most of the cases, even at the level of individual buyers and sellers, the Vickrey discounts and Vickrey surpluses obtained are close to the VCG values as computed by the direct solution approach. This shows that our approach is approximately strategy proof.

The results presented in Tables V and VI for the case of the decomposition approach use heuristic 1. Similar results are obtained if heuristic 2 is used instead.

### E. Computational Savings

Note that the clearance of a single-item, multiunit exchange by the direct method will involve solving $(m + n + 1)$ NP-hard problems in the worst case, where $m$ is the number of buyers and $n$ is the number of sellers. By using the decomposition approach, this complexity is reduced to that of solving the following three problems.

1) Determining a trading quantity, for which we have provided two heuristics. Before applying these heuristics, we first sort the bids from buyers and sellers, which has worst-case running time $O(max(m \log m, n \log n))$. Heuristic 1 has a worst-case running time of $O((U - L)N)$, where $N = n + m$ and $U$ and $L$ are as described in Section IV-A and Heuristic 2 has a worst case running time of $O(max(m \log m, n \log n))$.

2) Reverse auction using an FPTAS algorithm.

TABLE VII
COMPARISON OF COMPUTATION TIME (IN SECONDS) OF DECOMPOSITION
APPROACH (USING HEURISTIC 2) WITH THAT OF EXACT APPROACH

| m | n | $T_o$ | $T_d$ | $S_o$ | $S_d$ |
|---|---|---|---|---|---|
| 700 | 700 | 72 | 6 | 1.27e04 | 1.27e04 |
| 800 | 800 | 78 | 6 | 1.46e04 | 1.44e04 |
| 900 | 900 | 132 | 5 | 1.63e04 | 1.63e04 |
| 1000 | 1000 | 167 | 8 | 1.81e04 | 1.81e04 |
| 1100 | 1100 | 224 | 6 | 2.01e04 | 2.00e04 |
| 1200 | 1200 | *** | 9 | *** | 1.16e04 |
| 1300 | 1300 | *** | 21 | *** | 2.34e04 |
| 1400 | 1400 | *** | 27 | *** | 2.62e04 |
| 1500 | 1500 | *** | 30 | *** | 2.67e04 |
| 5000 | 5000 | *** | 164 | *** | 9.26e04 |

3) Forward auction using an FPTAS algorithm.

Since all of the above steps have polynomial time complexity, the decomposition approach will lead to significant savings in running time, compared to the direct method. Table VII compares the solution time of the decomposition approach with that of the exact approach. Here, $m$ is the number of buyers and $n$ is the number of sellers participating in the exchange. $T_o$ and $T_d$ denote the computation time in seconds of the direct approach (using ILOG CPLEX solver) and the decomposition approach, respectively. $S_o$ and $S_d$ denote the total surplus obtained using the direct approach and the decomposition approach, respectively. For each choice of $m$ and $n$, the experiment was conducted 20 times and the computation time reported is an average over these 20 replications. To make the problem interesting from a computational viewpoint, we introduced an additional business constraint in this experiment, namely, that no single buyer is to be allocated greater than 50% of the total quantity traded. We used heuristic 2 to compute the trading quantity in the decomposition approach. The $***$ entry in the table indicates that the ILOG CPLEX solver was unable to solve the instance even in 3600 s (1-h computing time). The table clearly shows the tremendous speedups achieved by the decomposition approach for large problem instances. Also, the surplus values computed by the decomposition approach are quite close to the optimal values (wherever the optimal values could be computed). Notice the nonmonotonicity in the sequence 6, 6, 5, 8, 6. This is a trend observed for small values of $m$ and $n$. Monotonicity is observed for higher values of $m$ and $n$. In fact, the computation time starts rising sharply only after $m = n = 1300$.

The results presented in Table VII for the case of decomposition approach use heuristic 2. Similar results are obtained if heuristic 1 is used instead.

## V. SUMMARY AND FUTURE WORK

In this paper, we have used a simple, natural method of decomposing a multiunit, single-item exchange problem into forward auction and reverse auction problems. We have presented two heuristics for determining the quantity to be traded which is required for solving the forward auction and reverse auction problems independently. We have used known fully polynomial time approximate algorithms for solving these individual problems. Our specific contributions in this paper are as follows:

- establishing that the decomposition approach is an attractive approach to clear single-item, multiunit exchanges with numerical experimentation;
- polynomial time heuristics for determining trading quantity to be used in the decomposition approach.

There is plenty of scope for further work in several directions: 1) we have looked at single-item exchanges here. The next immediate problem would be to look at multiunit combinatorial exchanges using the decomposition-based approach; 2) the strategy proofness properties of the mechanism when the decomposition approach is used needs to be formally investigated; 3) formal error bounds on the value of the trading quantity when heuristic 1 and heuristic 2 are used need to be investigated; and 4) formal error bounds on the value of the objective function, when the decomposition approach in conjunction with the heuristics is used, also need to be investigated.

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