POLYMERS WITH RANDOMNESS: PHASES AND PHASE TRANSITIONS

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Abstract

We discuss various aspects of the randomly interacting directed polymers with emphasis on the phases and phase transition. We also discuss the behaviour of overlaps of directed paths in a random medium.

I. INTRODUCTION

Polymers with randomness can be classified into two categories: (1) random medium, and (2) random interaction, analogous to the random field and random bond problems for magnets. These random problems remain notoriously difficult [1]. Attention has therefore gradually shifted to simpler models, and for the past ten years directed polymers played a crucial role in unraveling various issues concerning disordered systems. This model is also important because of its rich behavior, and connection with nonequilibrium surface growth problem with noise, flux line pinning in dirty samples etc [2]. We like to give an overview of the problem of randomly interacting directed polymers.

The pure problem is discussed in the second section. The randomly interacting model and its field theoretic study can be found in section III. An exact real space renormalization group approach to study the finite size effect is given in section IV. The question of overlap of two paths in a random medium that can be recast as a problem of interacting directed polymers in a random medium is explored in section V. A summary is given in section VI.

II. PURE PROBLEM: BINDING-UNBINDING

Directed polymers (DP) are lines or polymers with a preferred direction. For m such DPs a general Hamiltonian with many body interaction can be written as:

$$H_m = \frac{1}{2}\sum_{i=1}^{m}\int_0^N dz \ \dot{\mathbf{r}}_i^2 + \int_0^N dz \ v_m \prod_{i=1}^{m-1} \delta^d(\mathbf{r}_{ii+1}(z)) + \sum_{i=1}^{m}\int_0^N dz \ V(\mathbf{r}_i, z), \tag{1}$$

where $\dot{\mathbf{r}}_i = \partial \mathbf{r}_i/\partial z$, and $\mathbf{r}_{ij}(z) = \mathbf{r}_i(z) - \mathbf{r}_j(z)$, $\mathbf{r}_i(z)$ being the d dimensional transverse coordinate of a point at z on the contour of the ith chain [3–6]. The first term is the elastic energy, taking care of the connectivity of the chains. The polymers interact with a coupling constant v_m if all the m chains meet at a point. There can also be an external potential $V(\mathbf{r}, z)$ which in the random potential problem is a stochastic variable. For most of this paper we will consider only $m = 2$ and $m = 3$ cases. The external random potential problem will be considered in the last section.

With $V = 0$, the polymers undergo a binding unbinding transition as the strength of the potential is varied. For the two body ($m = 2$) problem, the transition takes place at $v_2 = 0$ for $d < 2$, while a minimum strength of attraction is needed for binding at $d > 2$. This is reflected in the renormalization group (RG) approach through the unstable fixed point for the β function for the coupling constant. This β function tells us the flow of the coupling as the system is probed at a bigger length scale. It is now well known that the field theoretic RG can be implemented exactly for this class of many body problems. Introducing a dimensionless coupling constant $u_2 = v_2 L^{2-d}$, the exact β function is given by

$$\beta(u) \equiv L\frac{\partial u}{\partial L} = 2\epsilon' u \left(1 - \frac{u}{4\pi\epsilon'}\right), \tag{2}$$

where u is the renormalized dimensionless coupling constant [3]. Note that $2\epsilon' = (2 - d)$.

The flow diagram for the dimensionless coupling constant u is shown in Fig.1a. The fact that for $d < 2$ any small attractive interaction is able to form a bound state is reflected by

the flow to the nonperturbative regime for any negative u. The repulsive or the positive u region is dominated by the stable, "fermionic" fixed point $u^*(= 4\pi\epsilon')$. For $d > 2$ there exists a nontrivial unstable fixed point $u = u^*$ which separates the bound and the unbound states for the two polymers. In short, the unstable fixed point represents the critical point for the binding-unbinding transition. The correlation length, $\xi_{\parallel}$ for the transition describes the average separation of two contacts along the chain, and it diverges as the critical point is approached with an exponent $\nu = 1/|\epsilon'|$ for $1 \leq d < 4$, except for $d = 2$, where the correlation length diverges exponentially as $\exp(1/v_2)$. Other approaches seem to suggest that $d = 4$ is the upper critical dimension for this problem [7,8], however RG is yet to give us that result. We would like to point out that the exact β function of Ed. 2 is obtained by absorbing the poles at $d = 2$ in a dimensional regularization scheme. This, of course, leaves the poles at $d = 4$ untreated. Is it the signal for an upper critical dimension at $d = 4$?

The stable fixed point describes, in this problem, the unbound phase. A way of characterizing the phase is to look at the asymptotic behaviour of the reunion partition function, $\mathcal{Z}_R(N)$. This partition function describes the situation where the chains are tied together at both the ends, and the end points can be anywhere in space. The asymptotic behaviour of $\mathcal{Z}_R(N) \sim N^{-\Psi}$ was studied long ago for $d = 1$ in a different context [9,10]. It is known that $\Psi = 3/2$ for two chains in 1 dimension with repulsive two body interaction. It follows from an exact random walk analysis or from a use of fermionic nature of the quantum particles. These methods are restricted to one dimension only. RG is the appropriate framework to obtain the asymptotic behaviour for other d, and the exponent follows form the multiplicative renormalization constant for the partition function. In this framework, the Huse-Fisher 1 dimensional exponent can be recovered as an $O(\epsilon')$ result which is, in fact, exact [11,12]. Furthermore, this RG analysis also shows that because of the marginality of the coupling at $d = 2$, there is a special log correction to the Gaussian behaviour, and $\mathcal{Z}_R(N) \sim N^{-1}(\log N)^{-2}$. More general results can be found in Ref. [12]. This log correction in two dimensions has recently been recovered by Guttmann and Essam in an exact lattice calculation [13].

If we now go to the three body interaction, then again a similar exact analysis can be carried out [4]. We, in this paper, however, restrict ourselves only to $d = 1$ which turns out to be the marginal case for v_3. The critical exponent for unbinding transition is $\nu = 2/|1 - d|$, except for $d = 1$ where the correlation length diverges like $\exp(1/v_3)$. The three chain reunion partition function will have a log correction, identical to the marginal two chain case, namely, $\mathcal{Z}_R(N) \sim N^{-2}(\log N)^{-2}$. The similarity in the behaviour of the many-body interactions, if proper variables are used, is a novel feature of the multicritical directed polymers, and has been termed "Grand Universality" [5]. We will see that such a *grand universality* is preserved also for the random case.

Since there is only one fixed point at $d = 2(d = 1)$ for the two (three) chain problem, one can identify the fixed point at zero as the critical point while the approach to the fixed point as describing the phase of the system.

III. RANDOM INTERACTION

We now consider the random version where the polymers interact with a random coupling constant and there is no external potential. For simplicity we consider randomness to be

dependent only on z. A physical realization would be a random distribution of monomers (or charges) along the backbone. The interaction is given by

$$\int_0^N dz \; v_0 \; [1 + b(z)] \; \delta(\mathbf{r}_{12}(z)), \tag{3}$$

where the randomness is introduced through $b(z)$. We take uncorrelated disorder with a Gaussian distribution

$$P(b(z)) = (2\pi\Delta)^{-1/2} \; \exp[-b(z)^2/(2\Delta)], \tag{4a}$$

$$\langle b(z)\rangle = 0, \text{ and } \langle b(z_1)b(z_2)\rangle = \Delta \; \delta(z_1 - z_2). \tag{4b}$$

For the many body interaction problem, the random Hamiltonian would be similar to Eq. 1 except that the coupling constant v_m is to be replaced by $v_m(1 + b(z))$ inside the integral.

The approach we take is to study the various cumulants of the partition function. The first cumulant, as one might expect, behaves like the pure problem but with a shifted critical point. Since there are sites with attractive interaction, the critical point for unbinding occurs not at zero average interaction but at a certain nonzero repulsion. It would also mean that even if the chains are on the average repulsive, (i.e., $v_2 > 0$), a binding-unbinding transition can be induced by tuning the disorder or changing the temperature. Such a thermal unbinding is not possible in the pure case for $d \leq 2$. Apart from that, the critical behaviour remains the same. The situation is different for the second cumulant.

When we consider the second cumulant of the partition function, we require four (six) chains for the two (three) body case. On averaging over the disorder, an interaction ("inter replica" interaction) is generated that couples the original chains with the replica. For example, for the two body problem, the interaction is of the type

$$H_{\text{rep}} = -\bar{r}_0 \int_0^N dz \; \delta(\mathbf{r}_{12}(z)) \; \delta(\mathbf{r}_{34}(z)), \tag{5}$$

with $\bar{r}_0 = v_0^2\Delta$. This interaction, different from the four body multicritical interaction of Eq. 1, is a correlation effect. If chains 1 and 2 meet at length z, then it is favourable for 3 and 4 also to have a contact at that same z though not necessarily at the same transverse space coordinate. The importance of the disorder can therefore be understood if we know the flow of this interaction as the probing length scale is changed. If we are at the critical point of the average interaction, then RG can be implemented exactly [14,15]. Defining the dimensionless coupling constant r_0 through an arbitrary length scale L as $r_0 = \bar{r}_0 L^{2\epsilon}(4\pi)^{-d}$, $\epsilon = 1 - d$, and denoting the renormalized coupling as r, the RG β function is given by

$$\beta(r) \equiv L\frac{\partial r}{\partial L} = 2(\epsilon r + r^2). \tag{6}$$

There are two fixed points: (i) $r = 0$ and (ii) $r^* = -\epsilon$. See Fig. 1b. The bare coupling constant r_0 which originates from $v_0^2\Delta$, where Δ, the variance of the distribution, is strictly positive, requires a positive r. Therefore, the nontrivial fixed point for $d < 1$ in negative r is unphysical. It however moves to the physical domain for $d > 1$. The main feature that comes from the analysis is that the disorder is marginally relevant at $d = 1$. This means

that even a small disorder will change the nature of the unbinding transition and take the critical system to a "strong" coupling regime. There is however no perturbative fixed point to describe the strong coupling regime. A marginally relevant variable means that a new critical feature would appear for higher dimensions. This is reflected in the new unstable fixed point. For small enough disorder, the β function for $d > 1$ takes Δ to zero, reproducing a pure type behaviour. If however, the starting disorder is higher than the fixed point value, then it goes to the strong coupling regime. The unbinding transition is therefore pure type for small disorder ("weak" disorder) and beyond a critical disorder, in the "strong" disorder regime, a new critical behaviour is expected.

The exact nature of the RG is lost if $v_2 \neq 0$. In a one loop approach, there are indications of the existence of a fixed point for the stable fixed point of v_2 [16]. Since the flows are different on the two sides, one wonders whether this indicates a new phase also.

The exact β function of Eq. 6 tells us also that around the critical disorder, the relevant length scale exponent is $1/ \mid 1 - d \mid$ along the chain. In one dimension, the length scale diverges exponentially around $r = 0$. These exponents have been verified numerically.

A dynamic renormalization group approach has also been developed for the two chain problem. In this approach, instead of looking at the moments of the partition function, the free energy is probed directly. This approach yields the same results and shows that there is no special fluctuation exponent for the free energy. [17]

IV. REAL SPACE RG

Due to the absence of any fixed point for the strong coupling regime, it is necessary to study the problem using nonperturbative methods. One such method is the real space RG (RSRG), which can be implemented exactly on hierarchical lattices [18,19] as shown in Fig 2. To avoid unnecessary approximations, we work with these lattices from the beginning. As per construction, we consider the partition function for two chains tied at the both ends and with a random site interaction. We want to study the various moments of the partition function.

It is again clear that for the nth moment, we require $2n$ chains and they will be coupled by the disorder. An effect of this is that there is an analog of the binding-unbinding transition for each moment, and the higher the moment (n) the higher is the transition temperature. In the high temperature phase for the nth moment, the free energy is expected to approach the free entropy of $2n$ chains. Subtracting out this free part, the free energy approaches zero in the thermodynamic limit for high temperatures while, it has an $O(1)$ value per bond in the low temperature phase. Let us define $\mathcal{Z}_\mu(n) = \overline{Z}_\mu^n / S_\mu^{2n}$, where μ is the generation, S_μ is the entropy of a single chain of length $L_\mu = 2^{\mu-1}$, the overline representing the disorder average. If we keep the temperature fixed (above the critical point for the first moment), then there exists a critical n, n_c, so that for $n < n_c$ the moments are in the high temperature phase. We want to study the finite size correction to the moments of as $n \to n_c-$.

The approach to the thermodynamic limit can be written as

$$\mathcal{Z}_\mu(n) = \mathcal{Z}^*(n) + B_z(n) \ L_\mu^{-\psi} + ..., \tag{7}$$

where $\mathcal{Z}^*(n)$ is the thermodynamic limit ($\mu \to \infty$), and $B_z(n)$ is the amplitude of the finite size correction.

For a given realization of the disorder, the partition function can be written as (see Fig 2)

$$Z_{\mu+1} = bZ_\mu^{(\mathcal{A})} y Z_\mu^{(\mathcal{B})} + b(b-1)S_\mu^2. \tag{8}$$

The first term originates from the configurations that require the two DPs to meet at C, while the second term counts the "no encounter" cases. There are no energy costs at the two end points. The Boltzmann weight is random, and for a Gaussian distribution of energy, $\overline{y^m} = \overline{y}^{m^2}$. Note also that $S_\mu = b^{L_\mu - 1}$, where $L_\mu = 2^\mu$ is the length of DP.

The moments of the partition function, from Eq. 8, are

$$\mathcal{Z}_{\mu+1}(n) = b^{-n} \sum_{m=0}^{n} P_{nm} \mathcal{Z}_\mu^2(m), \tag{9}$$

where $P_{nm} = \binom{n}{m}(b-1)^{n-m}\overline{y^m}$, with the initial condition $\mathcal{Z}_0(n) = 1$ for all the moments because there is no interaction in the zeroth generation (one single bond). By iterating these recursion relations, the moments are calculated exactly and the finite size correction is estimated. As shown in Fig. 2, the amplitude has a power law divergence as n_c is approached, namely

$$B_z(n) \sim (n_c - n)^{-r}, \quad \text{for } n \to n_c-, \tag{10}$$

with $r = .71 \pm .02$. The exponent is independent of temperature but depends on the effective dimensionality of the system. We call this a scar left by disorder in the high temperature phase.

V. OVERLAP IN A RANDOM MEDIUM

Much has been achieved in the problem of directed polymer in a random medium (DPRM). Unlike the random interaction case, carrying out a systematic RG beyond one loop is extremely hard [20]. On the other hand a remarkable extra gain in the DPRM case is the exact knowledge of the nontrivial geometric and thermal properties at least at $d = 1$. It is known through the mapping of DPRM problem to the nonlinear noisy growth equation of a surface (KPZ equation) [21] that at $d = 1$ the transverse extension of the polymer and the free energy fluctuation scale as $\overline{<x>^2}^{1/2} \sim N^{\zeta=2/3}$ and $f \sim x^{\chi=1/2}$. These results can be proved to be exact through the fluctuation dissipation theorem. At $d = 1$ this new value of $\zeta \neq 1/2$ persists for all temperatures and the system is always at the strong disorder or "low temperature" phase. For $d > 2$ there is a transition from a high temperature phase (free polymer) to a low temperature ('spin glass' type) phase. Though the unstable fixed point is well under control, the strong disorder fixed point is not reachable through perturbation. Numerical approaches intensified the controversies about the strong disorder phase. Another unsolved question is the existence of an upper critical dimension (UCD) which, in some approaches, seems to be 4. There is a hope that if RG can resolve the question of UCD for the pure interacting DP problem, as mentioned in section II, then the UCD problem for DPRM can also understood.

Here we discuss how the overlap behaves in this problem. Since the low temperature phase is a spin glass type phase we expect the overlap to serve as an order parameter. The m chain overlap is in general defined as

$$q_m = -\frac{1}{t} \int_0^t dz < \overline{\prod_{i=1}^{m-1} \delta(\mathbf{x}_{i,i+1}(z))} >, \tag{11}$$

where $\mathbf{x}_{i,i+1}(z) = \mathbf{x}_i(z) - \mathbf{x}_{i+1}(z)$, bar and the angular brackets denote the disorder and thermal average respectively. In the replica language, this overlap describes the common configurations of the valleys (i.e. common to m) of a rugged free energy landscape. The overlap comes from the common path of m chains in the same random medium. The behaviour of the overlap is nontrivial because the disorder induces an attraction among the replicas. We consider only the two chain overlap here.

The main key to solve the problem is to introduce a 2 body interaction in the Hamiltonian of DPRM problem and use a mapping that leads to a KPZ type nonlinear equation [22]. Defining the quenched free energy as $f_2(v_2, z)$, the overlap can be obtained as $q_2 = -z^{-1} \, df_2(v_2, z)/dv_2 \mid_{v_2=0}$. With the scaling hypothesis for the free energy, $f_2 = z^{\chi/\zeta} \mathsf{f}(v_2 t^{-\phi\zeta})$ we find the scaling of the overlap as $q_2 \sim z^{\Sigma}$ with $\Sigma = (\chi - \phi - 1)\zeta$. Our interest here is in finding out Σ.

The working Hamiltonian is therefore similar to Eq. 1 with $m = 2$. For convenience, we introduce a line tension γ so that the elastic part of Eq. 1 is of the type $\frac{\gamma \dot{\mathbf{r}}_i^2}{2}$ and also choose the coupling constant as $\lambda v_2/(2\gamma)$ and $\lambda V/(2\gamma)$ as the random potential. The random potential is with zero mean and $\langle V(\mathbf{r}, \tau)V(\mathbf{r}', \tau')\rangle = \Delta \delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau')$. The free energy defined as $h(\{\mathbf{x_j}\}, t) = (2\gamma/\lambda) \ln Z(\{\mathbf{x_j}\}, t)$, where $Z(\{\mathbf{x_j}\}, t)$ is the partition function, satisfies

$$\frac{\partial h}{\partial t} = \sum_{j=1}^2 (\gamma \nabla_j^2 h + \frac{\lambda}{2}(\nabla_j h)^2) + g_0, \tag{12}$$

where $g_0 = \sum_{j=1}^2 V(\mathbf{x}_j, t) + v_2 \delta(\mathbf{x}_{12}(t))$. In order to bring out the similarity with growth equation we use t instead of z as the variable for the special direction. A dynamic renormalization group calculation upto $O(\lambda^2)$ and $O(v_2)$ for the above equation in the Fourier space leads to the following crucial results [22]. (**i.**) There is no change in the single chain behavior. The renormalization of the single chain propagator, and the nonlinearity is the same as that of a single chain in a random medium. (**ii.**) The coupling constant gets renormalized even at $O(v_2)$ as given below

$$v_{2R} = v_2 + 8(-\frac{\lambda}{2})^2(2v_2\Delta) \int_{p,\Omega} p^4 G_0(\mathbf{p}, \Omega)[G_0(-\mathbf{p}, -\Omega)]^2 G_0(\mathbf{p}, -\mathbf{p}, 0). \tag{13}$$

Here G_0 is the bare propagator, defined as $G_0(\{\mathbf{k}_j\}, \omega) = [\gamma \sum_j k_j^2 - i\omega]^{-1}$, $(\mathbf{k}, \omega)$ being the Fourier conjugate variables for $(\mathbf{r}, t)$. A short hand notation $\int_{\mathbf{p}\omega} \sim \int d\mathbf{p} d\omega/(2\pi)^{d+1}$ is used. In the above equation only the nonzero momentum vectors are written explicitly as the arguments of G_0.

The next obvious point is the presence of an anomalous dimension in the coupling constant as is apparent from the recursion relation

$$\frac{dv_2}{dl} = [z - \chi - d + U]v_2. \tag{14}$$

Here the term proportional to $U = K_d \lambda^2 \Delta / (2\gamma^3)$ is the anomalous part from the RG and $K_d = (2\pi)^{-d} S_d$, with S_d being the surface area of the unit d dimensional sphere. Since we find that the single chain properties remain unaffected, we use in the following the KPZ fixed point value for U. Let us first consider $d = 1$. It has been shown that the KPZ fixed point is a stable one and indicates a glassy behavior at all temperatures. By substitution $U^* = 1$, we find that $\Sigma = 0$, which one would expect in the low temperature phase, as one finds numerically [23]. For $d = 2 + \epsilon$ the KPZ fixed point $U^* = 2\epsilon$, being an unstable fixed point, corresponds to the spin glass transition. The exponent for the overlap can be readily obtained at this fixed point as $\Sigma = -[d + \eta]$, where $\eta = -2\epsilon$. The analysis can be extended to m chain overlap also.

VI. CONCLUSION

Randomly interacting directed polymers exhibit a weak to strong disorder transition for $d > 1$. This can be established by an exact renormalization group approach. Real space renormalization group approach for hierarchical lattices reveals a diverging finite size correction. This might indicate the existence of a phase with no counterpart in the pure system. Could it be a Griffiths phase? This is an open question. Using interacting directed polymers in a random medium, we have calculated the decay of overlap at the spin glass transition point in $2 + \epsilon$ dimensions. In one dimension, this analysis recovers the result based on fluctuation arguments and numerical simulation.

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FIGURES

FIG. 1. Flow diagrams for (a) the pure coupling u for two chains, and (b) the disorder r_m for m chains. Here $d_m = 1/(m-1)$. In both cases, * denotes nontrivial fixed points.

Plot of $B_z(n)^{-1/r}$ vs n for $b = 4$ and various temperatures. (a) $\log \overline{y} = 0.065$, and $r = 0.73$, (b) $\log \overline{y} = 0.04$, and $r = .73$ (c) $\log \overline{y} = 0.03$, and $r = 0.72$ (d) $\log \overline{y} = 0.02$, and $r = 0.72$. Inset shows the construction of a hierarchical lattice with $b = 2$. .