Sādhanā Vol. 33, Part 2, April 2008, pp. 131–174. © Printed in India

Foundations of mechanism design: A tutorial Part 2 – Advanced concepts and results

DINESH GARG[1], Y NARAHARI[2*] and SUJIT GUJAR[2]

[1]IBM India Research Laboratory, Bangalore 560 071
[2]Electronic Commerce Laboratory, Department of Computer Science and Automation, Indian Institute of Science, Bangalore 560 012
e-mail: dingarg2@in.ibm.com; {hari,sujit}@csa.iisc.ernet.in

MS received 6 November 2006; revised 16 December 2007

Abstract. Mechanism design, an important tool in microeconomics, has found widespread applications in modelling and solving decentralized design problems in many branches of engineering, notably computer science, electronic commerce, and network economics. In the first part of this tutorial on mechanism design (Garg *et al* 2008), we looked into the key notions and classical results in mechanism design theory. In the current part of the tutorial, we build upon the first part and undertake a study of several other key issues in mechanism design theory.

Keywords. Mechanism design; game theory; social choice functions; auctions.

1. Introduction

In the second half of the twentieth century, game theory and mechanism design have found widespread use in a gamut of applications in engineering. More recently, game theory and mechanism design have emerged as an important tool to model, analyse, and solve *decentralized design problems* in engineering involving multiple autonomous agents that interact strategically in a *rational* and *intelligent* way. The first part of this tutorial dealt with key notions and classical results in mechanism design theory (Garg *et al* 2008). The importance of mechanism design in the current context can be seen by the fact that the Nobel Prize in Economic Sciences for the year 2007 was jointly awarded to three economists, Leonid Hurwicz, Eric Maskin, and Roger Myerson *for having laid the foundations of mechanism design theory* (The Nobel Foundation 2007). Earlier, in 1996, William Vickrey, the inventor of the famous Vickrey auction had been awarded the Nobel Prize in Economic Sciences in 1996.

The current paper is the second part of our tutorial introduction to mechanism design. Recall that the following issues were covered in Part 1 of this tutorial:

- We started by introducing the notion of a *social choice function* through several examples: bilateral trade, auctions for selling a single indivisible item, and a combinatorial auction. These examples are used throughout the rest of the paper for bringing out

*For correspondence

important insights. Next, we introduced the concept of a *mechanism* and brought out the difference between direct revelation mechanisms and indirect mechanisms. Following this, we focused on the notion of *implementation* of a social choice function by a mechanism. We introduced two key notions of implementation, namely dominant strategy implementation and Bayesian implementation.

- Next, we described the desirable properties of a social choice function, which included *ex-post efficiency, non-dictatorialness, dominant strategy incentive compatibility* and *Bayesian–Nash incentive compatibility*. We then stated and proved a fundamental result in mechanism design theory, the *revelation theorem*.
- Following this, we described a landmark result — the *Gibbard–Satterthwaite* impossibility theorem, which says that under fairly general conditions, no social choice function can satisfy the three properties — ex-post efficiency, non-dictatorial, and dominant strategy incentive compatibility simultaneously. This impossibility theorem is a special case of the celebrated Arrow's impossibility theorem, which was also presented.
- The Gibbard–Satterthwaite theorem, while ruling out implementability of certain desirable mechanisms, suggests two alternative routes to design useful mechanisms. The first of these two routes is to restrict the utility functions to what is known as a *quasi-linear environment*. We introduced the quasi-linear environment and showed that ex-post efficiency is equivalent to a combination of two properties, namely, *allocative efficiency* and *budget balance*. Later, we studied the celebrated *VCG (Vickrey–Clarke–Groves)* social choice functions (also known as VCG mechanisms), which are non-dictatorial, dominant strategy incentive compatible, and allocatively efficient.
- Finally, we explored the second route suggested by the Gibbard–Satterthwaite impossibility theorem by looking at Bayesian incentive compatibility instead of dominant strategy incentive compatibility. We showed that a class of social choice functions, known as *dAGVA* social choice functions are ex-post efficient, non-dictatorial, and Bayesian–Nash incentive compatible. We then developed a characterization of Bayesian incentive compatible social choice functions in linear environment.

In this part 2, we delve into a few deeper notions, mechanisms, and results. We also make an attempt to look at some recent advances in the area of mechanism design. This paper is organized as follows.

- In § 2, we discuss the *Revenue Equivalence Theorem* which implies that the expected revenue produced by the English auction, Dutch auction, first price auction, and second price auction is the same in the context of a single indivisible item. This is a well known result in auction theory and uses many of the concepts and results that we learnt in Part 1 of this tutorial.
- This is followed, in § 3, by a discussion of the concept of individual rationality and the well known *Myerson–Satterthwaite* impossibility theorem.
- In §4, we discuss a class of Groves mechanisms studied by Moulin (2007). These are allocatively efficient, individually rational, dominant strategy incentive compatible, and are almost budget balanced.
- Section 5 is devoted to optimal mechanisms. Here, we first define the notion of optimal mechanism design and then describe the seminal work of Myerson (1981) on design of optimal auctions. We also discuss several recent extensions of Myerson's work.
- In § 6, we look at a characterization of dominant strategy incentive compatible social choice functions. The major result we describe is that of Roberts (1979).

- Section 7 is devoted to dominant strategy implementation of Bayesian incentive compatible mechanisms, as described by Mookherjee & Reichelstein (1992).
- In § 8, we discuss the notion of implementation in ex-post Nash equilibrium which, while still weaker than dominant strategy implementation, is stronger than Bayesian implementation.
- In § 9, we discuss mechanism design where the types of the agents are not independent but are interdependent.
- In § 10, we provide a brief glimpse of some advanced topics such as: implementation theory, dynamic mechanisms, iterative mechanisms, stochastic mechanisms, cost sharing mechanisms, robust mechanisms, and virtual implementation.
- Finally, in § 11, we suggest several pointers to the literature to probe further into mechanism design theory.

2. Revenue equivalence of auctions

In this §, we prove two results that show the revenue equivalence of certain classes of auctions. The first theorem is a general result that shows the revenue equivalence of two auctions that satisfy certain conditions. The second result is a more specific result that shows the revenue equivalence of four different types of auctions (English auction, Dutch auction, first price auction, and second price auction) in the special context of auction of a single indivisible item. The proof of the second result crucially uses the first result.

2.1 *Revenue equivalence of two auctions*

Assume that $y_i(\theta)$ is the probability of agent i getting the object when the vector of announced types is $\theta = (\theta_1, \ldots, \theta_n)$. Expected payoff to the buyer i with a type profile $\theta = (\theta_1, \ldots, \theta_n)$ will be $y_i(\theta)\theta_i + t_i(\theta)$. The set of allocations is given by

$$K = \left\{ (y_1, \ldots, y_n) : y_i \in [0, 1] \forall i = 1, \ldots, n; \sum_{i=1}^{n} y_i \leq 1 \right\}.$$

As stated earlier, let $\overline{y_i}(\hat{\theta}_i) = E_{\theta_{-i}}[y_i(\hat{\theta}_i, \theta_{-i})]$ be the probability that agent i gets the object conditional to announcing his type as $\hat{\theta}_i$, with the rest of the agents announcing their types truthfully. Similarly, $\overline{t_i}(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ denotes the expected payment received by agent i conditional to announcing his type as $\hat{\theta}_i$, with the rest of the agents announcing their types truthfully. Let $\overline{v_i}(\hat{\theta}_i) = \overline{y_i}(\hat{\theta}_i)$. Then,

$$U_i(\theta_i) = \overline{y_i}(\theta_i)\theta_i + \overline{t_i}(\theta_i),$$

denotes the payoff to agent i when all the buying agents announce their types truthfully. We now state and prove an important proposition.

Theorem 2.1. *Consider an auction scenario with:*

(i) n *risk-neutral bidders (buyers)* $1, 2, \ldots, n$
(ii) The valuation of bidder i $(i = 1, \ldots, n)$ *is a real interval* $[\underline{\theta_i}, \overline{\theta_i}] \subset R$ *with* $\underline{\theta_i} < \overline{\theta_i}$
(iii) The valuation of bidder i $(i = 1, \ldots, n)$ *is drawn from* $[\underline{\theta_i}, \overline{\theta_i}]$ *with a strictly positive density* $\phi_i(\cdot) > 0$. *Let* $\Phi_i(\cdot)$ *be the cumulative distribution function.*
(iv) The bidders' types are statistically independent.

Suppose that a given pair of Bayesian–Nash equilibria of two different auction procedures are such that:

- *For every bidder i, for each possible realization of* $(\theta_1, \ldots, \theta_n)$*, bidder i has an identical probability of getting the good in the two auctions.*
- *Every bidder i has the same expected payoff in the two auctions when his valuation for the object is at its lowest possible level.*

Then the two auctions generate the same expected revenue for the seller.

Before proving the theorem, we elaborate on the first assumption above, namely risk neutrality. A bidder is said to be:

- *risk-averse* if his utility is a *concave* function of his wealth; that is an increment in the wealth at a lower level of wealth leads to an increment in utility that is higher than the increase in utility due to an identical increment in wealth at a higher level of wealth;
- *risk-loving* if his utility is a *convex* function of his wealth; that is an increment in the wealth at a lower level of wealth leads to an increment in utility that is lower than the increase in utility due to an identical increment in wealth at a higher level of wealth; and
- *risk-neutral* if his utility is a *linear* function of his wealth; that is an increment in the wealth at a lower level of wealth leads to the same increment in the utility as an identical increment would yield at a higher level of wealth.

Proof. By the revelation principle, it is enough we investigate two Bayesian incentive compatible social choice functions in this auction setting. It is enough we show that two Bayesian incentive compatible social choice functions having (a) the same allocation functions $(y_1(\theta), \ldots, y_n(\theta))\ \forall \theta \in \Theta$, and (b) the same values of $U_1(\underline{\theta_1}), \ldots, U_n(\underline{\theta_n})$ will generate the same expected revenue to the seller.

We first derive an expression for the seller's expected revenue given any Bayesian incentive compatible mechanism. Expected revenue to the seller

$$= \sum_{i=1}^{n} E_\theta[-t_i(\theta)]. \tag{1}$$

Now, we have:

$$E_\theta[-t_i(\theta)] = E_{\theta_i}[-E_{\theta_{-i}}[t_i(\theta)]]$$

$$= \int_{\underline{\theta_i}}^{\overline{\theta_i}} [\overline{y_i}(\theta_i)\theta_i - U_i(\theta_i)]\phi_i(\theta_i)d\theta_i$$

$$= \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left[[\overline{y_i}(\theta_i)\theta_i - U_i(\underline{\theta_i})] - \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \right] \phi_i(\theta_i)d\theta_i.$$

The last step is an implication of Myerson's characterization of Bayesian incentive compatible functions in linear environment. The above expression is now equal to

$$= \left[\int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\overline{y_i}(\theta_i)\theta_i - \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \right) \phi_i(\theta_i)d\theta_i \right] - U_i(\underline{\theta_i}).$$

Now, applying integration by parts with $\int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds$ as the first function, we get

$$\int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \right) \phi_i(\theta_i)d\theta_i = \int_{\underline{\theta_i}}^{\overline{\theta_i}} \overline{y_i}(\theta_i)d\theta_i - \int_{\underline{\theta_i}}^{\overline{\theta_i}} \overline{y_i}(\theta_i)\Phi_i(\theta_i)d\theta_i$$

$$= \int_{\underline{\theta_i}}^{\overline{\theta_i}} \overline{y_i}(\theta_i)[1 - \Phi_i(\theta_i)]d\theta_i.$$

Therefore, we get

$$E_{\theta_i}[-\overline{t_i}(\theta_i)] = -U_i(\underline{\theta_i}) + \left[\int_{\underline{\theta_i}}^{\overline{\theta_i}} \overline{y_i}(\theta_i) \left\{ \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right\} \phi_i(\theta_i)d\theta_i \right]$$

$$= -U_i(\underline{\theta_i}) + \left[\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_i}} y_i(\theta_1, \ldots, \theta_n) \right.$$

$$\left. \times \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \left(\prod_{j=1}^{n} \phi_j(\theta_j) \right) d\theta_n \ldots d\theta_1 \right]$$

since

$$\overline{y_i}(\theta_i) = \int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} y_i(\theta_1, \ldots, \theta_n) \quad \underbrace{d\theta_n \ldots d\theta_1}_{\text{without } d\theta_i}.$$

Therefore, the expected revenue of the seller

$$= \left[\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \sum_{i=1}^{n} y_i(\theta_1, \ldots, \theta_n) \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \right]$$

$$\times \left(\prod_{j=1}^{n} \phi_j(\theta_j) \right) d\theta_n \ldots d\theta_1 - \sum_{i=1}^{n} U_i(\underline{\theta_i}).$$

By looking at the above expression, we see that any two Bayesian incentive compatible social choice functions that generate the same functions $(y_1(\theta), \ldots, y_n(\theta))$ and the same values of $(U_1(\underline{\theta_1}), \ldots, U_n(\underline{\theta_n}))$ generate the same expected revenue for the seller.

As an application of the above theorem, we now state and prove a revenue equivalence theorem for a single indivisible item auction. Q.E.D.

2.2 *Revenue equivalence of four classical auctions*

There are four basic types of auctions when a single indivisible item is to be sold:

(i) English auction: This is also called oral auction, open auction, open cry auction, and ascending bid auction. Here, the price starts at a low level and is successively raised until only one bidder remains in the fray. This can be done in several ways: (a) an auctioneer announces prices, (b) bidders call the bids themselves, or (c) bids are submitted electronically. At any point of time, each bidder knows the level of the current best bid. The winning bidder pays the last going price.

(ii) Dutch auction: This is also called a descending bid auction. Here, the auctioneer announces an initial (high) price and then keeps lowering the price iteratively until one of the bidders accepts the current price. The winner pays the current price.
(iii) First price sealed bid auction: Recall that in this auction, potential buyers submit sealed bids and the highest bidder is awarded the item. The winning bidder pays the price that he has bid.
(iv) Second price sealed bid auction: This is the classic Vickrey auction. Recall that potential buyers submit sealed bids and the highest bidder is awarded the item. The winning bidder pays a price equal to the second highest bid (which is also the highest losing bid).

When a single indivisible item is to be bought or procured, the above four types of auctions can be used in a *reverse* way. These are called *reverse auctions* or *procurement auctions*. In this section, we would be discussing the revenue equivalence theorem as it applies to selling. The procurement version can be analysed on similar lines.

2.2a *The benchmark model:* There are four assumptions underlying the derivation of the revenue equivalence theorem: (i) risk neutrality of bidders, (ii) bidders have independent private values, (iii) bidders are symmetric, (iv) payments depend on bids alone. These are described below in more detail.

Risk neutrality of bidders: It is assumed in the benchmark model that all the bidders are risk neutral. This immediately implies that the utility functions are linear.

Independent private values model: In the independent private values model, each bidder knows precisely how highly he values the item. He has no doubt about the true value of the item to him. However, each bidder does not know anyone else's valuation of the item. Instead, he perceives any other bidder's valuation as a draw from some known probability distribution. Also, each bidder knows that the other bidders and the seller regard his own valuation as being drawn from some probability distribution. More formally, let $N = \{1, 2, \ldots, n\}$ be the set of bidders. There is a probability distribution Φ_i from which bidder i draws his valuation v_i. Only bidder i observes his own valuation v_i, but all other bidders and the seller know the distribution Φ_i. Any one bidder's valuation is statistically independent from any other bidder's valuation.

An apt example of this assumption is provided by the auction of an antique in which the bidders are consumers buying for their own use and not for resale. Another example is Government contract bidding when each bidder known his own production cost if he wins the contract.

A contrasting model is the *common value model*. Here, if V is the unobserved true value of the item, then the bidders' perceived values $v_i, i = 1, 2, \ldots, n$ are independent draws from some probability distribution $H(v_i|V)$. All the bidders know the distribution H. An example is provided by the sale of an antique that is being bid for by dealers who intend to resell it. The item has one single objective value, namely its market price. However, no one knows the true value. The bidders, perhaps having access to different information, have different guesses about how much the item is objectively worth. Another example is that of sale of mineral rights to a particular tract of land. The objective value here is the amount of mineral actually lying beneath the ground. However no one knows its true value.

Suppose a bidder somehow learns another bidder's valuation. If the situation is described by the common value model, then the above provides useful information about the likely true value of the item and the bidder would probably change his own valuation in the light of this. If the situation is described by the independent private value model, the bidder knows his own mind and learning about others valuation will not cause him to change his own valuation (although he may, for strategic reasons, change his bid).

Real world auction situations are likely to contain aspects of both the independent private values model and the common value model. It is assumed in the benchmark model that the independent private values assumption holds good.

Symmetry: This assumption implies that all the bidders have the same set of possible valuations and further they draw their valuations using the same probability density ϕ. That is $\phi = \phi_1 = \phi_2 = \cdots = \phi_n$.

Dependence of payments on bids alone: It is assumed that the payment to be made by the winner to the auctioneer is a function of bids alone.

Theorem 2.2 (Revenue equivalence theorem for single indivisible item auctions). *Consider a seller or an auctioneer trying to sell a single indivisible item in which n bidders are interested. For the benchmark model (bidders are risk neutral, bidders have independent private values, bidders are symmetric, and payments depend only on bids), all the four basic auction types (English auction, Dutch auction, first price auction, and second price auction) yield the same average revenue to the seller.*

The result looks counter intuitive: for example, it might seen that receiving the highest bid in a first price scaled bid auction must be better for the seller than receiving the second highest bid, as in second price auction. However, it is to be noted that bidders act differently in different auction situations. In particular, they bid more aggressively in a second price auction than in a first price auction.

Proof. The proof proceeds in three parts. In Part 1, we show that the first price auction and the second price auction yield the same expected revenue in their respective equilibria. In Part 2, we show that the Dutch auction and the first price auction produce the same outcome. In Part 3, we show that the English auction and the second price auction yield the same outcome. Q.E.D.

Part 1: Revenue equivalence of first price auction and second price auction: The first price auction and the second price auction satisfy the conditions of the theorem on revenue equivalence of two auctions.

- In both the auctions, the bidder with the highest valuation wins the auction
- bidders' valuations are drawn from $[\underline{\theta_i}, \overline{\theta_i}]$ and a bidder with valuation at the lower limit of the interval has a payoff of zero in both the auctions.

Thus the theorem can be applied to the equilibria of the two auctions: Note that in the case of first price auction, it is a Bayesian–Nash equilibrium while in the case of second price auction, it is a weakly dominant strategy equilibrium. In fact, it can be shown in any *symmetric* auction setting (where the bidders' valuations are independently drawn from identical distributions) that the conditions of the above proposition will be satisfied by any Bayesian–Nash equilibrium of first price auction and the weakly dominant strategy equilibrium of the second price scaled bid auction.

Part 2: Revenue equivalence of Dutch auction and first price auction: To prove this, consider the situation facing a bidder in these two auctions. In each case, the bidder must choose how high to bid without knowing the other bidders' decisions. If he wins, the price he pays equals his own bid. This result is true irrespective of which of the assumptions in the benchmark model apply. Note that the equilibrium in the underlying Bayesian game in the two cases here is a Bayesian–Nash equilibrium.

Part 3: Revenue equivalence of English auction and second price auction: First we analyse the English auction. Note that a bidder drops out as soon as the going price exceeds his valuation. The second last bidder drops out as soon as the price exceeds his own valuation. This leaves only one bidder in the fray and he wins the auction. Note that the winning bidder's valuation is the highest among all the bidders and he earns some payoff in spite of the monopoly power of the seller. Only the winning bidder knows how much payoff he receives because only he knows his own valuation. Suppose the valuations of the n bidders are $v_{(1)}, v_{(2)}, \ldots, v_{(n)}$. Since the bidders are symmetric, these valuations are drawn from the same distribution and without loss of generality, assume that these are in descending order. The winning bidder gets a payoff of $v_{(1)} - v_{(2)}$.

Next, we analyse the second price auction. In the second price auction, the bidder's choice of bid determines only whether or not he wins; the amount he pays if he wins is beyond his control. Suppose the bidder considers lowering his bid below his valuation. The only case in which this changes the outcome occurs when this lowering of his bid results in his bid now being lower than someone else's. Because he would have earned non-negative payoff if he won, lowering his bid cannot make him better off. Suppose the bidder considers raising his bid above his valuation. The only case in which this changes the outcome occurs when some other bidder has submitted a bid higher than the first bidder's valuation but lower than his new bid. Thus raising the bid causes this bidder to win but he must pay more for the item than it is worth to him; raising his bid beyond his valuation cannot make him better off. Thus, each bidder's equilibrium best response strategy is to bid his own valuation for the item. The payment here is equal to the actual valuation of the bidder with the second highest valuation (i.e. realization of the second order statistic). Thus, the expected payment and payoff are the same in English auction and the second price auction. This establishes Part 3 and therefore proves the revenue equivalence theorem.

Note that the outcomes of the English auction and the second price auction satisfy a weakly dominant strategy equilibrium. That is, each bidder has a well-defined best bid regardless of how high he believes his rivals will bid. In the second price auction, the weakly dominant strategy is to bid time valuation. In the English auction, the weakly dominant strategy is to remain in the bidding until the price reaches the bidder's own valuation.

2.2b *Some observations:* We now make a few important observations.

- The theorem does not imply that the outcomes of the four auction forms are always exactly the same. They are only equal on average. Note that in the English auction or the second price auction, the price exactly equals the valuation of the bidder with the second highest valuation, $v_{(2)}$. In Dutch auction or the first price auction, the price is the expectation of the second highest valuation conditional on the winning bidder's own valuation. The above two prices will be equal only by accident, however they are equal on average.
- Bidding logic is very simple in the English auction and second price auction. In the former, a bidder remains in bidding until the price reaches his valuation. In the latter, he submits a sealed bid equal to his own valuation.

- On the other hand, the bidding logic is quite complex in the Dutch auction and the first price auction. Here the bidder bids some amount less than his true valuation. Exactly how much less depends upon the probability distribution of the other bidders' valuations and the number of competing bidders. Finding the Nash equilibrium bid is a non-trivial computational problem.
- The revenue equivalence theorem for the single indivisible item is devoid of empirical predictions about which type of auction will be chosen by the seller in any particular set of circumstances. However, when the assumptions of the benchmark model are relaxed, particular auction forms emerge as being superior.
- The variance of revenue is lower in English auction or second price auction than in Dutch auction or first price auction. Hence if the seller were risk averse, he would choose English or second price rather than Dutch or first price.

For more details on the revenue equivalence theorems, the reader is referred to the papers by Myerson (1981), McAfee & McMillan (1987), Klemperer (2003), and the books by Milgrom (2004) and Krishna (2002).

3. Concept of individual rationality

Note that if a social choice function is BIC then each agent i finds it in his best interest to tell truth if all the other agents are also doing so. However, note that neither agent i nor the social planner has any control over the types that are being revealed by the agents other than i. Therefore, agent i may wonder what if any one or more of his rival agents announce untruthful types. In such a situation, telling truth should not result in any kind of loss to the agent i. Otherwise, either agent i will also start lying or, alternatively, he may quit the mechanism itself because participation in the mechanism is voluntary. Thus, in order to avoid such a situation, the social planner needs to ensure that each agent, despite what the rival agents are reporting, will be better off telling truth than not participating in the mechanism. These constraints are known as *participation constraints* or *individual rationality constraints*. Thus, individual rationality adds one more dimension to the desirable properties of a social choice function.

There are three stages at which participation constraints may be relevant in any particular application.

3.1 *Ex-post individual rationality constraints*

These constraints become relevant when any agent i is given a choice to withdraw from the mechanism at the ex-post stage that arise after all the agents have announced their types and an outcome in X has been chosen. Let $\overline{u_i}(\theta_i)$ be the utility that agent i receives by withdrawing from the mechanism when his type is θ_i. Then, to ensure agent i's participation, we must satisfy the following *ex-post participation (or individual rationality) constraints*

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq \overline{u_i}(\theta_i) \;\; \forall \; (\theta_i, \theta_{-i}) \in \Theta.$$

3.2 *Interim individual rationality constraints*

Let the agent i be allowed to withdraw from the mechanism only at an interim stage that arises after the agents have learned their type but before they have chosen their actions in the mechanism. In such a situation, the agent i will participate in the mechanism only if his interim expected utility $U_i(\theta_i|f) = E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i]$ from social choice function $f(\cdot)$, when his type is θ_i, is greater than $\overline{u_i}(\theta_i)$. Thus, *interim participation (or individual rationality) constraints* for agent i require that

$$U_i(\theta_i|f) = E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i] \geq \overline{u_i}(\theta_i) \;\; \forall\, \theta_i \in \Theta_i.$$

3.3 *Ex-ante individual rationality constraints*

Let agent i be allowed to refuse to participate in the mechanism only at ex-ante stage that arises before the agents learn their type. In such a situation, the agent i will participate in the mechanism only if his ex-ante expected utility $U_i(f) = E_\theta[u_i(f(\theta_i, \theta_{-i}), \theta_i)]$ from social choice function $f(\cdot)$ is greater than $E_{\theta_i}[\overline{u_i}(\theta_i)]$. Thus, *ex-ante participation (or individual rationality) constraints* for agent i require that

$$U_i(f) = E_\theta[u_i(f(\theta_i, \theta_{-i}), \theta_i)] \geq E_{\theta_i}[\overline{u_i}(\theta_i)].$$

The following proposition establishes a relationship among the three different participation constraints discussed above.

PROPOSITION 3.1.

For any social choice function $f(\cdot)$, we have

$$f(\cdot) \textit{ is ex-post IR} \Rightarrow f(\cdot) \textit{ is interim IR} \Rightarrow f(\cdot) \textit{ is ex-ante IR}.$$

The next proposition investigates the individual rationality of Clarke mechanism. First, we provide two definitions.

DEFINITION 3.1 (Choice set monotonicity).

We say that a mechanism $\mathscr{M}$ is choice set monotone if the set of feasible outcomes X (weakly) increases as additional agents are introduced into the system. An implication of this property is $K_{-i} \subset K \; \forall\, i = 1, \ldots, n$.

DEFINITION 3.2 (No negative externality).

Consider a choice set monotone mechanism $\mathscr{M}$. We say that the mechanism $\mathscr{M}$ has no negative externality if for each agent i, each $\theta \in \Theta$, and each $k^*_{-i}(\theta_{-i}) \in B^*(\theta_{-i})$, we have

$$v_i(k^*_{-i}(\theta_{-i}), \theta_i) \geq 0.$$

It is easy to verify that the mechanisms in all the previous examples —*fair bilateral trade, first-price sealed bid auction, Vickrey auction*, and *GVA* satisfy all the three properties described above. We now state and prove a proposition which provides a sufficient condition for the ex-post individual rationality of the Clarke mechanism (discussed in Section 10·3 of Part 1). Recall that the Clarke mechanism is a special case of the Groves mechanism and includes the GVA and the Vickrey auction as special cases.

PROPOSITION 3.2 (Ex-post individual rationality of Clarke mechanism).

Let us consider a Clarke mechanism in which

(i) $\overline{u_i}(\theta_i) = 0 \; \forall \theta_i \in \Theta_i; \;\; \forall\, i = 1, \ldots, n$

(ii) The mechanism satisfies choice set monotonicity property
(iii) The mechanism satisfies no negative externality property.

Then the Clarke mechanism is ex-post individual rational.

Proof. Recall that utility $u_i(f(\theta), \theta_i)$ of an agent i in Clarke mechanism is given by

$$u_i(f(\theta), \theta_i) = v_i(k^*(\theta), \theta_i) + \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] - \left[\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)\right]$$
$$= \left[\sum_{j} v_j(k^*(\theta), \theta_j)\right] - \left[\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)\right].$$

By virtue of choice set monotonicity, we know that $k^*_{-i}(\theta_{-i}) \in K$. Therefore, we have

$$u_i(f(\theta), \theta_i) \geq \left[\sum_{j} v_j(k^*_{-i}(\theta_{-i}), \theta_j)\right] - \left[\sum_{j \neq i} v_j(k^*_{-i}(\theta_{-i}), \theta_j)\right]$$
$$= v_i(k^*_{-i}(\theta_{-i}), \theta_i)$$
$$\geq 0 = \overline{u_i}(\theta_i).$$

The last step follows due to the fact that mechanism has no negative externality. Q.E.D.

Let us now investigate the individual rationality of the social choice functions of the examples discussed earlier.

3.4 *Individual rationality in Fair bilateral trade*

Let us consider the example of fair bilateral trade. Let us assume that utility $\overline{u_i}(\theta_i)$ derived by the agents i from not participating into the trade, when his type is θ_i, is as follows.

$$\overline{u_1}(\theta_1) = \theta_1 \ \ \forall\, \theta_1 \in [\underline{\theta_i}, \overline{\theta_i}]$$
$$\overline{u_2}(\theta_2) = 0 \ \ \forall\, \theta_2 \in [\underline{\theta_i}, \overline{\theta_i}].$$

In view of the above definitions, it is now easy to see that the SCF used in this example is ex-post IR.

3.5 *Individual rationality in first-price sealed bid auction*

Let us consider the example of first-price sealed bid auction. If for each possible type θ_i, the utility $\overline{u_i}(\theta_i)$ derived by the agents i from not participating into the auction is 0, then it is easy to see that the SCF used in this example would be ex-post IR.

3.6 *Individual rationality in Vickrey auction*

Let us consider the example of second-price sealed bid auction. If for each possible type θ_i, the utility $\overline{u_i}(\theta_i)$ derived by the agents i from not participating into the auction is 0, then it is easy to see that the SCF used in this example would be ex-post IR. Moreover, the ex-post IR of this example also follows directly from Proposition 3.2 because this is a special case of Clarke mechanism satisfying all the required conditions in the proposition.

Table 1. Properties of social choice functions in quasi-linear environment.

SCF	AE	BB	DSIC	Ex-Post IR
Fair bilateral trade	✓	✓	×	✓
First-price auction	✓	×	×	✓
Vickrey auction	✓	×	✓	✓
GVA	✓	×	✓	✓

3.7 *Individual rationality in GVA*

Let us consider the example of generalized Vickrey auction. If for each possible type θ_i, the utility $\overline{u_i}(\theta_i)$ derived by the agents i from not participating into the auction is 0, then by virtue of Proposition 3.2, we can claim that SCF used here would be ex-post IR.

By including the above facts regarding individual rationality in tables 1 and 2 of part 1 (Garg *et al* 2008) of this tutorial, we get tables 1 and 2.

3.8 *The Myerson–Satterthwaite theorem*

In the previous section, we saw that we did not even have a single example where we have all the desired properties in a SCF - AE, BB, BIC, and IR. This provides a motivation to study the feasibility of having all these properties in a social choice function.

The Myerson–Satterthwaite theorem is one disappointing news in this direction, which tells us that in a bilateral trade setting, whenever the gains from the trade are possible but not certain, then there is no SCF which satisfies AE, BB, BIC, and Interim IR all together. The precise statement of the theorem is as follows.

Theorem 3.1 (Myerson–Satterthwaite impossibility theorem). *Consider a bilateral trade setting in which the buyer and seller are risk neutral, the valuations θ_1 and θ_2 are drawn independently from the intervals $[\underline{\theta_1}, \overline{\theta_1}] \subset \mathbb{R}$ and $[\underline{\theta_2}, \overline{\theta_2}] \subset \mathbb{R}$ with strict positive densities, and $(\underline{\theta_1}, \overline{\theta_1}) \bigcap (\underline{\theta_2}, \overline{\theta_2}) \neq \emptyset$. Then there is no Bayesian incentive compatible social choice function that is ex-post efficient and gives every buyer type and every seller type non-negative expected gains from participation.*

For proof of the above theorem, refer to Proposition 23·E·1 of Mas-Colell *et al* (1995).

4. Moulin mechanisms

Recall that the Green–Laffont theorem rules out the possibility of a DSIC mechanism that satisfies both allocative efficiency and strict budget balance. So we have to, in the least,

Table 2. Properties of social choice functions in linear environment.

SCF	AE	BB	DSIC	BIC	Ex-post IR
Fair bilateral trade	✓	✓	×	×	✓
First-price auction	✓	×	×	×	✓
Vickrey auction	✓	×	✓	✓	✓

compromise on one of these three properties. Either we can achieve budget balance and DSIC or achieve allocative efficiency as well as budget balance but settle for Bayesian incentive compatibility. The Green–Laffont theorem says that the search for mechanisms which are DSIC and allocative efficient ends with the Groves mechanisms. As we have seen, the Groves mechanisms are not necessarily budget balanced. Moulin (2007) has designed a Groves mechanism, which minimizes the budget imbalance. He has proposed the redistribution of surplus created in a way that the mechanism remains a Groves mechanism and achieves minimum budget imbalance.

4.1 *Almost budget balanced assignments*

Suppose there are p identical objects and n agents ($n > p$) claim one unit each of these objects. For example, we might be interested in distributing p tickets to a movie among n friends. Obviously, not all claims can be satisfied. As $p < n$, the idea is to have a fair allocation, that is, allocate the p objects to those p agents, who value the object most. If the Groves mechanism is used, we will have dominant strategy truthful revelation and efficient assignments. But the Groves Mechanism inherently possesses the problem of budget imbalance, resulting in a deficit or a surplus. This budget imbalance depends upon the type profiles of the agents. It would be nice to have a third party that would perform the role of a residual claimant, supplying money if needed and burning out the surplus, if generated.

Moulin (2007) defines the notion of *efficiency loss*. The surplus loss of a mechanism is measured as the ratio of budget surplus to efficient surplus. The efficiency loss of a mechanism as defined as the worst case value of such ratio over all possible profiles of valuations. It is denoted by $L(n, p)$ in the above allocation problem.

If the imbalance of a mechanism is negative, that is, the surplus is negative, somebody has to supply the deficit amount and such a mechanism is said to be *infeasible*. For a mechanism to be feasible, the surplus therefore should be positive. One more desirable property to satisfy would be *Voluntary Participation* (VP), that is, no participant ends in loss at any profile of valuation. This is the same as ex-post individual rationality discussed in § 3. Let $\hat{L}(n, p)$ be the smallest efficiency loss possible over all feasible Groves mechanisms. Moulin has designed a Groves mechanism which achieves this efficiency loss. He has also found the smallest efficiency loss $L^*(n, p)$ possible over all feasible Groves mechanisms which also satisfies the VP constraints. He has designed a feasible Groves mechanism satisfying VP and achieving efficiency loss $L^*(n, p)$.

4.2 *The model*

The model under consideration and the notation used in the rest of this section are described in table 3. Recall that a general Groves mechanism is described by n arbitrary functions h_i defined on $\mathbb{R}^{N\setminus\{i\}}$. The payment by agent i is given by

$$t_i(\theta) = h_i(\theta_{-i}) - \sum_{j\neq i, j=1}^{p} \theta^{*j}.$$

Therefore, the utility of agent i is:

$$u_i(\theta) = S_p(\theta) - h_i(\theta_{-i}).$$

Table 3. Notation for Moulin mechanisms.

p	Number of identical objects
N	Set of agents $N = \{1, 2, \dots, n\}$
θ_i	Valuation of agent i
θ	Type profile (profile of valuations) of all the agents $\theta \in \mathbb{R}^n_+$
θ^*	Permutation of type profile θ, where coordinates are arranged in decreasing order $\theta^{*1} \geq \theta^{*2} \geq \cdots \geq \theta^{*n}$
θ_{-i}	Profile θ obtained by deleting i^{th} coordinate
θ^*_{-i}	Permutation of θ_{-i} where coordinates are arranged in decreasing order
$S_p(\theta)$	Efficient surplus given p objects and profile of valuations θ $S_p(\theta) = \theta^{*1} + \theta^{*2} + \cdots + \theta^{*p}$
$S_p(\theta_{-i})$	$S_p(\theta_{-i}) = \theta^{*1}_{-i} + \theta^{*2}_{-i} + \cdots + \theta^{*p}_{-i}$
$t_i(\theta)$	Payment made by the agent i
$u_i(\theta)$	Utility of an agent i at profile θ
$\Delta(\theta)$	Budget imbalance $\Delta(a) = \sum_{i \in N} t_i(\cdot)$
$L(n, p)$	Efficiency loss

The budget imbalance of this mechanism is

$$\Delta(\theta) = \sum_{i \in N} t_i(\theta) = \sum_{i \in N} h_i(\theta_{-i}) - (n-1)S_p(\theta).$$

In most Groves mechanisms, the sign of Δ is arbitrary. Money can flow in or flow out. A residual claimant will absorb the surplus (if the sign is positive) or cover any deficit (if the sign is negative). This measure has to be normalized for the comparison among different mechanisms. Moulin has defined the performance index of mechanism by

$$L(n, p) = \max \frac{\text{MoneyImbalance}}{\text{Efficientsurplus}} = \max_{\theta \in \mathbb{R}^N_+ \setminus \{0\}} \frac{|\Delta(\theta)|}{S_p(\theta)}.$$

If $\Delta(\theta) > 0$, the residual claimant receives a part of the efficient surplus in cash, so we can interpret $\frac{|\Delta(\theta)|}{S_p(\theta)}$ as a relative efficiency loss. This interpretation is not valid if $\Delta(\theta) < 0$, because in this case the residual claimant subsidizes the participants. We then regard $\frac{|\Delta(\theta)|}{S_p(\theta)}$ as the relative cost of running the mechanism, and $\frac{S_p(\theta)}{|\Delta(\theta)|}$ as the multiplicative effect of the subsidy.

We will focus our attention on self-sufficient mechanisms where the money only flows out. We call these mechanisms feasible:

$$\text{Feasibility: } \Delta(\theta) \geq 0 \Leftrightarrow \sum_{i \in N} h_i(\theta_{-i}) \geq (n-1)S_p(\theta) \;\; \forall \;\; \theta \in \Theta. \tag{2}$$

Under feasibility, we call the index $L(n, p)$ the worst efficiency loss, or simply the efficiency loss of the mechanism:

$$L(n, p) = \max \frac{\text{outflow of money}}{\text{efficient surplus}} = \max_{\theta \in \mathbb{R}^N_+ \setminus \{0\}} \frac{\Delta(\theta)}{S_p(\theta)}.$$

Voluntary participation can be expressed as:

$$\text{Voluntary Participation } (VP) : u_i(\theta) \geq 0 \;\; \forall \, \theta, \forall \, i \tag{3}$$

$$\Leftrightarrow h_i(\theta_{-i}) \leq S_p(\theta_{-i}) \;\; \forall \;\; \theta_{-i}, \forall \, i.$$

Note that F and VP imply $0 \leq L(n, p) \leq 1$. Indeed, the inequality (3) gives

$$h_i(\theta_{-i}) \leq S_p(\theta_{-i}) \leq S_p(\theta)\forall i,$$

which sums to $\Delta(\theta) \leq S_p(\theta)$.

4.2a *Pareto superiority to Vickrey auction:* We now show that the Vickrey auction has the largest efficiency loss among all feasible and voluntary Groves mechanisms. First, we observe that the *Vickrey auction* is basically the Clarke mechanism applied to a combinatorial auction. In the Vickrey auction, the residual claimant 'owns' the objects and sells them at the $(p+1)^{\mathrm{st}}$ highest price. Thus,

$$h_i^{\mathrm{vick}}(\theta_{-i}) = S_p(\theta_{-i}) \quad \text{and}$$

$$u_i^{\mathrm{vick}}(\theta) = S_p(\theta) - S_p(\theta_{-i}) \;\; \forall \;\; i \text{ and } \forall \;\; \theta. \tag{4}$$

The Vickrey auction is feasible and individually rational: p highest bidders get

$$u_i^{\mathrm{vick}}(\theta) = \theta_i - \theta^{*(p+1)}.$$

If $\theta^{*1} = \cdots = \theta^{*(p+1)}$, then $u_i(\theta) = 0 \;\; \forall \;\; i \in N$. The residual claimant will capture the whole surplus implying $L^{\mathrm{vick}}(n, p) = 1$. Thus, the Vickrey auction has the largest efficiency loss among all feasible and voluntary Groves mechanisms. In view of Definition (4), inequality (3) reads

$$u_i^{\mathrm{vick}}(\theta) \leq u_i(\theta) \;\; \forall \;\; i \text{ and } \theta.$$

This means a Groves mechanism is voluntary if and only if it is Pareto superior to the Vickrey auction.

For the sake of convenience, we will write the functions $h_i(\theta_{-i})$ as

$$h_i(\theta_{-i}) = S_p(\theta_{-i}) - r_p(i; \theta_{-i}).$$

The function $r_p(i; \theta_{-i})$ is a *rebate* function. We can interpret it as the rebate given to an agent i, which in turn will help to reduce the budget imbalance. Note that the rebate of agent i depends upon the type profile of other agents and not on the valuation of agent i. Thus by giving a rebate, we are not destroying the structure of the Groves payments, ensuring that allocative efficiency and DSIC properties are retained. Thus, the utility to agent i in Moulin's mechanism is

$$u_i(\theta) = S_p(\theta) - S_p(\theta_{-i}) + r_p(i; \theta_{-i}) = u_i^{vick}(\theta) + r_p(i; \theta_{-i}) \;\; \forall \;\; \theta \in \mathbb{R}_+^N,$$

When voluntary participation holds, $r_p(i; \theta_{-i}) \geq 0$ and then, we interpret $r_p(i; \theta_{-i})$ as agent i's share of the seller's revenue in the Vickrey auction. For the sake of simplicity, we will drop the first i and will simply write $r_p(\theta_i)$ for rebate function.

4.3 *Optimal feasible mechanisms: Voluntary and unvoluntary*

The word *optimal* here refers to optimality with respect to efficiency loss. This is not to be confused with the *optimal mechanisms* that we will see in detail in § 5. Define:

$$B_s(t, u) = \sum_{k=t}^{u} \binom{s}{k},$$

Moulin (2007) has proved the following results:

Theorem 4.1. *Under feasibility and voluntary participation, the smallest efficiency loss is given by,*

$$L^*(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}(p, n-1)}$$

The following linear rebate functions define an optimal mechanism:

$$r_p^*(\theta_{-i}) = \sum_{k=p+1}^{n-1} (-1)^{k-p-1} \frac{pL^*(n,p)}{kL^*(n,k)} \theta_{-i}^{*k} \ \text{ if } \ p \leq n-2; r_{n-1}(\theta_{-i}) = 0$$

The corresponding budget surplus is

$$\Delta^*(\theta) = pL^*(n,p) \left\{ \sum_{k=p+1}^{n} (-1)^{k-p-1} \theta^{*k} \right\}.$$

Theorem 4.2. *Under feasibility, the smallest efficiency loss, $\hat{L}(n,p)$ is given by,*

$$\hat{L}(n,1) = L^*(n,1); \hat{L}(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}(p, n-1) + \frac{n}{p} B_{n-2}(0, p-2)}$$

The following linear rebate functions define an optimal mechanism.

$$\hat{r_1}(\theta_{-i}) = r_1^*(\theta_{-i})$$

$$\hat{r_p}(\theta_{-i}) = \hat{L}(n,p) \left\{ \sum_{k=1}^{p-1} \gamma_k \theta_-^{*k} i \right\} + (1 - \frac{\hat{L}(n,p)}{L^*(n,p)}) \theta_-^{*p} i$$

$$+ \sum_{k=p+1}^{n-1} (-1)^{k-p-1} \frac{p\hat{L}(n,p)}{kL^*(n,k)} \theta_-^{*k} i \tag{5}$$

$$\gamma_k = -\frac{n}{n-1} \frac{B_{n-2}(n-k, n-2)}{\binom{n-2}{n-k-1}} - \frac{1}{n-1} \text{ *if p-k is odd;*}$$

$$\gamma_k = \frac{n}{n-1} \frac{B_{n-2}(n-k, n-2)}{\binom{n-2}{n-k-1}}$$

Note that the right summation in Equation (5) is zero if $p = n-1$. The budget surplus is

$$\hat{\Delta}(\theta) = \hat{L}(n,p) \left\{ \sum_{k=1,3,\ldots}^{\leq p-1} (p-k)(\theta^{*(p-k)} - \theta^{*(p-k+1)}) + p \sum_{k=p+1}^{n} (-1)^{k-p-1} a^{*k} \right\}$$

We will see this with an example.
Let us compute $L^*(n, p), r_p^*(\theta_{-i})$ for $n = 3, p = 1$.

$$L^*(3, 1) = \frac{\binom{2}{1}}{B_2(1, 2)} = \frac{2}{\binom{2}{1} + \binom{2}{2}} = \frac{2}{3}$$

$$r_1^*(\theta_{-i}) = \sum_{k=2}^{2} (-1)^{k-2} \frac{L^*(3, 1)}{kL^*(3, k)} \theta_{-i}^{*k} = \frac{\frac{2}{3}}{2} a_{-i}^{*k} = \frac{1}{3} a_{-i}^{*2}.$$

Consider the type profile $(1, 1, 0)$. With this type profile, agent 1 wins. Agent i has rebate 0 and pays $1 - 0 = 1$. So its utility is 0. Agent 2 has rebate 0, which is also its utility. Agent 3's rebate is $1/3$ which means it pays $0 - 1/3$, that is, it receives $1/3$. Budget imbalance is $1 - \frac{1}{3} = \frac{2}{3}$. Efficient surplus is 1. So surplus loss at profile $(1, 1, 0)$ is $\frac{2}{3}$. Note this is precisely $L^*(3, 1)$.

4.4 *Some observations*

- The rebate functions in Theorem 4.1 and Theorem 4.2 are not the only rebate functions achieving respective efficiency losses. Moulin has shown that if we restrict the rebate functions to be symmetric, these are the only choices for the rebate functions achieving corresponding efficiencies.
- We discuss first the case $p = 1$. Here, voluntary participation comes free: the optimal linear rebates under F define a voluntary mechanism, therefore the optimal efficiency loss under F is also the optimal loss under F and VP:

$$\hat{L}(n, 1) = L^*(n, 1) = \frac{n-1}{2^{n-1} - 1} \simeq \frac{2n}{2^n}.$$

- Consider the situation when $p = n - 1$. It is easy to see that $L^*(n, n-1) = 1$ and $r_{n-1}^*(\theta_{-i}) = 0$. This clearly indicates that we cannot improve upon the Vickrey mechanism when $p = n - 1$. However, the optimal feasible (unvoluntary) mechanism achieves an efficiency loss even smaller than in the one object case:

$$\hat{L}(n, n-1) = \frac{n-1}{n2^{n-2} - 1'} \simeq \frac{4}{2^n}.$$

- In general, Moulin mechanisms achieve budget balance asymptotically with respect to n. The summary of asymptotic behaviour is as follows:
  - (i) $L^*(n, p)$ increases strictly in p and decreases strictly in n.
  - (ii) $\hat{L}(n, p)$ increases in n for $p \leq n \leq 2p - 1$, $\hat{L}(n, p)$ decreases in n if $2p \leq n$
  - (iii) $\hat{L}(n, p)$ increases in p for $1 \leq p \leq \left\{\frac{n}{2}\right\}$ decreases in p if $\left\{\frac{n}{2}\right\} \leq p \leq n$. $\left\{\frac{n}{2}\right\} = \frac{n}{2}$ if n is even and $= \frac{2n-1}{2}$ otherwise.
  - (iv) Loosely speaking, $\hat{L}(n, p)$ and $L^*(n, p)$ converge exponentially fast to zero in n if $\frac{p}{n} < \frac{1}{2}$ and as $\frac{1}{\sqrt{n}}$ if $\frac{p}{n} \simeq \frac{1}{2}$.
  - (v) If $\frac{p}{n} > \frac{1}{2}$, unvoluntary mechanisms still allow exponentially fast efficiency while voluntary ones preclude asymptotic efficiency altogether.

- Though the Moulin mechanisms achieve the least efficiency loss, that is, the least worst case budget imbalance, they inherently have some problems. Consider a mechanism with F and VP conditions. Let the type profile be $(\underbrace{1, 1, \ldots, 1}_{p+1 \text{ times}}, 0, \ldots, 0)$. The $p < n - 1$ objects are given to the first p agents. The rebate to $p + 1$ agents having valuation 1 is 0 and so is the utility. The rebate to remaining agents is $\frac{1-L^*(n,p)}{n-p-1} > 0$. The agents having least preferences will have strict positive utility. Obviously, the agents having higher valuation will envy the agents having least valuation. In general, in Moulin's mechanisms under F and VP conditions, the utilities are not in the same order as type values of the agents. So this mechanism in which this property holds true is called an envy free mechanism. For example, Vickrey auctions are envy free. Moulin's mechanism are not envy free. It is noteworthy that the Vickrey auction is the only envy free auction among all feasible Groves mechanisms which have bounded efficiency loss.
- For the Vickrey auction, $u_i(\theta)$ is weakly increasing in p and weakly decreasing in n. Both properties have been violated by mechanisms given in Theorem 4.1.

4.5 *Space of mechanisms*

We have seen a variety of mechanisms having different sets of properties. Figure 1 depicts different classes of mechanisms we have studied so far and their inter-relationships.

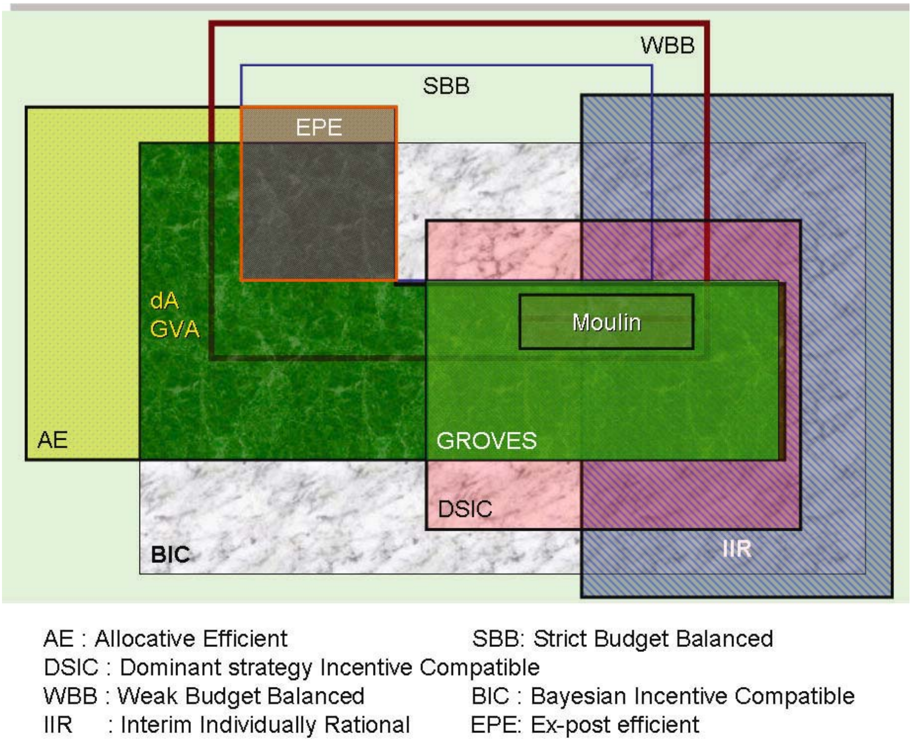

Figure 1. Design space of mechanisms with quasi-linear utilities.

5. Optimal mechanisms

An obvious problem that faces a social planner is to decide which direct revelation mechanism (or equivalently, social choice function) is *optimal* for a given problem. In the rest of this paper, our objective is to familiarize the reader with a couple of techniques which social planner can adopt to design an optimal direct revelation mechanism for a given problem at hand.

One notion of optimality in multi-agent systems is that of *Pareto efficiency*. We now define three different notions of efficiency: ex-ante, interim, and ex-post. These notions were introduced by Holmstorm and Myerson (1983).

DEFINITION 5.1 (Ex-ante efficiency).

For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is ex-ante efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties

$$E_\theta[u_i(\hat{f}(\theta), \theta_i)] \geq E_\theta[u_i(f(\theta), \theta_i)] \;\; \forall\, i = 1, \ldots, n$$

$$E_\theta[u_i(\hat{f}(\theta), \theta_i)] > E_\theta[u_i(f(\theta), \theta_i)] \;\; \text{for some } i.$$

DEFINITION 5.2 (Interim efficiency).

For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is interim efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties

$$E_{\theta_{-i}}[u_i(\hat{f}(\theta), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(f(\theta), \theta_i)|\theta_i] \;\; \forall\, i = 1, \ldots, n, \;\; \forall\, \theta_i \in \Theta_i$$

$$E_{\theta_{-i}}[u_i(\hat{f}(\theta), \theta_i)|\theta_i] > E_{\theta_{-i}}[u_i(f(\theta), \theta_i)|\theta_i] \;\; \text{for some } i \text{ and some } \theta_i \in \Theta_i.$$

DEFINITION 5.3 (Ex-post efficiency).

For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is ex-post efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties

$$u_i(\hat{f}(\theta), \theta_i) \geq u_i(f(\theta), \theta_i) \;\; \forall\, i = 1, \ldots, n, \;\; \forall\, \theta \in \Theta$$

$$u_i(\hat{f}(\theta), \theta_i) > u_i(f(\theta), \theta_i) \;\; \text{for some } i \text{ and some } \theta \in \Theta.$$

Using the above definition of ex-post efficiency, we can say that a social choice function $f(\cdot)$ is ex-post efficient in the sense of definition 5·1 in (Garg *et al* 2008) if and only if it is ex-post efficient in the sense of definition 5.3 when we take $F = \{f : \Theta \to X\}$.

The following proposition establishes a relationship among these three different notions of efficiency.

PROPOSITION 5.1.

Given any set of feasible social choice functions F and $f(\cdot) \in F$, we have

$$f(\cdot) \textit{ is ex-ante efficient} \Rightarrow f(\cdot) \textit{ is interim efficient} \Rightarrow f(\cdot) \textit{ is ex-post efficient.}$$

For proof of the above proposition, refer to Proposition 23·F·1 of Mas-Colell (1995). Also, compare the above proposition with the Proposition 3.1.

With this set-up, we now try to formalize the design objectives of a social planner. For this, we need to define the concept known as *social utility function*.

DEFINITION 5.4 (Social utility function).

A social utility function is a function $w : \mathbb{R}^n \to \mathbb{R}$ that aggregates the profile $(u_1, \dots, u_n) \in \mathbb{R}^n$ of individual utility values of the agents into a social utility.

Consider a mechanism design problem and a direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ proposed for it. Let $(\theta_1, \dots, \theta_n)$ be the actual type profile of the agents and assume for a moment that they will all reveal their true types when requested by the planner. In such a case, the social utility that would be realized by the social planner for every possible type profile θ of the agents is given by:

$$w(u_1(f(\theta), \theta_1), \dots, u_n(f(\theta), \theta_n)). \tag{6}$$

However, recall the implicit assumption behind a mechanism design problem, namely, that the agents are autonomous and they would report a type as dictated by their rational behaviour. Therefore, the assumption that all the agents will report their true types is not true in general. In general, rationality implies that the agents report their types according to a strategy suggested by a Bayesian–Nash equilibrium $s^*(\cdot) = (s_1^*(\cdot), \dots, s_n^*(\cdot))$ of the underlying Bayesian game. In such a case, the social utility that would be realized by the social planner for every possible type profile θ of the agents is given by

$$w(u_1(f(s^*(\theta)), \theta_1), \dots, u_n(f(s^*(\theta)), \theta_n)). \tag{7}$$

In some instances, the above Bayesian–Nash equilibrium may turn out to be a dominant strategy equilibrium. Better still, truth revelation by all agents could turn out to be a Bayesian–Nash equilibrium or a dominant strategy equilibrium.

5.1 *Optimal mechanism design problem*

In view of the above notion of social utility function, it is clear that the objective of a social planner would be to look for a social choice function $f(\cdot)$ that would maximize the expected social utility for a given social utility function $w(\cdot)$. However, being the social planner, it is always expected of him to be fair to all the agents. Therefore, the social planner would first put a few fairness constraints on the set of social choice functions which he can probably choose from. The fairness constraints may include any combination of all the previously studied properties of a social choice function, such as ex-post efficiency, incentive compatibility, and individual rationality. This set of social choice functions is known as *set of feasible social choice functions* and is denoted by F. Thus, the problem of a social planner can now be cast as an optimization problem where the objective is to maximize the expected social utility and the constraint is that the social choice function must be chosen from the feasible set F. This problem is known as the *optimal mechanism design* problem and the solution of the problem is some social choice function $f^*(\cdot) \in F$ which is used to define the optimal mechanism $\mathscr{D}^* = ((\Theta_i)_{i \in N}, f^*(\cdot))$ for the problem that is being studied.

Depending on whether the agents are loyal or autonomous entities, the optimal mechanism design problem may take two different forms.

$$\underset{f(\cdot) \in F}{\text{maximize}} \; E_{\theta}\left[w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n))\right] \tag{8}$$

$$\underset{f(\cdot) \in F}{\text{maximize}} \; E_{\theta}[w(u_1(f(s^*(\theta)), \theta_1), \ldots, u_n(f(s^*(\theta)), \theta_n))]. \tag{9}$$

The problem (8) is relevant when the agents are loyal and always reveal their true types whereas the problem (9) is relevant when the agents are rational. At this point of time, one may ask how to define the set of feasible social choice functions F. There is no unique definition of this set. The set of feasible social choice functions is a subjective judgement of the social planner. The choice of the set F depends on what all fairness properties the social planner would wish to have in the optimal social choice function $f^*(\cdot)$. If we define

$$\begin{aligned}
F_{DSIC} &= \{f : \Theta \to X | f(\cdot) \text{ is dominant strategy incentive compatible}\}\\
F_{BIC} &= \{f : \Theta \to X | f(\cdot) \text{ is Bayesian incentive compatible}\}\\
F_{ExPostIR} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-post individual rational}\}\\
F_{IntIR} &= \{f : \Theta \to X | f(\cdot) \text{ is interim individual rational}\}\\
F_{ExAnteIR} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-ante individual rational}\}\\
F_{Ex-AnteEff} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-ante efficient}\}\\
F_{IntEff} &= \{f : \Theta \to X | f(\cdot) \text{ is interim efficient}\}\\
F_{Ex-PostEff} &= \{f : \Theta \to X | f(\cdot) \text{ is ex-post efficient}\}.
\end{aligned}$$

The set of feasible social choice functions F may be either any one of the above sets or intersection of any combination of the above sets. For example, the social planner may choose $F = F_{BIC} \bigcap F_{IntIR}$. In the literature, this particular feasible set is known as *incentive feasible set* due to Myerson (1997). Also, note that if the agents are loyal then the sets F_{DSIC} and F_{BIC} will be equal to the whole set of all the social choice functions.

If the environment is quasi-linear, then we can also define the set of allocatively efficient social choice functions F_{AE} and the set of budget balanced social choice functions F_{BB}. In such an environment, we will have $F_{Ex-PostEff} = F_{AE} \bigcap F_{BB}$.

5.2 *Myerson's optimal auction: An example of optimal mechanism*

Let us consider Example 2·1 in (Garg *et al* 2008), of single unit–single item auction without reserve price and discuss an optimal mechanism developed by Myerson (1981). The objective function here is to maximize the auctioneer's revenue.

Recall that each bidder i's type lies in an interval $\Theta_i = [\underline{\theta_i}, \overline{\theta_i}]$. We impose the following additional conditions on the environment.

(i) The auctioneer and the bidders are risk neutral
(ii) Bidders' types are statistically independent, that is, the joint density $\phi(\cdot)$ has the form $\phi_1(\cdot) \times \cdots \times \phi_n(\cdot)$
(iii) $\phi_i(\cdot) > 0 \; \forall \; i = 1, \ldots, n$

(iv) We generalize the outcome set X relative to that considered in Example 2·1 in Garg *et al* (2008), by allowing a random assignment of the good. Thus, we now take $y_i(\theta)$ to be buyer i's probability of getting the good when the vector of announced types is $\theta = (\theta_1, \ldots, \theta_n)$. Thus, the new outcome set is given by

$$X = \Bigg\{ (y_0, y_1 \ldots, y_n, t_0, t_1, \ldots, t_n) | y_0 \in [0, 1], t_0 \geq 0, y_i \in [0, 1],$$

$$t_i \leq 0 \; \forall \; i = 1, \ldots, n, \sum_{i=1}^{n} y_i \leq 1 \sum_{i=0}^{n} t_i = 0 \Bigg\}.$$

[1] Recall that the utility functions of the agents in this example are given by

$$u_i(f(\theta), \theta_i) = u_i(y_0(\theta), \ldots, y_n(\theta), t_0(\theta), \ldots, t_n(\theta), \theta_i)$$

$$= \theta_i y_i(\theta) + t_i(\theta) \; \forall \; i = 1, \ldots, n.$$

Thus, viewing $y_i(\theta) = v_i(k(\theta))$ in conjunction with the second and third conditions above, we can claim that the underlying environment here is linear.

In the above example, we assume that the auctioneer is the social planner and he is looking for an optimal direct revelation mechanism to sell the good. Myerson's (1981) idea was that the auctioneer must use a social choice function which is Bayesian incentive compatible and interim individual rational and at the same time fetches the maximum revenue to the auctioneer. Thus, in this problem, the set of feasible social choice functions is given by $F = F_{BIC} \bigcap F_{InterimIR}$. The objective function in this case would be to maximize the total expected revenue of the seller which would be given by

$$E_\theta \left[w(u_1(f(\theta), \theta_1), \ldots, u_n(f(\theta), \theta_n)) \right] = -E_\theta \left[\sum_{i=1}^{n} t_i(\theta) \right].$$

Note that in the above objective function we have used $f(\theta)$ not $f(s^*(\theta))$. This is because in the set of feasible social choice functions we are considering only BIC social choice functions and for these functions we have $s^*(\theta) = \theta \;\; \forall \; \theta \in \Theta$. Thus, Myerson's optimal auction design problem can be formulated as the following optimization problem.

$$\underset{f(\cdot) \in F}{\text{maximize}} \; - E_\theta \left[\sum_{i=1}^{n} t_i(\theta) \right], \tag{10}$$

where

$$F = \{ f(\cdot) = (y_1(\cdot), \ldots, y_n(\cdot), t_1(\cdot), \ldots, t_n(\cdot)) | f(\cdot) \text{ is BIC and interim IR} \}.$$

By invoking Myerson's Characterization Theorem (Theorem 11·2 in Garg *et al* (2008)) for BIC SCF in linear environment, we can say that an SCF $f(\cdot)$ in the above context would be BIC if it satisfies the following two conditions

(i) $\overline{y_i}(\cdot)$ is non-decreasing for all $i = 1, \ldots, n$

[1] $\sum_{i=1}^{n} y_i < 1$ when there is no trade.

(ii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \ \ \forall\, \theta_i \in \Theta_i;\ \forall\, i = 1, \ldots, n.$

Also, we can invoke the definition of interim individual rationality to claim that an SCF $f(\cdot)$ in the above context would be interim IR if it satisfies the following conditions

$$U_i(\theta_i) \geq 0 \ \ \forall \theta_i \in \Theta_i;\ \forall\, i = 1, \ldots, n,$$

where

- $\overline{t_i}(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ be bidder i's expected transfer given that he announces his type to be $\hat{\theta}_i$ and that all the bidders $j \neq i$ truthfully reveal their types.
- $\overline{y_i}(\hat{\theta}_i) = E_{\theta_{-i}}[y_i(\hat{\theta}_i, \theta_{-i})]$ is the probability that bidder i would receive the object given that he announces his type to be $\hat{\theta}_i$ and all bidders $j \neq i$ truthfully reveal their types.
- $U_i(\theta_i) = \theta_i \overline{y_i}(\theta_i) + \overline{t_i}(\theta_i)$[2].

In view of the above paraphernalia, problem (10) can be rewritten as follows.

$$\underset{(y_i(\cdot),\, U_i(\cdot))_{i \in N}}{\text{maximize}} \ \sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{y_i}(\theta_i) - U_i(\theta_i)\right) \phi_i(\theta_i) d\theta_i \tag{11}$$

subject to

(i) $\overline{y_i}(\cdot)$ is non-decreasing $\ \forall\, i = 1, \ldots, n$
(ii) $y_i(\theta) \in [0, 1], \sum_{i=1}^{n} y_i(\theta) \leq 1 \ \forall i = 1, \ldots, n, \forall\, \theta \in \Theta$
(iii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds \ \ \forall\, \theta_i \in \Theta_i;\ \forall\, i = 1, \ldots, n$
(iv) $U_i(\theta_i) \geq 0 \ \forall\, \theta_i \in \Theta_i;\ \forall\, i = 1, \ldots, n.$

We first note that if constraint (iii) is satisfied then constraint (iv) will be satisfied if $U_i(\underline{\theta_i}) \geq 0 \ \ \forall\, i = 1, \ldots, n$. As a result, we can replace the constraint (iv) with

(iv′) $U_i(\underline{\theta_i}) \geq 0 \ \forall\, i = 1, \ldots, n.$

Next, substituting for $U_i(\theta_i)$ in the objective function from constraint (iii), we get

$$\sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i \overline{y_i}(\theta_i) - U_i(\underline{\theta_i}) - \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds\right) \phi_i(\theta_i) d\theta_i.$$

Integrating by parts the above expression, the auctioneer's problem can be written as one of choosing the $y_i(\cdot)$ functions and the values $U_1(\underline{\theta_1}), \ldots, U_n(\underline{\theta_n})$ to maximize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} y_i(\theta_i) J_i(\theta_i)\right] \left[\prod_{i=1}^{n} \phi_i(\theta_i)\right] d\theta_n \ldots d\theta_1 - \sum_{i=1}^{n} U_i(\underline{\theta_i})$$

[2]We can take unconditional expectation because types are independent

subject to constraints (i), (ii), and (iv'), where

$$J_i(\theta_i) = \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}\right) = \left(\theta_i - \frac{\overline{\Phi_i}(\theta_i)}{\phi_i(\theta_i)}\right),$$

where, we define $\overline{\Phi_i}(\theta_i) = 1 - \Phi_i(\theta_i)$. It is evident that solution must have $U_i(\underline{\theta_i}) = 0$ for all $i = 1, \dots, n$. Hence, the auctioneer's problem reduces to choosing functions $y_i(\cdot)$ to maximize

$$\int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_n}}^{\overline{\theta_n}} \left[\sum_{i=1}^{n} y_i(\theta_i) J_i(\theta_i)\right] \left[\prod_{i=1}^{n} \phi_i(\theta_i)\right] d\theta_n \dots d\theta_1$$

subject to constraints (i) and (ii).

Let us ignore constraint (i) for the moment. Then inspection of the above expression indicates that $y_i(\cdot)$ is a solution to this relaxed problem if for all $i = 1, \dots, n$, we have

$$y_i(\theta) = \begin{cases} 0 & : \text{ if } J_i(\theta_i) < \max\{0, \max_{h \neq i} J_h(\theta_h)\} \\ 1 & : \text{ if } J_i(\theta_i) > \max\{0, \max_{h \neq i} J_h(\theta_h)\} \end{cases}. \tag{12}$$

Note that $J_i(\theta_i) = \max\left\{0, \max_{h \neq i} J_h(\theta_h)\right\}$ is a zero probability event.

In other words, if we ignore the constraint (i) then $y_i(\cdot)$ is a solution to this relaxed problem if the good is allocated to a bidder who has highest non-negative value for $J_i(\theta_i)$. Now, recall the definition of $\overline{y_i}(\cdot)$. It is easy to write down the following expression

$$\overline{y_i}(\theta_i) = E_{\theta_{-i}}\left[y_i(\theta_i, \theta_{-i})\right]. \tag{13}$$

Now, if we assume that $J_i(\cdot)$ is non-decreasing in θ_i then it is easy to see that above solution $y_i(\cdot)$, given by (12), will be non-decreasing in θ_i, which in turn implies, by looking at expression (13), that $\overline{y_i}(\cdot)$ is non-decreasing in θ_i. Thus, the solution to this relaxed problem actually satisfies constraint (i) under the assumption that $J_i(\cdot)$ is non-decreasing. Assuming that $J_i(\cdot)$ is non-decreasing, the solution given by (12) seems to be the solution of the optimal mechanism design problem for single unit–single item auction. The condition that $J_i(\cdot)$ is non-decreasing in θ_i is met by most of the distribution functions such as uniform and exponential.

So far we have computed the allocation rule for the optimal mechanism and now we turn out attention towards the payment rule. The optimal payment rule $t_i(\cdot)$ must be chosen in such a way that it satisfies

$$\overline{t_i}(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] = U_i(\theta_i) - \theta_i \overline{y_i}(\theta_i) = \int_{\underline{\theta_i}}^{\theta_i} \overline{y_i}(s)ds - \theta_i \overline{y_i}(\theta_i). \tag{14}$$

Looking at the above formula, we can say that if the payment rule $t_i(\cdot)$ satisfies the following formula (15), then it would also satisfy the formula (14).

$$t_i(\theta_i, \theta_{-i}) = \int_{\underline{\theta_i}}^{\theta_i} y_i(s, \theta_{-i})ds - \theta_i y_i(\theta_i, \theta_{-i}) \quad \forall\, \theta \in \Theta. \tag{15}$$

The above formula can be rewritten more intuitively, as follows. For any vector θ_{-i}, let we define

$$z_i(\theta_{-i}) = \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_i(\theta_i) \geq J_j(\theta_j)\ \forall\ j \neq i\right\}.$$

Then $z_i(\theta_{-i})$ is the infimum of all winning bids for bidder i against θ_{-i}, so

$$y_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & : \text{ if } \theta_i > z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i < z_i(\theta_{-i}) \end{cases}.$$

This gives us

$$\int_{\underline{\theta_i}}^{\theta_i} y_i(s, \theta_{-i}) ds = \begin{cases} \theta_i - z_i(\theta_{-i}) & : \text{ if } \theta_i \geq z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i < z_i(\theta_{-i}) \end{cases}.$$

Finally, the formula (15) becomes

$$t_i(\theta_i, \theta_{-i}) = \begin{cases} -z_i(\theta_{-i}) & : \text{ if } \theta_i \geq z_i(\theta_{-i}) \\ 0 & : \text{ if } \theta_i < z_i(\theta_{-i}) \end{cases}.$$

That is bidder i must pay only when he gets the good, and then he pays the amount equal to his lowest possible winning bid.

A few interesting observations are worth mentioning here.

(i) When the various bidders have differing distribution function $\Phi_i(\cdot)$ then, the bidder who has the largest value of $J_i(\theta_i)$ is *not* necessarily the bidder who has bid the highest amount for the good. Thus Myerson's optimal auction need not be allocatively efficient and therefore, need not be ex-post efficient.

(ii) If the bidders are symmetric, that is,

- $\Theta_1 = \cdots = \Theta_n = \Theta$
- $\Phi_1(\cdot) = \cdots = \Phi_n(\cdot) = \Phi(\cdot)$

then the allocation rule would be precisely the same allocation rule of first-price and second-price auctions. In such a case the object would be allocated to the highest bidder. In such a situation, the optimal auction would also become allocatively efficient. Also, note that in such a case the payment rule that we described above would coincide with the payment rules in second-price auction. In other words, the second price (Vickrey) auction would be the optimal auction when the bidders are symmetric. Therefore, many a time, the optimal auction is also known as *modified Vickrey auction*.

Riley and Samuelson (1981) also have studied the problem of design of an optimal auction for selling a single unit of a single item. They assume the bidders to be symmetric. Their work is less general than that of Myerson (1981).

5.3 *Extensions to Myerson's auction*

5.3a *Efficient optimal auctions:* Krishna and Perry (1998) have argued in favour of an auction which will maximize the revenue subject to allocative efficiency (AE) and also DSIC and IIR constraints. The Green–Laffont theorem (Theorem 10·2 in Garg *et al* (2008)) tells us that any DSIC and AE mechanism is necessarily a VCG mechanism. So, we have to look for a VCG mechanism which will maximize the revenue to the seller. Krishna and Perry (1998) define, *social utility* as the value of an efficient allocation:

$$SW(\theta) = \sum_{j=1}^{j=n} v_j(k^*(\theta), \theta_j)$$

$$SW_{-i}(\theta) = \sum_{j \neq i} v_j(k^*(\theta), \theta_j).$$

With these functions, we can write the payment rule in Clarke's pivotal mechanism as

$$t_i(\theta) = SW_{-i}(0, \theta_{-i}) - SW_{-i}(\theta).$$

That is, payment by the agent i is the externality he is imposing by reporting type to be θ_i rather than zero. The authors of Krishna and Perry (1998) generalize it. Fix a vector, $s = (s_1, s_2, \ldots, s_n) \in \Theta$ called as *basis* because, it defines the payment rule . The *VCG mechanism with basis s* is defined by

$$t_i(\theta|s_i) = SW(s_i, \theta_{-i}) - SW_{-i}(\theta).$$

It can be seen that this new mechanism is also DSIC. Now choosing an appropriate basis, one can always find an optimal auction in the class of VCG mechanisms. Krishna and Perry (1998) have shown that the classical Vickrey auction is an optimal and efficient auction for a single indivisible item. They have also shown that the Vickrey auction is an optimal one among VCG mechanisms for multi-unit auctions, when all the bidders have downward sloping demand curves.

5.3b *Armstrong's two item optimal auction:* Armstrong (2000) has considered an optimal auction for two objects. There is a single seller who wishes to sell two items. Armstrong has designed an optimal auction for such setting. He has considered two different models. In the first model, he assumes that the valuation is binary and has developed an optimal auction which is efficient as well. In the second model, he relaxes the assumption of binary valuation and designed an optimal auction. This auction may not be efficient.

Let A and B be the two objects that the seller wishes to sell. There are n bidders who have valuations for object $k \in \{A, B\}$. The valuation θ^k of bidder k belongs to the binary set $\{\theta_L^k, \theta_H^k\}$. Thus the number of possible types of each bidder is 4. Let us refer to the four types as $\{LL, LH, HL, HH\}$. When we say a bidder has type LH, we mean, his type for object A is θ_L^A and for B is θ_H^B.

Armstrong defines three auctions.

(i) *Independent auction*: Each object is allocated to the bidder who has highest valuation for that object (and is allocated fairly and randomly in the event of tie).
(ii) *Bundling auction*: Consider a modification to the above auction: if there is one or more bidder with a high value for an object, then as above that object is allocated to that bidder (or allocated randomly if there is more than one such bidder). In the case of object A, if there is no bidder with a high value for the object, then the object is definitely allocated to a type LH bidder if such a bidder exists, and if all bidders have type LL then the object is randomly allocated to these bidders. A similar rule is followed for object B. Thus, compared to the *independent auction*, the difference here is that a type LH bidder always wins object A against a type LL bidder.
(iii) *Mixed auction*: After the types have been announced, the seller allocates the object k using the independent auction above with probability $1 - \gamma$ and using the bundling auction with probability γ.

Armstrong has found an optimal auction for this setting. Depending upon the correlation between the valuations of objects A and B, he has shown that an optimal auction is one of the above three auctions or a combination of the above auctions. Readers are referred to Armstrong (2000) for technical details.

Table 4. Notation for optimal combinatorial auctions with single minded bidders.

M	Set of items the seller is interested in selling $\{1, 2, \ldots, m\}$
θ_i	True valuation of agent i for its bundle interest, $\theta_i \in [\underline{\theta_i}, \bar{\theta_i}]$
b_i	Bid of the buyer i
b	Bid vector, $(b_1, b_2, \ldots, b_n)$
b_{-i}	Bid vector without the agent i, i.e. $(b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$
$t_i(b)$	Payment by the agent i when submitted bid vector is b
$\overline{t_i}(b_i)$	Expected payment by the buyer i when he submits bid b_i Expectation is taken over all possible values of b_{-i}
$y_i = y_i(b)$	1, if the agent i gets his bundle, 0 otherwise
$\phi_i(\theta_i)$	Probability density function of (θ_i)
$\Phi_i(\theta_i)$	Cumulative distribution function of $f_i(\theta_i)$
$J_i(\theta_i)$	Virtual cost function for the buyer i, $J_i(\theta_i) = \theta_i + \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}$
K	Set of all feasible allocations of m objects among n agents. Each agent gets either its bundle of interest or gets nothing $K \subseteq \{0, 1\}^N$
k	$k \in K, k = (k_1, k_2, \ldots, k_n)$ $k_i = 1$ means i^{th} agent gets his bundle interest. 0 otherwise.

5.3c *Optimal combinatorial auctions in the presence of single minded bidders:* In combinatorial auctions, it may happen that a bidder may not be willing to bid on all possible subsets of the items. When a bidder is interested in bidding only on a specific bundle of the items, we say the bidder is single minded. This scenario is realistic as shown by the FCC (Federal Communications Commission) (Cramton 2005) auctions where a majority of the bidders are interested in a specific bundle of spectrum wavelengths. Recently Ledyard (2007) has characterized an optimal combinatorial auction when the bidders are single minded. Table 4 provides the notation used in this section.

Using BIC constraints, Ledyard has shown that an optimal auction for the seller is given by

$$\max_{k\in K} \int \left[\sum_{i=1}^{i=n} \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) k_i d\Phi \right],$$

where K is set of all feasible allocations of m objects among n agents in which each agent either gets his bundle of interest or doesn't get any object. Define the following virtual cost function:

$$J_i(\theta_i) = \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}.$$

Consider the following regularity condition: $J_i(\theta_i)$ is non-decreasing in θ_i. The optimal auction when the regularity condition is satisfied is given by,

$$\max_{k\in K} \sum_{i=1}^{n} J_i(b_i) y_i(b_i), \tag{16}$$

where b_i is the bid submitted by the agent i, with $y_i \in \{0, 1\}$ indicating whether the agent i gets its bundle of interest or not. The payment made by the agent i if it gets the bundle of interest is given by

$$t_i(b_i, b_{-i}) = \text{ infimum } \{b_i' | k_i(b_i', b_{-i}) = 1 \text{ in } (16)\}.$$

It can be noted that it is a weakly dominant strategy for each agent i to bid truthfully when the regularity condition holds true.

5.3d *An optimal auction for multi-unit procurement with capacitated bidders:* Here we look at a procurement situation where a buyer is interested in procuring multiple units of the same object and present an optimal auction (Gautam *et al* 2007). In practice, a seller is usually capacitated, that is, there is an upper bound on the quantity of the object that the seller can supply. The capacity of a seller is typically private information in addition to production cost. Thus, the problem of incentive compatibility becomes two-dimensional. The sellers might be able to do better by misreporting their capacities. For example, if each seller is paid an amount equal to the first losing bid, a seller might report a capacity that is less than the actual capacity, therefore increasing the value of the losing bid leading to more profit. In most situations, it can be safely assumed that the seller will never inflate the capacity, as it can be detected. We make this assumption.

Let $y_i(b)$ be the quantity to be procured from the seller i and $\overline{y_i}(b_i)$, the expected value of $y_i(b)$ (expectation taken over all possible values of b_{-i}). The buyer has to offer incentive to the sellers to bid truthfully. Suppose the following incentive is chosen:

$$\forall i \in N, \rho_i(b_i) = \overline{t_i}(b_i) - \hat{c}_i \overline{y_i}(b_i), \text{ where } b_i = (\hat{c}_i, \hat{q}_i).$$

Note that in this case, each bidder will be bidding the capacity $(\hat{q}_i)$ and unit cost $(\hat{c}_i)$. We assume that the true cost to the seller i, c_i belongs to $[\underline{c_i}, \bar{c_i}]$ and $q_i \in [\underline{q_i}, \bar{q_i}]$. Kumar and Iyengar (2006) have proved that:

Theorem 5.1. *Any mechanism in the presence of the capacitated sellers is BIC and IR if*

(i) $\rho_i(b_i) = \rho_i(\bar{c_i}, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c_i}} \overline{y_i}(s, \hat{q}_i) ds$
(ii) $\rho_i(b_i)$ *non-negative, and non-decreasing in* $\hat{q}_i \;\; \forall \, \hat{c}_i \in [\underline{c_i}, \bar{c_i}]$
(iii) The quantity which the seller i *is asked to supply,* $\overline{y_i}(c_i, q_i)$*, is non-increasing in* c_i*,* $\forall q_i \in [\underline{q_i}, \bar{q_i}]$.

Define a virtual cost of bidder i by

$$J_i(c_i, q_i) = c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)},$$

where $F_i(c_i|q_i)$ is the conditional CDF for agent i and $f_i(c_i|q_i)$ is the conditional PDF for agent i. Assume that $J_i(c_i, q_i)$ is non-increasing in q_i and non-decreasing in c_i. This regularity assumption is similar to Myerson's (1981) regularity assumption. Under this assumption, the following auction for multi-unit procurement can be shown to be optimal:

(i) Collect the bids from the sellers
(ii) Sort the bidders in increasing order of their virtual costs
(iii) If the capacity of the first seller is greater than the required quantity, order the required quantity
(iv) Else, order the seller to supply his full capacity

(v) Remove the above seller from the list. Reduce the required quantity by the amount to be supplied by the just deleted seller
(vi) Go to step 3.

At the end of the auction, each agent is paid according to Theorem 5.1. It can be shown for each seller i that the best response is to bid truthfully irrespective of whatever the other sellers are bidding. Thus, this mechanism satisfies dominant strategy incentive compatibility. The above property is a direct consequence of the result proved by Mookherjee and Reichelstein (1992) who have provided the monotonicity conditions for DSIC implementation of a BIC mechanism. Under these regularity assumptions, y_i satisfies these conditions. So we have a DSIC mechanism.

5.4 *Optimal auctions — A network perspective*

Malakhov and Vohra (2005) have presented a novel perspective for an optimal auction in the presence of capacitated bidders. They have used a network approach to solve this problem when the type set of each bidder is finite. They provide an innovative interpretation to the optimal auction design problem as an instance of the parametric shortest path problem on a lattice.

As the type set is finite, we can index it, that is, we can write $\Theta = \{\theta_1, \ldots, \theta_k\}$ as the set of all possible types for all agents. Each $\theta_i \in \Theta$ can be multi-dimensional. Let $\overline{T_i}$ be the payment made by any agent when it reports the valuation to be θ_i. Incentive compatibility requires:

$$T_i - T_j \leq E[v(k(\theta_i, .)|\theta_i)] - E[v(k(\theta_j, \cdot)|\theta_i)].$$

Now we can draw a directed graph with the types as vertices. A length $E[v(k(\theta_i, \cdot)|\theta_i)] - E[v(k(\theta_j, \cdot)|\theta_i)]$ is assigned to a directed edge (j, i). One more vertex, call it a dummy type, θ_0 is added. Utility to this type is 0 at all allocations. Now, the optimization problem of an optimal auction in this finite multi-dimensional type set reduces to determining the shortest path in this network. For more details about this, the article by Malakhov and Vohra (2005) may be consulted.

6. Characterization of DSIC mechanisms

In the previous part of the tutorial, we have seen the notion of DSIC mechanisms. In our study of DSIC mechanisms, due to the power of the revelation principle, we can restrict our attention to direct revelation mechanisms and hence to DSIC SCFs. In this section, we will present some important results on the characterization of dominant strategy incentive compatible SCFs.

We have seen that a direct revelation mechanism is specified as $\mathscr{D} = ((\Theta_i)_{i\in N}, f(\cdot))$, where f is the underlying social choice function and Θ_i is the type set of agent i. A valuation function of each agent i, $v_i(\cdot)$, associates a value of the allocation chosen by f to agent i, that is, $v_i : K \to \mathbb{R}$.

In the case of an auction for selling a single unit of a single item, suppose each agent i has a valuation for the object $\theta_i \in [\underline{\theta_i}, \overline{\theta_i}]$. If agent i gets the object, $v_i(\cdot, \theta_i) = \theta_i$. Otherwise, the valuation is zero. Thus for the agent i, the set of valuation functions over the set of allocations K, can be written as $\Theta_i = [\underline{\theta_i}, \overline{\theta_i}]$. Thus Θ_i is single-dimensional in this environment.

In a general setting Θ_i, may not be single-dimensional. If we are considering all real valued functions on X and allowing each user to have a valuation function to be any of these functions, we say Θ_i is unconstrained. In this case, $\Theta_i = \mathbb{R}^K$. Suppose $|K| = m$, then $\theta_i \in \Theta_i$ is an m-dimensional vector:

$$\theta_i = (\theta_{i_1}, \ldots, \theta_{i_j}, \ldots, \theta_{i_m}).$$

Note that θ_{i_j} will be the valuation of agent i if the j^{th} allocation from K is selected. In other words, $v_i(j) = \theta_{i_j}$. With such unconstrained type sets/valuation functions, we will study a characterization of DSIC SCFs. Due to the Green–Laffont theorem (Theorem 10·2 in Garg *et al* (2008)), any DSIC SCF in this setting which is also allocatively efficient, is necessarily Groves or VCG mechanism. Roberts (1979) characterizes all DISC SCFs in this environment. We will see characterization of payment rule for DSIC implementation in the subsection 6.1 and characterization of SCF in the subsection 6.2. Then we will see Roberts theorem in 6.3. We will see some variant of quasi-linear environment in the subsection 6.4.

A detailed treatment of the results presented here can be found in Nisan (2007), Lavi *et al* (2004) and Roberts (1979).

6.1 *Direct characterization*

PROPOSITION 6.1.

A social choice function f is DSIC if $\forall i \in N$,

(a) The payment rule is of the form

$$t_i : K \times \Theta_{-i} \rightarrow \mathbb{R}$$

That is, t_i depends only on the allocation and valuations of the other agents and not on θ_i. In other words, $\forall i \in N,\quad \forall \theta_i in \Theta_i \quad s.t.\ f(\theta_i, \theta_{-i}) = k$,

$$t_i(\theta_i, \theta_{-i}) = t_k.$$

(b) The mechanism maximizes the utility for each player. That is,

$$f(\theta_i, \theta_{-i}) \in \arg\max_{k \in K} (v_i(k) - t_k).$$

Proof. We first prove the sufficiency followed by necessity. Q.E.D.

Part 1: Sufficiency: Let θ_i be the true type of the player i and suppose θ_{-i} is the vector of types reported by players other than i. Let the SCF f choose $x = f(\theta_i, \theta_{-i})$ and $x' = f(\theta_i', \theta_{-i})$ with allocations k and k' respectively. Denote $t_k = t_i(\theta_i, \theta_{-i})$ and $t_{k'} = t_i(\theta_i', \theta_{-i})$. The utility of i when reporting the truth is $v_i(k) - t_k$. Even though the agent reports its type to be θ_i' instead of its true type θ_i, note that its valuation function will not change. Only the outcome will change from x to x'. As the SCF f satisfies statement (b) of the proposition, we have

$$v_i(k) - t_k \geq v_i(k') - t_{k'}.$$

Hence reporting truth is a best response to every agent i, irrespective of the types reported by the other agents. Thus the SCF f satisfying the above two properties is DSIC.

Part 2: Necessity: Let f be a DSIC SCF.

- Suppose, for some θ_i, θ_i', the SCF $f(\cdot)$ chooses the same outcome from X, that is, $f(\theta_i, \theta_{-i}) = f(\theta_i', \theta_{-i})$, but, $t_i(\theta_i, \theta_{-i}) > t_i(\theta_i', \theta - i)$. Then an agent with type θ_i will increase its utility by declaring θ_i'. Thus, if the payment rule is of not the form defined in statement (a) of the theorem, f may not be incentive compatible.
- If $f(\theta_i, \theta_{-i})$ chooses allocation rule k and $k \notin \arg\max_{k \in K}(v_i(k) - t_k)$, then fix k' as:

$$k' \in \arg\max_{a \in K}(v_i(a) - t_a)$$

in the range of allocation rules selected by $f(\cdot, \theta_{-i})$, and thus k' will be the allocation rule for some θ_i'. Now an agent with type θ_i will increase its utility by declaring θ_i'. This is a contradiction to the fact that f is DSIC. □

The above proposition is a characterization of the payment rule for DSIC implementation of a social choice function. Now we will provide a characterization that involves the social choice function in the next subsection.

6.2 *Weak monotonicity*

DEFINITION 6.1.

A social choice function f satisfies Weak monotonicity (WMON) property if for all $i \in N$, for all $\theta_{-i} \in \Theta_{-i}$, we have that $f(\theta_i, \theta_{-i}) = x \neq y = f(\theta_i', \theta_{-i})$ implies that $v_i(x) - v_i(y) \geq v_i'(x) - v_i'(y)$.

The WMON property means that if the social choice function chooses a different outcome when an agent changes its valuation, then it must be because the agent derives more value out of the new outcome relative to the value derived out of the original outcome.

Theorem 6.1. *If the direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$ is dominant strategy incentive compatible, then f satisfies the WMON property. If Θ_is are convex subsets of the Euclidean spaces and f satisfies the WMON property, then there exists a social choice function f' having the same allocation rule as f and payments $t_1, t_2, \ldots, t_n$, such that f' is DSIC.*

Using proposition 6.1, one can prove that the DSIC property implies the WMON property. Proving the second statement of the theorem is quite involved and we refer the reader to Mas-Colell (1995) for a proof.

6.3 *Roberts' theorem*

We have seen the Gibbard–Satterthwaite impossibility theorem (Theorem 7·1 in Garg *et al* (2008)). In this subsection, we will see another impossibility theorem under an unrestricted domain of preferences. This theorem is due to Roberts (1979). It turns out that in this environment, that is, when type set Θ_i is unconstrained for all i, the only DSIC mechanisms are variations of the VCG mechanism. These variants are often referred to as the weighted VCG mechanisms. In a wighted VCG mechanism, we are allowed to give weights to the agents and weights to the alternatives. The resulting SCF is said to be an *affine maximizer*. The notion of an affine maximizer is defined below. Next we state the Roberts' theorem.

DEFINITION 6.2.

A social choice function f is called an affine maximizer if for some subrange $A' \subset X$, for some agent weights $w_1, w_2, \ldots, w_n \in \mathbb{R}^+$, and for some outcome weights $c_x \in \mathbb{R}$, and for every $x \in A'$, we have that

$$f(\theta_1, \theta_2, \ldots, \theta_n) \in \arg\max_{x \in A'} \left(c_x + \sum_i w_i v_i(x) \right).$$

Theorem 6.2 (Roberts' theorem). *If $|X| \geq 3$ and for each agent $i \in N$, Θ_i is unconstrained, then any DSIC function f has non-negative weights $w_1, w_2, \ldots, w_n$ (not all of them zero) and constants $\{c_x\}_{x \in X}$, such that for all $\theta \in \Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$,*

$$f(\theta) \in \arg\max_{x \in X} \left\{ \sum_{i=1}^{i=n} w_i v_i(x) + c_x \right\}.$$

For a proof of this important theorem, we refer the reader to the article by Roberts (1979). Lavi, Mu'alem, and Nisan have provided two more proofs for the theorem — interested readers might refer to their paper (Lavi *et al* 2004) as well.

There have been positive results for DSIC SCFs for single-dimensional domains. Roberts showed that such results are impossible in unrestricted domains. In an unconstrained domain, the DSIC SCFs have to be necessarily affine maximizers. For domains which lie between single-dimensional and unrestricted domains, there are no significant possibility or impossibility results.

6.4 *A variant of quasi-linear environment*

In this section, we consider a variant of the quasi-linear environment which is commonly encountered in applications. In this environment, we make the following change in the settings of quasi-linear environment: we assume that $K = \mathbb{R}$. Note, this new set of project choices is no more a compact set. Next, we assume that for each agent i and each of its types $\theta_i \in \Theta_i$, we have

(i) $v_i(\cdot, \theta_i) : K \to \mathbb{R}$ is a twice continuously differentiable function

(ii) $\frac{\partial^2 v_i(k, \theta_i)}{\partial k^2} < 0$

(iii) $\frac{\partial^2 v_i(k, \theta_i)}{\partial k \, \partial \theta_i} > 0$

(iv) $\theta_i \in \left[\underline{\theta}_i, \overline{\theta_i}\right] \subset \mathbb{R}$.

It is easy to verify that even in this modified environment, there is no special choice function which is dictatorial. We now define a special class $\mathscr{C}$ of the social choice functions in this environment.

$$\mathscr{C} = \left\{ f : \Theta \to \mathbb{R}^{n+1} |\; f \text{ is a continuously differentiable function} \right\}.$$

The social choice functions of this class are commonly encountered in applications and they have some very interesting properties. In what follows we characterize the social choice functions of this class $\mathscr{C}$ from the perspective of various properties satisfied by them.

PROPOSITION 6.2 (Characterization of AE SCFs).

If an SCF $f \in \mathscr{C}$ is AE then $\forall\ i = 1, \ldots, n$ and $\forall\ \theta \in \Theta$, $k(\theta)$ is non-decreasing in θ_i.

PROPOSITION 6.3 (Characterization of DSIC SCFs).

An SCF $f \in \mathscr{C}$ is DSIC if $\forall\ i = 1, \ldots, n$ and $\forall\ \theta \in \Theta$, we have

(i) $k(\theta)$ *is non-decreasing in* θ_i

(ii) $t_i(\theta_i, \theta_{-i}) = t_i(\underline{\theta}_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(k(s,\theta_{-i}),s)}{\partial k} \frac{\partial k(s,\theta_{-i})}{\partial s} ds.$

PROPOSITION 6.4 (Characterization of AE+DSIC SCFs).

An AE SCF $f \in \mathscr{C}$ is DSIC if it satisfies the Groves payment scheme, that is, $\forall\ i = 1, \ldots, n$ and $\forall\ \theta \in \Theta$, we have

$$t_i(\theta) = \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] + h_i(\theta_{-i}) \ \ \forall\ i = 1, \ldots, n \tag{17}$$

where $h_i(\cdot)$ is any arbitrary function of θ_{-i}.

The proofs of this propositions can be found in Mas-Colell (1995), Section 23·C. In the part one Garg *et al* (2008) of this tutorial, in Theorem 10·2, we have seen that, under fairly general setting, Groves mechanism are the only allocative efficient and DSIC. The above three propositions state that under differentiable case, that is SCF $f \in \mathscr{C}$ which is allocatively efficient and DSIC should necessarily be Groves.

7. Dominant strategy implementation of BIC rules

Due to the impossibility theorems by Gibbard–Satterthwaite and Roberts, there is no hope for dominant strategy implementation of social choice functions in unrestricted or in most general settings. But, by restricting our attention to only quasi-linear environments, we can have DSIC mechanisms, for example, the Groves mechanisms.

Also, we have shown that, by settling for Bayesian incentive compatibility, we can implement a wider class of SCFs. However, BIC implementation has a few striking drawbacks. Bayesian implementation assumes the private information structure to be common knowledge. It also assumes that the social planner knows a common prior distribution. In some cases, this requirement might be quite demanding. Also, slight mis-specification of the common prior may lead the equilibrium to shift discontinuously. These may cause the mechanism to incur significant losses when compared to the optimal revenue or optimal cost that is obtainable with an exact specification of the common prior distribution. DSIC implementation overcomes these problems in a simple way since the equilibrium strategy does not depend upon the common prior distribution. We would therefore always wish to have a DSIC implementation. However, because of the restricted nature of SCFs that are possible in the DSIC domain, generally a mechanism designer would be forced to look for BIC SCFs to achieve desired goals. At this stage, we can ask the question: Can we implement a BIC SCF as a DSIC rule with the same expected interim utilities to all the players? Mookherjee and Reichelstein (1992) have answered this question by characterizing BIC rules which can be equivalently implemented in dominant strategies.

7.1 *Some definitions*

The environment is similar to the one we described in § 10 of part 1 of this tutorial, but is modelled slightly differently. Each agent i observes private signal $\theta_i \in [\underline{\theta_i}, \bar{\theta_i}]$. Depending upon the type profile θ of all agents, the social planner makes public decision $k \in K$. The monetary transfer to each agent is $t_i(k, \theta)$. When rule k is implemented agent i incurs cost $v_i(k, \theta_i)$ unlike rest of the paper. In rest of the paper, agent i has value $v_i(k, \theta_i)$ when allocation rule is k. To compensate this cost, principal agent pays him $t_i(k, \theta)$. Hence, in this setting, agent i's utility is $u_i(k, \theta_i) = t_i(k) - v_i(k, \theta_i)$. And social choice function is,

$$f : \Theta \to K \times \mathbb{R}^n.$$

DEFINITION 7.1.

A BIC SCF $f(\cdot) = (k(\cdot), t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot))$ can be equivalently implemented in dominant strategies by another SCF $f'(\cdot) = (k(\cdot), \bar{t_1}(\cdot), \ldots, \bar{t_n}(\cdot))$, if the SCF f' is DSIC and

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - \bar{t_i}(\theta_i, \theta_{-i})|\theta_i] = 0 \quad \forall\, \theta_i \in \Theta_i,\ \forall\, i \in N.$$

The equivalent implementation means that the allocation $k(\cdot)$ is unchanged at all type profiles in the new SCF and the original payment transfers, $t_i(\cdot)$ are replaced with $\bar{t_i}(\cdot)$. In the new SCF, reporting truth is a dominant strategy and the interim utility of every agent is unchanged.

DEFINITION 7.2.

We say the allocation rule $k(\cdot)$ is implementable in dominant strategies if there exist payment rules $(t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot))$ such that $(k(\cdot), t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot))$ is a DSIC SCF.

DEFINITION 7.3 (Weak single crossing property).

The valuation function $v_i(\cdot)$ satisfies the weak single crossing property if, for any two allocation rules $k_1, k_2 \in K, \exists\, \tilde{\theta_i} \in \Theta_i$ s.t

$$(\partial/\partial\theta_i)v_i(k_1, \tilde{\theta_i}) > (\partial/\partial\theta_i)v_i(k_2, \tilde{\theta_i}) \quad \text{then}$$

$$(\partial/\partial\theta_i)v_i(k_1, \theta_i) > (\partial/\partial\theta_i)v_i(k_2, \theta_i) \ \ \forall\theta_i \in \Theta_i.$$

7.2 *Conditions for equivalent dominant strategy implementation*

We now state some conditions for a BIC rule to admit equivalent dominant strategy implementation, as given by Mookherjee and Reichelstein (1992).

PROPOSITION 7.1.

A BIC SCF $(k(\cdot), t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot))$ can be equivalently implemented in dominant strategies if the allocation rule $k(\cdot)$ is implementable in dominant strategies.

This proposition characterizes the BIC SCFs that are dominant strategy implementable. In the next proposition, we will give a sufficient condition under which an allocation rule $k(\cdot)$ is implementable in dominant strategies.

PROPOSITION 7.2.

An allocation rule $k(\cdot)$ is dominant strategy implementable if

$$\gamma_i(\theta_{-i}, t|\theta_i) \equiv \frac{\partial}{\partial \theta_i} v_i(k(\theta_{-i}, t), \theta_i) \tag{18}$$

is decreasing in t for all $\theta_{-i} \in \Theta_{-i},\ \theta_i \in \Theta_i\ i \in N$.

The above proposition provides a sufficient condition for DSIC implementation of allocation rule $k(\cdot)$. If $k(\cdot)$ satisfies the weak single crossing property, then the monotonicity condition in Equation (18) becomes necessary.

PROPOSITION 7.3.

Suppose the valuation functions of the agents satisfy the weak single crossing property. Then the BIC allocation rule $(k(\cdot), t_1(\cdot), t_2(\cdot), \ldots, t_n(\cdot))$ can be equivalently implemented in dominant strategies if the monotonicity condition (18) is satisfied.

For a proof of the above results, we refer the reader to the paper by Mookherjee and Reichelstein (1992).

8. Implementation in ex-post Nash equilibrium

Dominant strategy implementation and Bayesian implementation are widely used for implementing a social choice function. There exists another notion of implementation, called ex-post Nash implementation, which is stronger than Bayesian implementation but weaker than dominant strategy implementation. We discuss this briefly here.

Hurwicz (1972) introduced the notion of incentive compatibility in 1972. Gibbard (1973) was the first to describe the revelation principle for dominant strategy implementation. This motivated mechanism design researchers to look for dominant strategy incentive compatible mechanisms. Impossibility results such as the Gibbard–Satterthwaite theorem (Theorem 7·1 in part 1 of this tutorial) spurred researchers to look for implementations other than dominant strategy implementation (Groves & Ledyard 1977, Hurwicz 1979, Maskin 1999). Maskin (1999) formalized the notion of *Nash Equilibrium* implementation. This is now known as *Ex-post Nash* implementation. Dasgupta *et al* (1979) generalized this to Bayesian–Nash implementation. We have already discussed Bayesian implementation in detail in the section 11 of part 1 (Garg *et al* 2008) of this tutorial.

DEFINITION 8.1.

A profile of strategies $(s_1^*(\cdot), s_2^*(\cdot), \ldots, s_n^*(\cdot))$ is an ex-post Nash equilibrium if for every $\theta = (\theta_1, \ldots \theta_n) \in \Theta$, the profile $(s_1^*(\theta_1), \ldots s_n^*(\theta_n))$ is a Nash equilibrium of the complete information game defined by $(\theta_1, \ldots, \theta_n)$. That is $\forall\, i \in N,\ \forall\, \theta \in \Theta$, we have $\forall \theta \in \Theta$ and $\forall s_i' \in S_i,$ we have

$$u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i) \geq u_i(s_i'(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i)\ \forall\ s_i' \in S_i.$$

This ex-post Nash equilibrium notion is stronger than Bayesian–Nash equilibrium. In Bayesian–Nash equilibrium, the equilibrium is strategy is played by the agents after observing their own private types and computing an expectation over others' types. It is an equilibrium only in expected sense. In ex-post Nash equilibrium, even after the players are informed with the types of the other players, it is still a Nash equilibrium for each agent i to play an action according to s_i^*. This is called *lack of regret* feature. That is, even if agents come to know about the others' types, the agent need not regret playing this action. Bayesian–Nash equilibrium may not have this feature since the agents may like to revise their strategies after knowing the types of the other agents.

DEFINITION 8.2.

We say that the mechanism $\mathscr{M} = ((S_i)_{i \in N}, g(\cdot))$ implements the social choice function $f(\cdot)$ in ex-post Nash equilibrium if there is a pure strategy ex-post Nash equilibrium $s^*(\cdot) = \left(s_1^*(\cdot), \ldots, s_n^*(\cdot)\right)$ of the game Γ^b induced by $\mathscr{M}$ such that

$$g\left(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)\right) = f(\theta_1, \ldots, \theta_n) \;\forall\, (\theta_1, \ldots, \theta_n) \in \Theta.$$

For example, consider the first price sealed bid auction with two bidders. Let, $\Theta_1 = \Theta_2 = [0, 1]$ and θ_1 denote the valuation of the first agent and θ_2 that of the agent 2. It can be shown that it is a Bayesian–Nash equilibrium for each bidder to bid according to the strategy $(b_1^*(\theta_i), b_2^*(\theta_2)) = (\frac{\theta_1}{2}, \frac{\theta_2}{2})$. Now suppose agent 1 is informed that the other agent values the object at 0·6. If agent 1 has a valuation of 0·8, say, it is not a Nash equilibrium for him to bid 0·4 even if agent 2 is still following Bayesian–Nash strategy.

Though ex-post implementation is stronger than Bayesian–Nash implementation, it is still much weaker than dominant strategy implementation. Also note that, Bayesian–Nash equilibrium is only in the expected sense and hence ex-post Nash equilibrium implies Bayesian–Nash equilibrium. In the next section, we will present an example of an auction in which it is an ex-post Nash equilibrium (but not a dominant strategy) for each agent to report its true type.

9. Interdependent values

We have so far assumed that the private values or signals observed by the agents are independent of one another. This is a reasonable assumption in many situations. However, in the real world, there are environments where the valuation of agents might depend upon the information available or observed by the other agents. We will look at two examples.

Example 1. Consider an auction for an antique painting. There is no guarantee that the painting is an original one or a plagiarized version of some nice art work. If all the agents know that the painting is not an original one, they will have a very low value for it independent of one another, whereas on the other hand, they will have a high value for it when it is a genuine piece of work. But suppose they have no knowledge about its authenticity. In such a case, if a certain bidder happens to get information about its genuineness, the valuations of all the other agents will naturally depend upon this signal (indicating the authenticity of the painting) observed by this agent.

Example 2. Consider an auction for oil drilling rights. At the time of the auction, buyers usually conduct geological tests and their private valuations depending on the results of these tests. If a prospective bidder knew the results of the tests of the others, his own willingness to pay for the drilling rights would be modulated suitably based on the information available.

The interdependent private value models have also been studied in the mechanism design literature. Among the various interdependent private value models, there exists a popular model called the *common value model* (which we have already seen in § 4).

Table 5. Notation for Cremer and McLean Auction.

Θ_i	Type set of the agent $i = 1, 2$. $= \{1, 2\}$
θ	(θ_1, θ_2), true type profile of the bidders.
$\hat{\theta}$	$(\hat{\theta_1}, \hat{\theta_2})$, profile of types reported by the bidders.
$y_i(\hat{\theta})$	Probability of agent i winning the good when agents report the types as $\hat{\theta}$
$v_i(\theta)$	Valuation of a good for the agent i, if the agent gets it when true type profile is θ. $v_i(\theta) = \theta_1 * \theta_2$
$t_i(\hat{\theta})$	Payment made by the agent i, when the profile of reported types is $(\hat{\theta})$
$u_i(\hat{\theta}, \theta)$	Utility to agent i when the reported types are $\hat{\theta_1}, \hat{\theta_2}$ and the actual types are θ_1, θ_2 $= y_i(\hat{\theta}) * \theta_1 * \theta_2 - t_i(\theta_1, \theta_2)$
$p(\theta_1, \theta_2)$	Probability that the agents will have type profile (θ_1, θ_2).

9.1 *An optimal auction with interdependent demands*

Cremer and McLean (1985) consider a situation when a seller is trying to sell an indivisible good or a fixed quantity of a divisible good. The value of the received good for the bidders depends upon each others' private signals. Also, the private signals observed by the agents are interdependent of specified properties. In such a scenario, Cremer and McLean (1985) have designed an auction that extracts a revenue from the bidders, which is equal to what could have been extracted when the actual signals of the bidders are known. In this auction, it is an ex-post Nash equilibrium for the agents to report their true types. This auction is interim individually rational but may not be ex-post individually rational. We will illustrate this auction with the following example.

A seller is trying to sell a single unit of a indivisible item. There are two bidders. Table 5 provides the notation.

Let, $p(1, 1) = \frac{1}{3}$, $p(1, 2) = \frac{1}{6}$, $p(2, 1) = \frac{1}{6}$ and $p(2, 2) = \frac{1}{3}$. Assign the object to the agent who reports higher type. If a tie occurs, award the object randomly to any of the agents. Define the payment rule as: $t_1(1, 1) = t_2(1, 1) = \frac{1}{3}$, $t_1(1, 2) = t_2(2, 1) = \frac{1}{3}$, $t_1(2, 1) = t_2(1, 2) = \frac{4}{3}$, and $t_1(2, 2) = t_2(2, 2) = \frac{7}{3}$. It can be verified that it is an ex-post Nash equilibrium for each agent to report its type truthfully. As an example, we will verify it for the type profile (2, 2).

$$
\begin{aligned}
u_1((2, 2), (2, 2)) &= y_1(2, 2) * 2 * 2 - t_1(2, 2) \\
&= \frac{1}{2} * 4 - \frac{7}{3} \\
&= -\frac{1}{3} \\
&= u_1((1, 2), (2, 2)) \\
&\geq u_1((1, 2), (2, 2)).
\end{aligned}
$$

Similarly, $u_2((2, 2), (2, 2)) \geq u_2((2, 1), (2, 2))$. Thus reporting true type is a Nash equilibrium even if the agents are informed of the types of the other agents. However, it is not a dominant strategy equilibrium. If agent 1 is not using Nash equilibrium strategy, that is, reports false type '1' when the actual type is '2', then agent 2's best response is to report its type to be '1', even if its type were '2'.

Now we will show that the ex-ante revenue to the seller in this auction is the same as the ex-ante revenue in a complete information setting. The expected revenue to the seller in the complete information setting is:

$$\frac{1}{3} * 1 + \frac{1}{6} * 2 + \frac{1}{6} * 2 + \frac{1}{3} * 4 = \frac{7}{3}.$$

The expected revenue to the seller in the above defined payment rules is:

$$\frac{1}{3} * \left(\frac{1}{3} + \frac{1}{3}\right) + \frac{1}{6} * \left(\frac{1}{3} + \frac{4}{3}\right) * 2 + \frac{1}{3} * \left(\frac{7}{3} + \frac{7}{3}\right) * 4 = \frac{7}{3}.$$

9.2 *Other results with interdependent types*

As we have already seen, the Vickrey auction is allocatively efficient under independent value model. However, in the common value model it is not efficient. Similarly, it has been shown that the generalized Vickrey auction will be efficient under the common value model only if the buyers' private information is single-dimensional. In general, if the buyers' private signals are multi-dimensional, efficiency is unattainable. In a broad class of cases, Dasgupta and Maskin (2000) have designed an auction which is constrained-efficient in the sense of being efficient subject to incentive constraints.

Jehiel and Moldavanu (2001) have given necessary conditions for Bayesian incentive compatibility and allocative efficient mechanisms. These conditions are satisfied by independent private value models. But, in general, these conditions may not be satisfied. Jehiel and Moldavanu (2001) have studied the linear environment. Bergemann and Valimaki (2006) have given necessary conditions for incentive compatible, efficient mechanisms in non-linear environment. They also give weaker sufficient conditions.

10. Other key topics in mechanism design

10.1 *Implementation theory*

Dominant strategy incentive compatibility ensures that reporting true types is a weakly dominant strategy equilibrium. Bayesian incentive compatibility ensures that reporting true types is a Bayesian–Nash equilibrium. Typically, the Bayesian game underlying a given mechanism may have multiple equilibria, in fact, could have infinitely many equilibria. These equilibria typically will produce different outcomes. Thus, it is possible that non-optimal outcomes are produced by truth telling.

The *implementation problem* addresses the above difficulty caused by multiple equilibria. The implementation problem seeks to design mechanisms in which all the equilibrium outcomes are optimal. This property is called the *weak implementation property*. If it also happens that every optimum outcome is also an equilibrium, we call the property as *full implementation property*. Maskin (1999) provided a general characterization of Nash implementable social choice functions using a monotonicity property, which is now called *Maskin Monotonicity*. Maskin's work shows that Maskin monotonicity, in conjunction with another property called *no-veto-power* will guarantee that all Nash equilibria will produce an optimal outcome. Maskin's results have now been generalized in many directions, for example, see the references in (The Nobel Foundation 2007).

10.2 *Dynamic mechanisms*

In the discussion so far, we have assumed that agents observe their private values and take actions depending on their values and the rules of mechanism. The social planner has to take the decision exactly once based on the actions by the agents. Such a mechanism can be categorized as a static implementation. Let us consider a dynamic scenario such as described below.

Consider an air carrier which wishes to sell the tickets for the flights. The buyers are arriving at different times and each buyer could have a unique demand. The valuation a buyer attaches to the air-ticket constitutes the buyer's private information. The air carrier's goal is to maximize revenue in such a dynamic environment.

To handle such scenarios, we need mechanisms which are *dynamic*. When a sequence of decisions is required to be made, a static mechanism cannot be employed if the parties receive information over time that should affect the decisions. For example, agents who are involved in a long-term relationship may need to make a sequence of trading and investment decisions in a changing environment. A procurement authority may wish to conduct a sequence of auctions, where bidders have serially correlated values or capacity constraints or learning-by-doing. In recent times, the design of mechanisms in a dynamic environment has emerged as an interesting area of research.

Athey and Segal (2007) have considered a dynamic environment in which agents observe a sequence of private signals over a number of periods/time slots. The number of periods can either be finite or countably infinite. In each time slot, the agents report their private signals. Based on the reported signals, the public decision is made. The probability distribution over future signals may depend on both past signals and past decisions. In such an environment, Athey and Segal have designed a BIC mechanism which is also efficient and budget balanced.

Contracts to operate public facilities such as airports or to use natural resources such as forests, are renewed periodically. The current winner of the contract receives additional information about it, learns most about the value of the resource. This enables this current winner to revise its valuation. In such dynamic settings, Bapna and Weber (2005) and Bergemann and Valimaki (2006) have designed a BIC and efficient mechanism in this settings.

10.3 *Iterative mechanisms*

In many situations, the decisions can be made sequentially. The information acquisition for making decisions can either be made in one go or sequentially over the length of the period. The sequential way proves to be more efficient. This approach is also useful even in settings in which the resource allocation decision is a one-time decision. Two outstanding examples of well known iterative mechanisms are the English auction and the Dutch auction.

The computational and communication costs of the agents may be reduced by making the decisions sequentially. For example, in case of combinatorial auctions, the total number of bundles is exponential and the agents need to find out valuations for all the bundles. In such scenarios, rather than having a static combinatorial auction, iterative auctions could be used to reduce the cost of computing valuations and allocations. Iterative auctions are also efficient in the case where the agents learn about their valuations over a period of time. So, initially they have to just bid low and can increase their bids as the auction progresses and progressively they come to know about their valuations. The survey paper by Parkes (2005) is an excellent source of material on iterative mechanisms.

10.4 *Stochastic mechanisms*

Consider a scenario in which the decision maker has to solve a stochastic optimization problem and the parameters for optimization are private to the agents. For example, consider the following situation discussed by Ieong *et al* (2006). There is a university book seller, who sells course material. The seller has two options: (1) Get it printed from a printing press, which costs less, but takes more time; (2) Photocopy the material with less turnaround time, but higher cost than the first option. If the seller has prior knowledge about how many students will be registering for the course, the seller can directly print the copies. The students are not sure, before semester starts, whether or not they would be taking the course. However, each student may have a probability distribution that gives the probability of this student taking the course. If the students report their distributions to the book seller truthfully, the book seller can simply solve a two stage stochastic optimization problem. However, the students may not reveal this distribution function truthfully. In such a scenario, mechanism design can be used. Ieong *et al* (2006) have proposed a two stage stochastic mechanism which is BIC. It is not known whether one can design a dominant strategy implementation of a stochastic mechanism.

10.5 *Cost sharing mechanisms*

If a set of users share a common resource, they have to share the cost for it as well. For example, suppose there is a common facility to pump water out of a well. A number of users may use this facility. There is a cost associated with the creation and running of the facility. This cost has to be shared among the users. There has to be some mechanism for sharing this cost. This type of mechanism is called a *cost sharing mechanism*. In a cost sharing mechanism, each user (simultaneously) demands a quantity of output after which the mechanism distributes the cost among users. Moulin (1999) has designed a cost sharing mechanism, namely *Serial Cost Sharing* which is DSIC.

10.6 *Robust mechanisms*

In his seminal work, Harsanyi (1967) introduced games with incomplete information. He suggested the modelling of these games by introducing notion as a type space and adding *nature* as a player in a game. The nature assigns a random variable to each player over the player's type space. It is assumed that all the players share a common prior distribution, from which using one's own type, each agent can compute the probability over the types of other players. But this requirement may not be satisfied in general. Also, mis-specification of the common prior distribution could lead to a drastic shift in equilibria. This may result in huge losses for the agents.

In such situations, a desirable line of attack is to develop models that are independent of the common prior distribution. Bergemann and Morris (2004) have tackled this problem by enhancing the type spaces. The authors also assume the knowledge of payoff functions of other players to be uncertain. It is assumed that a player knows only his own payoff type. Under this setting, the authors have studied the implementation of social choice correspondence over this newly constructed type spaces.

10.7 *Virtual implementation of a social choice function*

Due to the impossibility theorems on SCF implementation, either we have to restrict the domain of the SCFs or we have to weaken the solution concept from dominant strategies, to Bayesian implementation. If we want to implement an SCF in dominant strategies and also we want this class of SCFs to be rich, then a way out is *virtual implementation* of an SCF. In virtual implementation, the SCF may not be exactly implementable, but we have an approximate implementation of it.

An SCF f is ϵ *close* to f', if for all preference/type profiles of users, the outcome of f and f' are ϵ close to each other. An SCF f is said to be *virtually implementable* if, for every $\epsilon > 0$, there exists another SCF f', which is implementable and is ϵ close to f. For more details on virtual implementation, the readers are referred to Dilip & Hitoshi (1992), Chambers (2004).

11. To probe further

For a more detailed treatment of mechanism design, the readers are requested to refer to textbooks, such as the ones by Mas-Colell *et al* (1995), Green & Laffont (1979), and Laffont (1988). There is an excellent recent survey article by Nisan (2007). There are many other scholarly survey papers on mechanism design — for example by Myerson (1989) and by Jackson (2001, 2003). The Nobel Prize website has a highly readable technical summary of mechanism design theory (2007). The recent edited volume on *Algorithmic Game Theory* by Nisan *et al* (2007) also has valuable articles related to mechanism design.

The current paper is not to be treated as a survey on auctions in general. There are popular books (for example, by Milgrom (2004) and Krishna (2002)) and surveys on auctions (for example, (McAfee & McMillan 1987, Milgrom 1989, Klemperer Paul 2004, Wolfstetter 1996, Jayant & Parkes 2005)) which deal with auctions in a comprehensive way.

The current paper is also not to be treated as a survey on combinatorial auctions (currently an active area of research). Exclusive surveys on combinatorial auctions include the articles by de Vries and Vohra (2003, 2005), Pekec & Rothkopf (2003), and Narahari & Dayama (2005). Cramton (2005) has brought out a comprehensive edited volume containing expository and survey articles on varied aspects of combinatorial auctions.

For a more comprehensive treatment of mechanism design and its applications in network economics, the readers are referred to the forthcoming monograph by Narahari *et al* (2008).

We thank Professor Vivek Borkar, Former Editor, Sādhanā, for encouraging us to write this paper. Our grateful thanks to three anonymous referees for their insightful comments and helpful suggestions. The second author would like to acknowledge the support he has received from the Homi Bhabha Fellowships Council, Mumbai and from the Office of Naval Research, Washington, DC.

Notation

n	Number of agents
N	Set of agents: $\{1, 2, \ldots, n\}$
Θ_i	Type set of agent i
Θ	Set of all type profiles $= (\Theta_1 \times \cdots \times \Theta_n)$
Θ_{-i}	Set of all profiles of types of agents other than $i = (\Theta_1 \times \cdots \times \Theta_{i-1} \times \Theta_{i+1} \times \cdots \times \Theta_n)$
θ_i	Actual type of agent i, $\theta_i \in \Theta_i$

θ	A profile of actual types $= (\theta_1, \ldots, \theta_n) \in \Theta$
θ_{-i}	A profile of actual types of agents other than $i = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \in \Theta_{-i}$
$\hat{\theta}_i$	Reported type of agent i, $\hat{\theta}_i \in \Theta_i$
$\hat{\theta}$	A profile of reported types $= (\hat{\theta}_1, \ldots, \hat{\theta}_n) \in \Theta$
$\hat{\theta}_{-i}$	A profile of reported types of agents other than $i = (\hat{\theta}_1, \ldots, \hat{\theta}_{i-1}, \hat{\theta}_{i+1}, \ldots, \hat{\theta}_n) \in \Theta_{-i}$
$\Phi_i(\cdot)$	A cumulative distribution function (CDF) on Θ_i
$\phi_i(\cdot)$	A probability density (or mass) function (PDF) on Θ_i
$\Phi(\cdot)$	A cumulative distribution function on Θ
$\phi(\cdot)$	A probability density (or mass) function on Θ
X	Outcome set
x	A particular outcome, $x \in X$
$u_i(\cdot)$	Utility function of agent i
$f(\cdot)$	A social choice function
F	Set of social choice functions
$W(\cdot)$	A social welfare function
$\mathscr{M}$	An indirect mechanism
$\mathscr{D}$	A direct revelation mechanism
$g(\cdot)$	Outcome rule of an indirect mechanism
S_i	Set of actions available to agent i in an indirect mechanism
S	Set of all action profiles $= S_1 \times \cdots \times S_n$
S_{-i}	Set of all profiles of actions of agents other than agent i
b_i	A bid of agent i; $b_i \in S_i$
b	A profile of bids $= (b_1, \ldots, b_n) \in S$
b_{-i}	A profile of bids by agents other than $i = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$
$b^{(k)}$	k^{th} highest element in $(b_1, \ldots, b_n)$
$(b_{-i})^{(k)}$	k^{th} highest element in $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$
$s_i(\cdot)$	A strategy of agent i
$s(\cdot)$	A profile of strategies $= (s_1(\cdot), \ldots, s_n(\cdot))$
K	A Set of project choices
k	A particular project choice, $k \in K$
t_i	Monetary transfer to agent i
$v_i(\cdot)$	Valuation function of agent i
$U_i(\cdot)$	Expected utility function of agent i
X_f	Set of feasible outcomes
$\mathbb{R}$	Set of real numbers

References

Armstrong M 2000 Optimal multi-object auctions. *Rev. Econ. Studies* 67(3): 455–81

Athey S, Segal I 2007 An efficient dynamic mechanism. Technical report, UCLA Department of Economics

Bapna A, Weber T A 2005 Efficient dynamic allocation with uncertain valuations. *SSRN eLibrary*

Bergemann D, Morris S 2004 Robust mechanism design. Yale School of Management Working Papers ysm380, Yale School of Management

Bergemann D, Valimaki J 2006 Efficient dynamic auctions. Cowles Foundation discussion papers 1584, Cowles Foundation, Yale University

Chambers C P 2004 Virtual repeated implementation. *Econ. Lett.* 83(2): 263–268

Cramton P 2005 Simultaneous ascending auctions. In P Cramton, Y Shoham, R Steinberg, (eds.), *Combinatorial Auctions*, (Cambridge, Massachusetts, USA: The MIT Press) 99–114

Cremer J, McLean R P 1985 Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. *Econometrica* 53(2): 345–61

Dasgupta P, Hammond P, Maskin E 1979 The implementation of social choice rules: Some general results on incentive compatibility. *Rev. Econ. Studies* 46: 181–216

Dasgupta P, Maskin E 2000 Efficient auctions. *The Quarterly J. Econ.* 115(2): 341–388

de Vries S, Vohra R V 2003 Combinatorial auctions: A survey. *INFORMS J. Computing* 15(1): 284–309

de Vries S, Vohra R V 2005 Design of combinatorial auctions. In *Handbook of Quantitative Supply Chain Analysis: Modelling in the E-Business Era*. International Series in Operations Research and Management Science, (Norwell, MA, USA: Kluwer Academic Publishers) 247–292

Dilip A, Hitoshi M 1992 Virtual implementation in iteratively undominated strategies: Complete information. *Econometrica* 60(5): 993–1008

Garg D, Narahari Y, Gujar S 2008 Foundations of mechanism design: A tutorial - Part 1: Key concepts and classical results. *Sādhanā* 33(2): 83–130

Gautam Raghav, Hemachandra N, Narahari Y, Prakash Hastagiri 2007 Optimal auctions for multi-unit procurement with volume discount bidders. In *Proc. of IEEE Conference on Electronic Commerce (IEEE CEC-2007)* 21–28

Gibbard A 1973 Manipulation of voting schemes. *Econometrica* 41: 587–601

Green J R, Laffont J J 1979 *Incentives in Public Decision Making*. North-Holland Publishing Company, Amsterdam

Groves T, Ledyard J 1977 Optimal allocation of public goods: a solution to the 'free rider' dilemma. *Econometrica* 45: 783–811

Harsanyi John C 1967 Games with incomplete information played by Bayesian players. Part I: The basic model. *Management Sci.* 14: 159–182

Holmstrom Bengt, Myerson Roger B 1983 Efficient and durable decision rules with incomplete information. *Econometrica* 51(6): 1799–819

Hurwicz L 1972 On informationally decentralized systems. In *Decision and Organization*. Radner and McGuire (eds.) North-Holland, Amsterdam

Hurwicz L 1979 Outcome functions yielding walrasian and lindahl allocations at Nash equilibrium points. *Rev. Econ. Studies* 46(2): 217–25

Ieong S, Man-Cho So A, Sundararajan M 2006 Mechanism design for stochastic optimization problems. Technical report, Stanford University, Working Paper

Jackson M O 2001 A crash course in implementation theory. *Social Choice and Welfare* 18: 655–708

Jackson M O 2003 Mechanism theory. In Ulrich Derigs, (ed.) *Optimizations and Operations Res.* Oxford, UK

Jayant K and Parkes D C 2005 Auctions, bidding, and exchange design. In David Simchi-Levi, David S Wu, Max Shen, (eds.), *Handbook of Quantitative Supply Chain Analysis: Modelling in the E-Business Era*, Int. Series in Operations Research and Management Science. (Norwell, USA: Kluwer Academic Publishers)

Jehiel P, Moldovanu B 2001 Efficient design with interdependent valuations. *Econometrica* 69(5): 1237–1259

Klemperer Paul 2003 Why every economist should learn some auction theory. In M Dewatripont, L Hansen, S Turnovsky, (eds.), *Advances in economics and econometrics: Invited lectures to 8th World Congress of the Econometric Society*. Cambridge University Press

Klemperer Paul 2004 *Auctions: Theory and Practice. The Toulouse Lectures in Economics*. Princeton University Press

Krishna V, Perry M 1998 Efficient mechanism design. *SSRN eLibrary*

Krishna V 2002 *Auction Theory*. Academic Press

Kumar A, Iyengar G 2006 Optimal procurement auctions for divisible goods with capacitated suppliers. Technical report, Columbia University

Ledyard J O 2007 Optimal combinatoric auctions with single-minded bidders. In *EC'07: Proceedings of the 8th ACM conference on electronic commerce*, (New York, NY, USA: ACM Press) 237–242

Laffont J J 1988 *Fundamentals of Public Economics*. (Cambridge: The MIT Press)

Lavi R, Mu'alem A, Nisan N 2004 Two simplified proofs for Roberts' theorem. Technical report, School of Computer Science and Engineering, The Hebrew University of Jerusalem, Israel, Working paper

McAfee P R, McMillan J 1987 Auctions and bidding. *J. Econ. Literature* 25(2): 699–738

Malakhov A, Vohra R V 2005 An optimal auction for capacitated bidders: A network perspective. Discussion papers, Kellogg School of Management, Northwestern University

Mas-Colell A, Whinston M D, Green J R 1995 *Microeconomic Theory*. Oxford University Press, New York

Maskin E 1999 Nash equilibrium and welfare optimality. *Rev. Econ. Studies* 66: 23–38

Milgrom P 1989 Auctions and bidding: A primer. *J. Econ. Perspect.* 3(3): 3–22

Milgrom P 2004 *Putting Auction Theory to Work*. Cambridge University Press

Mookherjee D, Reichelstein S 1992 Dominant strategy implementation of bayesian incentive compatible allocation rules. *J. Econ. Theory* 56(2): 378–399

Moulin H 2007 Efficient, strategy-proof and almost budget-balanced assignment. Technical report, Northwestern University, Center for Mathematical Studies in Economics and Management Science

Moulin H 1999 Incremental cost sharing: Characterization by coalition strategy-proofness. *Social Choice and Welfare* 16(2): 279–320

Myerson R B 1989 Mechanism design. In J Eatwell, M Milgate, P Newman, (eds.), *The New Palgrave: Allocation, Information, and Markets*, Norton, New York, 191–206

Myerson R B 1981 Optimal auction design. *Math. Operations Res.* 6(1): 58–73

Myerson R B 1997 *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, Massachusetts

Narahari Y, Dayama P 2005 Combinatorial auctions for electronic business. *Sādhanā* 30(2–3): 179–212

Narahari Y, Garg D, Rama Suri N, Prakash H 2008 *Game Theoretic Problems in Network Economics and Mechanism Design Solutions*. Advanced Information and Knowledge Processing Series, (London: Springer) in press

Nisan N 2007 Introduction to mechanism design (for computer scientists). In E Tardos N Nisan, T Roughgarden and V Vazerani, (eds.), *Algorithmic Game Theory*, Cambridge University Press, 209–242

Nisan N, Roughgarden T, Tardos E, Vazerani V 2007 *Algorithmic Game Theory*, Cambridge University Press

Parkes D C 2005 Iterative combinatorial auctions. In P Cramton, Y Shoham, R Steinberg, (eds.), *Combinatorial Auctions*. (Cambridge, Massachusetts, USA: The MIT Press)

Pekec A, Rothkopf M H 2003 Combinatorial auction design. *Management Sci.* 49: 1485–1503

Riley J G, Samuelson W F 1981 Optimal auctions. *Am. Econ. Rev.* 71(3): 383–92

Roberts K 1979 The characterization of implementable choice rules. In *Aggregation and Revelation of Preferences*, 321–349

The Nobel Foundation 2007 The Sveriges Riksbank Prize in Economic Sciences in memory of Alfred Nobel 2007: Scientific Background. Technical report, The Nobel Foundation, Stockholm, Sweden

Wolfstetter E 1996 Auctions: An introduction. *Econ. Surv.* 10: 367–421