Is small-world network disordered?

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We study by extensive Monte Carlo simulations, the sample-to-sample fluctuation of various physical quantities in the critical region on ensembles of quenched Ising model on a small-world network. The system is found to be strongly self-averaging in the critical region in spite of relevant randomness. This is associated with the sharpening of the probability distribution of the inverse pseudo-critical temperatures. Single realisation finite-size data of various physical quantities show as good a data collapse(finite-size scaling) as the average.

Random long range bonds in a Euclidean Lattice(EL) leads to a small-world (SW) behaviour where any two points far away on the EL can be bridged by a finite number of connections[1, 2, 3] . If such bonds are introduced with a probability p, then the small world behaviour is obtained even for very small p if the average number of bonds introduced is large. Such networks are characterized by various statistical properties defined over the ensemble of networks. We address the question of influence of the stochastic nature of the small-world network on the behaviour of a physical system defined on it.

Let us think of the ferromagnetic Ising model with a spin $s_i = \pm 1$ at each site i of a network based on an Euclidean lattice of N points similar to [4, 5, 6, 7, 8, 9] . Its Hamiltonian is taken as

$$H(\{\mathcal{S}\}) = -J \sum_{\langle ij \rangle} s_i s_j - J \sum_{(ij)} s_i s_j, \qquad (1)$$

where $\langle ij \rangle$ are the nearest-neighbours on the lattice , (ij) are the long distance neighbours along the random bonds of $\{\mathcal{S}\}$ added for the network and $J > 0$. The Hamiltonian, being dependent on the set $\{\mathcal{S}\}$, is random for a given configuration of the spins $\{s_i\}$. Any physical property X of the model, therefore, requires an overall averaging over the ensemble of networks. It would suffice to have a description in terms of the average $[X]$ where [..] denotes averaging over networks ("sample averaging") provided the relative variance $R_X = V_X/[X]^2 \to 0$ for large N, where $V_X = [X^2] - [X]^2$. In such a case a single large system is enough to represent the ensemble. Such a quantity is called *self-averaging* (SA). Off criticality, if one builds up a larger lattice from smaller ones, then central limit theorem(CLT) implies that $R_X \sim N^{-1}$ ensuring self-averaging. In contrast, at a critical point, long range correlations mars the additivity requirement of the CLT. In this context, recent renormalization group and numerical studies have shown that if randomness or disorder is relevant (i. e. changes the critical behaviour of the pure system) then self-averaging property is lost and in particular, R_X at the critical point *approaches a constant as* $N \to \infty$. Unlike the SA case, even if the critical point is known exactly, statistics in numerical simulations cannot be improved by going over to larger lattices (large N).

Several studies have shown that the SW bonds lead to a mean field type transition. E.g. one-dimensional Ising model has $T_c = 0$ but SW bonds lead to a finite T_c and mean-field criticality. Thus the randomness introduced by the SW bonds in Eq. (1) is undoubtedly relevant. Therefore the particular question we would like to study is the behaviour of R_X for various X to determine if the randomness of the SWN leads to any non self-averaging behaviour.

The prediction of non self-averaging nature of critical quantities is an extremely significant result coming from general renormalization group arguments. This basic result of Ref. [10] and the hypothesis of Ref. [11] can be summarized as follows. According to finite size scaling, when the critical region sets in the size of the system is comparable to the correlation length ξ that grows as the critical point is approached. The appropriate scaling variable is N/N_c where $N_c = \xi^d$ is the correlation volume in d dimensions. At the critical point of a random system, there is an additional source of fluctuation from the variation in the transition temperature itself. Therefore, instead of the conventional finite size scaling (FSS), a sample dependent scaled variable is required. A reduced temperature is defined as $\tilde{t_i} = |T - T_c(i, N)|/T_c$ where $T_c(i, N)$ is a pseudo-critical temperature of sample i of N sites with T_c as the ensemble average of critical temperature in the $N \to \infty$ limit. In terms of this temperature, a critical quantity X is expected to show a sample dependent finite size scaling form

$$X_i(T, N) = N^{\rho} Q(\tilde{t}_i N^{1/\bar{\nu}}) \qquad (2)$$

where ρ characterizes the behaviour of $[X]$ at T_c.[††] Thus $\rho = \gamma/\bar{\nu}$ where $\bar{\nu} = d\nu$ when X is the magnetic susceptibility χ. The RG approach seems to validate this hypothesis especially the absence of any extra anomalous dimension in powers of N for R_X. Incidentally, this hypothesis, Eq. 2, excludes rare events of large pure type lattices for which pure $\bar{\nu}$ should be used. We are not

[††] Conventional notations of critical exponents are used: $C \sim t^{-\alpha}$, $\chi \sim t^{-\gamma}$, $\xi \sim t^{-\nu}$ where C, χ and ξ denote the specific heat, magnetic susceptibility and correlation length of the system. t is the temperature-like variable with the critical point at $t = 0$.

considering such cases dominated by these rare events (Griffiths' singularity). With this scaling form the relative variance R_X at the critical point or in the critical region is given by

$$R_X \sim [(\delta T_c)^2] N^{2/\bar{\nu}}, \quad (3)$$

where $[(\delta T_c)^2]$ is the sample average variance of the pseudo-critical temperature. A random system can have several temperature scales, namely $(T_c(N) - T_c)$ and $(T - T_c)$ in addition to the shift in the transition temperature itself. It is plausible that for a system with relevant disorder all these scales behave in the same way so that typical fluctuations in the pseudo-critical temperature is set by the correlation volume, yielding $[(\delta T_c)^2] \sim N^{-2/\bar{\nu}}$. An immediate consequence of this is that R_X approaches a constant as $N \to \infty$ indicating complete absence of self-averaging at the critical point in a random system. A finite size scaling form for R_X is

$$R_X(N, N_c) = N^{\kappa} \mathcal{R}(N/N_c), \quad (4)$$

where $\mathcal{R}(z)$ is a scaling function and $\kappa = 0$ for systems with relevant disorder. For a pure type critical point (irrelevant disorder) $-1 < \kappa = \alpha/\bar{\nu} < 0$ (weakly self-averaging). Accepting that off-critical $R_X \sim N^{-1}$, we have $\mathcal{R}(z) \sim z^{-1}$ for large z and $R_X\big|_{T_c} = \mathcal{R}(0)$. These predictions have been verified for various types relevant and irrelevant disorders and also with canonical ensemble of disorder (fixed concentration of disorder as opposed to grand canonical disorder) for cases with $\alpha < 0$ at the random critical point [11, 12, 13, 14] .

With this background we set to check the behaviour of R_X for various X for SWN. One of the major differences with respect to the previous studies of random systems is that in this particular case, it is well established that the shifted critical behaviour has $\alpha = 0$ and therefore $\bar{\nu} = 2$.

The Ising model has been studied extensively on a Small-World network [SWN] [4, 5, 6, 7, 8, 9] using techniques of disordered systems. The present work explicitly looks into the issue of sample-to-sample fluctuations to gauge the influence of the variations in the distribution of the long range bonds that make it a SWN.

We start with the Ising model on a SWN in 1D. In our model each site on the lattice with an Ising spin has random links to two distant spins such that no two spins are connected by more than one link. All links of equal strength. Thus we have a "canonical" scenario since the number of links at each site is fixed. Hence no extra normalisation factor is needed in the long range part of the Hamiltonian of Eq 1. In the present work we chose $J/k_B = 1$ where k_B is the Boltzmann constant.

Data were taken at $T = 2.85$ (close to the estimated critical temperature)and χ, C and the Binder cumulant,$U_m = [< m^4 >/< m^2 >^2] - 3$ were calculated, using the single histogram reweighting technique [15]. We examined lattice sizes : $N = 100, 500, 1000, 2000, 3000$ in

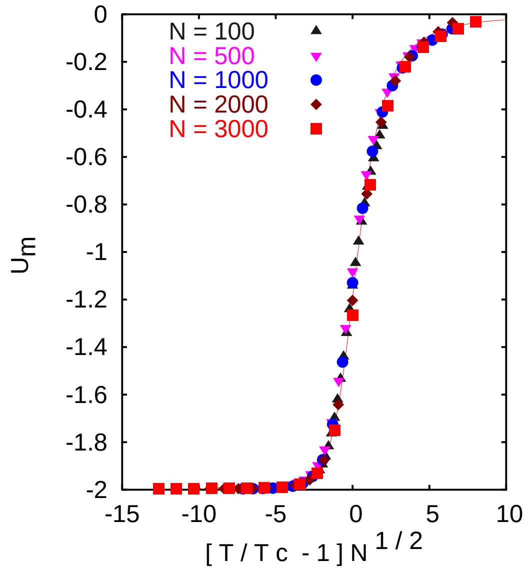

FIG. 1: Plot of the data-collapse of the Binder Cumulant versus scaled temperature for various N

our Monte Carlo simulations. We studied 1535 samples for $N = 100$ to 517 samples for $N = 3000$ using 10^3 equilibration and 10^6 MC steps for each N. Data were taken at intervals of 10^3 MC steps .

A data-collapse of U_m with finite size scaling variable $N^{1/\bar{\nu}}(T - T_c)/T_c$ would give T_c (the infinite lattice critical temperature) and $\bar{\nu}$. By using the data-collapse method of Ref. [16] we obtained $T_c = 2.804(1)$ and $\bar{\nu} = 2.00(4)$ [‡‡] The value of $\bar{\nu}$ is consistent with previous results [6]. The resulting collapse is shown in Fig. 1. We also investigated similar plots for χ and C after averaging over many realisations of disorder (not shown here).

Further proof of the mean-field nature of the transition comes from the comparison of the data with the mean-field form of U_m. To evaluate the mean field form of U_m we use the mean-field form of the magnetisation per spin m probability distribution in the critical region [17] :

$$P_N(m) \propto \exp\left[-N(a_1 t_N m^2 + a_2 m^4)\right] \quad (5)$$

with t_N being the critical temperature of a lattice of size N and a_1, a_2 being constants. By replacing $\hat{m} = (a_2 N)^{1/4} m$, we find U_m where the averages are obtained by integrating $\hat{m}$ from $-\infty$ to $+\infty$ with the weight

$$P_N(\hat{m}) \propto \exp\left[-b_1(x - b_2)\hat{m}^2 - \hat{m}^4\right] \quad (6)$$

[‡‡] It is striking that the value of T_c for the Ising model on a small world network (with $q-2$ extra bonds) is nearly the same as that from the Bethe Approximation (BA) for a lattice of co-ordination number q, $T_c = \frac{2}{\ln[q/(q-2)]}$. This is valid not only for our case with $q = 4$ but also for other q. For $q = 3$, from Ref. [6], $T_c = 1.82(2)$ which is to be compared with $T_c = 1.8204$ by BA. We have also verified that for an SWN with $q = 6$, $T_c = 4.92(1)$ whereas the corresponding BA value is $T_c = 4.9326$.

where $b_1 = a_1/{a_2}^{1/2}$, x is the finite size scaling variable $[(T - T_c)/T_c]N^{1/\bar{\nu}}$ with $b_1 b_2$ taking care of the finite size shift of the critical temperature. The solid curve in Fig. 1 is obtained with $b_1 = 1.7$, $b_2 = 0.72$.

We find good data collapse by using $\tilde{t} = T - T_c$ even though FSS is supposedly better with the use of $\tilde{t} = T - T_c(N)$ after finding out the $T_c(N)$ for every sample size[18]. This is because our system is SA and this method should be more pertinent for non SA or weakly SA systems.

To investigate the distribution of pseudo-critical temperature, $\beta_c(i, N)$ (the temperature at which the specific heat of sample i of size N is a maximum), data were taken at $T = 2.85$ and $\beta_c(i, N)$ for various N were calculated using the histogram method [15]. The distribution of $\beta_c(i, N)$ for $N = 10, 50, 100$ is constructed.We studied 3291 lattice samples for $N = 10$, 1645 lattice samples for $N = 50$ and 1535 lattice samples for $N = 100$. As in earlier works [11] we find that the inverse critical temperature $\beta_c(i, N)$, scales as

$$[\beta_c(i, N) - \beta_c(N)]^2 \sim N^{-2/\bar{\nu}}, \quad (7)$$

but one also needs a scale factor $N^{-1/2}$ for the probability distribution $P(\beta_c(i, N))$. Fig. 2 shows the data collapse of this distribution. The data-collapse is best achieved with $\bar{\nu} = 2.00(3)$ which is consistent with the value of $\bar{\nu}$ obtained in the data collapse shown in fig. 1. This is in marked contrast to the other cases of random systems studied so far[11, 12]. The fact that the peak scales inversely as the width shows that despite the fluctuation in pseudo-critical temperatures, the distribution approaches a $\delta-$function. As a result the critical temperature of a large N network can be thought of as the average of pseudo-critical temperatures of the small sub-networks. This averaging out is tantamount to self-averaging .

Whilst in the present work we have used a "canonical" ensemble with a fixed number of bonds it would be interesting to test the Ising model on a SWN in a "grand canonical" ensemble where the number of bonds can vary.

We then studied R_M, R_χ and V_C at T_c^∞ for the above lattice sizes. About 56440 lattice samples for N=10 to 1000 lattice samples for N=3000 were studied. For each sample we used 10^3 equilibration and 10^5 MC steps. Data were taken at intervals of 10^3 MC steps. The data is fitted to the form $R_X = A_X N^{\rho_X}$ where R_X is the relative variance for M and χ. The values obtained are $\rho_M = -0.96(9)$, and $\rho_\chi = -0.94(8)$. Thus χ and M are *strongly self-averaging* . The singular part of energy cannot be filtered out and hence the behaviour of V_E can not be predicted decisively. We see in fig. 3 that V_C is a constant as expected and hence C is also *strongly self-averaging* .

In case of a strongly self-averaging system, a typical sample should be a representative of the average. We observe good data collapse with even a single realisation of disorder (as shown in Fig. 4). Thus in such situations

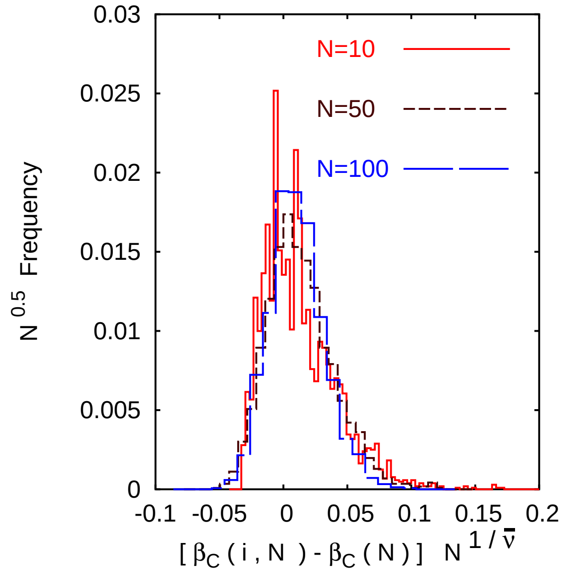

FIG. 2: Plot of the distribution of pseudocritical inverse temperatures $\beta_C(i, N)$ for various N

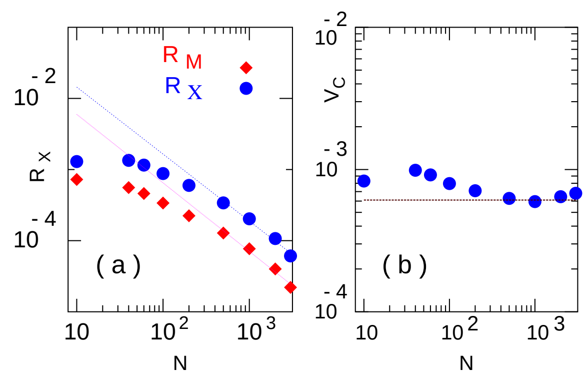

FIG. 3: (a) R_M, R_χ versus N at T_C and (b) V_C versus N at T_C.The straight lines show straight line fits to R_M and R_χ .

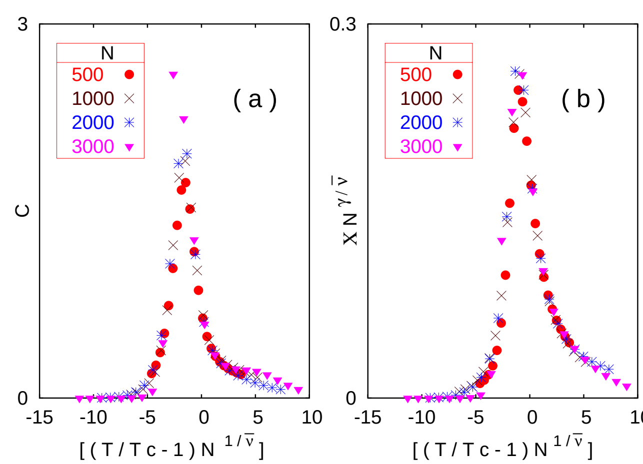

FIG. 4: Data collapse of C and χ on considering a single realisation of randomness

annealed averaging as done in ref [4] should work well. Consequently no extra order parameter (as required in replica approach) should be needed for networks.

It is not clear if this feature of strong self-averaging is a consequence of $\alpha = 0$, in which case it should be true for all relevant disorder problems with mean-field behaviour. An extension of the RG argument that predicted non self-averaging [10] to encompass situations with sharp limit of T_c distribution may shed light on this. Whether this result on the disorder aspect of a network is important in other real life situations like the railway network [19] needs further study.

To conclude, we investigated the self-averaging behaviour of the Ising model on a small world network in 1D. The distribution of $\beta_c(i, N)$ is found to become sharper as $N \to \infty$ with the fluctuation decaying as $[\delta\beta_c(i, N)]^2 \sim N^{-2/\bar{\nu}}$. The data collapse of various physical quantities both for a single realisation of disorder and after averaging over many disorder realisations showed no significant difference. At T_c^∞, the relative fluctuations R_M, R_χ for magnetization and susceptibility are found to behave as $R_M, R_\chi \sim N^{-1}$ while the variance V_C for the specific heat approaches a constant for large N. This shows that the system is strongly SA in the critical region in spite of relevant randomness.

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