ON LINEAR TRANSFORMATIONS OF BOUNDED SEQUENCES—I.

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PART I.

§ 1. *Introduction.*

IN a remarkable book devoted to Cesáro's summability method, Anderson[1] has derived a number of very interesting theorems relating to transformations corresponding to differences of any real order of a given sequence. Some theorems of a similar nature (generalizing the notion of monotonicity) are also given by Knopp.[2] It has occurred to me that, such theorems being special types of transformations of sequences, it would be desirable to study properties of transformations of a general nature of subject, of course, to suitable conditions. I have therefore considered here a class of linear transformations (T) given by the infinite matrix $\| a_{mn} \|$ characterised by four conditions, the reciprocals (T^{-1}) of these transformations, and the products of the T and T^{-1}.

Several very interesting results have emerged as a result of these considerations. The most interesting property of the class (T) is *that they all have unique reciprocals with regard to null sequences.* The conditions under which a set of operators, T_1, T_2, $T_3 \cdots$, and T_1^{-1}, T_2^{-1}, T_3^{-1}, $\cdots$ may be validly combined, and the equivalence of one combination with another when applied to bounded or null sequences are also discussed. For example,

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[1] A. F. Anderson, *Studier over Cesáro's summabilitet's methode* (Danish). See the second chapter entitled "*Om differences*".

[2] K. Knopp, "*Mehrfach monotone Zahlen,*" *Math. Zs.*, 1925, 22, 75–85.

if $\{x_n\}$ be a null sequence, $T_1 T_2 \{T_1^{-1}(x_n)\}$ has, in general, no meaning, whereas $T_2 T_1 \{T_1^{-1}(x_n)\} = T_2 (x_n)$. If we denote by (G) the class of the (T), (T^{-1}) and products of (T) and (T^{-1}), we can say that the *G's are, in general, non-commutative.* These results are comprised in parts one and two.

Part three of this series deals with a sub-class of (T) which we denote by $\bar{U}$, their reciprocals and products such as $\bar{U}_1\bar{U}_2\bar{U}_3\cdots$, $\bar{U}_1\bar{U}_2\bar{U}_3^{-1}\cdots$, etc. Denoting this class by (G_1) it is shown that, in contrast to the G's, *the G_1's are commutative.* Also, the transformations corresponding to differences of any real order form a sub-class of G_1 itself, so that *the theorems of Anderson and Knopp referred to above follow as particular cases of our general theorems* in parts one and two.

A type of transformation corresponding to (T) for application to functions of the continuous real variable is given and corresponding to the existence of a unique reciprocal for a T we have the unique solution of the integral equation

$$u(t) + \int_t^{\infty} u(t_1)\, k(t_1, t_1)\, dt_1 = \phi(t)$$

under the condition $u(t) \to 0$, $t \to \infty$ and the Kernel $K(t, t_1)$ being characterised by conditions similar to those imposed on a_{mn}.

§ 2. *Theorem on a Class of Infinite Matrices.*

We shall consider in this paper, the class of linear transformations (T) given by the infinite matrix $\| a_{mn} \|$ characterised by the following four conditions :—

$$\left.\begin{array}{llll} (a) & a_{nn} = 1 & & \\ (b) & a_{mn} = 0 & (n < m) & \\ (c) & a_{mn} \leqslant 0 & (n > m) & \\ (d) & -\sum\limits_{p=1}^{\infty} a_{n,\, n+p} \leqslant 1 & & \end{array}\right\} \qquad (2,\ 1)$$

Let $A = \| a_{mn} \|$ be the defining matrix, and $\| \delta_{mn} \|$ the unit matrix. We shall prove the *theorem that there exists a unique matrix B such that B.A.* $= \| \delta_{mn} \|$. Let

$$A = \begin{Vmatrix} 1 & a_{01} & a_{02} & \cdot & \cdot & \cdot \\ 0 & 1 & a_{12} & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix} = \begin{Vmatrix} 1 & -\alpha_{01} & -\alpha_{02} & \cdot & \cdot & \cdot \\ 0 & 1 & -\alpha_{12} & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix}$$

Conditions (2, 1) imply $\alpha_{mn} \geqslant 0$, and $\sum\limits_{p=1}^{\infty} \alpha_{n,n+p} \leqslant 1$.

Let $B = \| \beta_{mn} \|$. Then the condition $B \cdot A = 1$ gives

$$\sum_{r=0}^{\infty} \beta_{mr}\, a_{rn} = \delta_{mn}$$

i.e., $\sum_{r=0}^{n} \beta_{mr}\, a_{rn} = \delta_{mn}$, since, from (*b*) of (2, 1) $a_{rn} = 0$ for $r > n$.

Case 1. $n < m$: $(\delta_{mn} = 0)$.

$$\sum_{k=0}^{n} \beta_{mk}\, a_{kn} = \sum_{k=0}^{n-1} \beta_{mk}\, a_{kn} + \beta_{mn}\, a_{nn} = \delta_{mn} = 0.$$

Assuming $\beta_{m0}, \beta_{m1} \cdots \beta_{m,n-1}$ all to be zero, we have

$$\beta_{mn}\, a_{nn} = 0, \text{ i.e., } \beta_{mn} = 0; \quad \text{since } a_{nn} = 1 \text{ from } (2, 1);$$

but $\beta_{m0} = 0$, for $\beta_{m0} = \beta_{m0}\, a_{00} = \delta_{m0} = 0$. Hence by induction

$$\beta_{mn} = 0, \qquad (n < m) \tag{2, 2}$$

Case 2. $n = m$: $(\delta_{nn} = 1)$.

$$\beta_{nn} = \beta_{nn}\, a_{nn} = \sum_{r=0}^{n} \beta_{nr}\, a_{rn} = \delta_{nn}$$

$$\text{i.e., } \beta_{nn} = 1 \tag{2, 3}$$

Case 3. $n > m$: $(\delta_{mn} = 0)$. Let $n = m + \rho$, then

$$\sum_{r=0}^{n} \beta_{mr}\, a_{rn} = \sum_{r=m}^{r=n} \beta_{mr}\, a_{rn}$$

$$= \sum_{k=0}^{\rho} \beta_{m,m+k}\, a_{m+k,m+\rho}$$

$$= \beta_{m,m+\rho} + \sum_{k=0}^{\rho-1} a_{m+k,m+\rho}\, \beta_{m,m+k}$$

$$= \beta_{m,m+\rho} - \sum_{k=0}^{\rho-1} a_{m+k,m+\rho}\, \beta_{m,m+k}$$

and $\sum_{r=0}^{n} \beta_{mr}\, a_{rn} = \delta_{mn} = \delta_{m,m+\rho} = 0$, *i.e.*

$$\beta_{m,m+\rho} - \sum_{k=0}^{\rho-1} a_{m+k,m+\rho}\, \beta_{m,m+k} = 0 \tag{2, 4}$$

Solving for $\beta_{m,m+\rho}$ from the set of equations (2, 4) we get at once

$$\beta_{mm} = 1, \quad \beta_{m,m+1} = a_{m,m+1}$$

$$\beta_{m,m+\rho} = \begin{vmatrix} a_{m,m+\rho} & \cdots & \cdots & a_{m,m+\rho} \\ -1 & a_{m+1,m+2} & \cdots & a_{m+1,m+\rho} \\ 0 & -1 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & -1 & a_{m+\rho-1,m+\rho} \end{vmatrix} \tag{2, 5}$$

Expanding the determinant in terms of the first row, we get

$$\beta_{m,\, m+p} - \sum_{k=1}^{p} a_{m,\, m+k}\, \beta_{m+k.\, m+p} = 0 \qquad (2, 6)$$

which proves that

$$A \cdot B = \| \delta_{mn} \| = B.A.$$

We denote the transformation corresponding to B as T^{-1} where A defines T.

§ 3. *Illustrative Theorems.*

We give three theorems which, although quite obvious, are noted here for the sake of completeness of the exposition of the structure of the algebra of the transformations (T).

THEOREM 1: *If* $\{x_n\}$ *be a* $\left[\frac{\text{bounded}}{\text{null}}\right]$ *sequence, then* $y_n = T(x_n)$ *is*

a $\left[\frac{\text{bounded}}{\text{null}}\right]$ *sequence.*

Let $\bar{x}_{n_0}$ be the upper bound of $| x_n |$ for $n \geqslant n_0$, then

$$| y_n | = | T(x_n) | = \left| \sum_{\rho=0}^{\infty} a_{n,\, n+\rho}\, x_{n+\rho} \right|$$

$$\leqslant \bar{x}_n \left\{ 1 - \sum_{\rho=1}^{\infty} a_{n,\, n+\rho} \right\}$$

$$\leqslant 2\bar{x}_n, \text{ using } (d) \text{ of } (2, 1). \qquad (3, 1)$$

Hence the theorem is proved. Denoting the matrices corresponding to T_1, T_2, and (T_1, T_2) by A_1, A_2, and $A_1 \cdot A_2$ respectively, we have

THEOREM 2: *If* $\{x_n\}$ *be a* $\left[\frac{\text{bounded}}{\text{null}}\right]$ *sequence then* $(T_1\, T_2)\,(x_n)$, *and* $(T_2\, T_1)$

(x_n) *are* $\left[\frac{\text{bounded}}{\text{null}}\right]$ *sequences, and* $(T_1\, T_2)\,(x_n) = T_1\,[T_2\,(x_n)]$.

In general $(T_1\, T_2) \neq (T_2\, T_1)$.

Let $A_1 = \| a_{mn} \|$, $A_2 = \| \beta_{mn} \|$. If ${}_1A_1\, A_2 = \| C_{mn} \|$

$$C_{mn} = \Sigma a_{mj}\, \beta_{jn} = 0, \quad \text{if } n < m$$

$$= 1, \quad \text{if } m = n, \text{ and}$$

$$C_{n,\, n+\rho} = \sum_{r=0}^{\rho} a_{n,\, n+r}\, \beta_{n+r,\, n+\rho}.$$

$$| C_{n,\, n+\rho} | \leqslant \sum_{0}^{\rho} | a_{n,\, n+r} |\, | \beta_{n+r,\, n+\rho} |.$$

Now $$|(T_1 T_2)(x_n)| \leqslant \sum_{p=0}^{\infty} |C_{n, n+p}| \, |x_{n+p}|$$

$$\leqslant \bar{x}_{n_0} \cdot \sum_{p=0}^{\infty} \cdot \sum_{r=0}^{p} |a_{n, n+r}| \cdot |\beta_{n+r, n+p}|$$

$$\leqslant \bar{x}_{n_0} \cdot \sum_{r=0}^{\infty} |a_{n, n+r}| \cdot \sum_{p=r}^{\infty} |\beta_{n+r, n+p}|.$$

Since $\sum_{r=0}^{\infty} |a_{n, n+r}| \leqslant 2, \; \sum_{r=0}^{\infty} |\beta_{n, n+r}| \leqslant 2$

$$|(T_1 T_2)(x_n)| \leqslant 4\bar{x}_{n_0}, \tag{3, 2}$$

which proves the first portion of the theorem. Again,

$$T_1[T_2(x_n)] = \sum_{r=0}^{\infty} a_{n, n+r} \sum_{p=r}^{\infty} \beta_{n+r, n+p} \, x_{n+p}$$

$$= \sum_{p=0}^{\infty} \left(\sum_{r=0}^{p} a_{n, n+r} \, \beta_{n+r, n+p} \right) \cdot x_{n+p},$$

since the latter series is absolutely convergent. Therefore,

$$T_1[T_2(x_n)] = \sum_{p=0}^{\infty} c_{n, n+p} \, x_{n+p} = (T_1 T_2)(x_{n+p}).$$

This could be easily generalised to the product of any finite number of transformations. Also since the multiple series concerned are all absolutely convergent, we have

THEOREM 3 : $T_1 \{T_2 [T_3 (x)]\} = (T_1 \; T_2) [T_3 (x)] = T_1 [(T_2 \; T_3) \; (x)] = (T_1 \, T_2 \, T_3) \, (x)$, *where* $(T_1 \; T_2 \; T_3)$ *is defined by* $A_1 \cdot A_2 \cdot A_3$. (Proof being very similar to that of Theorem II).

Both the theorems 2 and 3 could be generalised to a finite number of T's. Since in general $\Sigma \; a_{n, n+r} \cdot \beta_{n+r, n+p} \neq \Sigma \; \beta_{n, n+r} \cdot a_{n+r, n+p}$ we have $(T_1 \; T_2) \neq (T_2 \; T_1)$. But if all the terms in any line parallel to the diagonal of A_1 have the same value ; *i.e.*,

$$\left.\begin{aligned} a_{01} &= a_{12} = \cdot \cdot = a_{n \; n+1} = \cdot \cdot \, , \\ a_{02} &= a_{13} = \cdot \cdot = a_{n, \, n+2} = \cdot \cdot \, , \end{aligned}\right\} \tag{3, 3}$$

and a similar condition holds for A_2, then $(T_1 \; T_2) = (T_2 \, T_1)$.

§ 4. *Preliminary Lemmas.*

Before we proceed to Theorem 4 about the existence of a unique reciprocal, we shall prove here a set of lemmas to be used later on.

Let T_0 correspond to $\|a_{mn}\|$. By definition $a_{nn} = 1$.
We will define $-a_{mn} = \alpha_{mn}$, so that $\alpha_{mn} \geqslant 0$, for $n \geqslant m+1$.

We shall prove later, in § 5, that $\| \beta_{mn} \|$ defines T_0^{-1}. For the present we shall collect together the following characterisations of the elements of $\| \beta_{mn} \|$.

By (2·2) $\qquad \beta_{mn} = 0 \ (n < m) \qquad$ (4, 1)

By (2·3) $\qquad \beta_{nn} = 1. \qquad$ (4, 2)

Since all the $\alpha_{n, n+\rho}$ are $\geqslant 0$, we have

$$\beta_{n, n+\rho} \geqslant 0 \ (\rho \geqslant 0) \quad \text{by (2·5).} \tag{4, 3}$$

Let us assume $\beta_{n, n+\rho} \leqslant 1$ for all n and $\rho = 0, 1, 2, \cdots \rho_0$. Then by (2, 6)

$$\beta_{n, n+\rho_0+1} = \sum_{r=1}^{r=1+\rho_0} \alpha_{n, n+r} \beta_{n+r, n+\rho_0+1} \leqslant \sum_{r=1}^{1+\rho_0} \alpha_{n, n+r} \leqslant 1.$$

Hence by induction for all n and all ρ, $\beta_{n, n+\rho} \leqslant 1$, since for $\rho = 0$ $\beta_{nn} = 1$. (4, 4)

Let $\| \beta^1_{mn} \|$ and $\| \beta^2_{mn} \|$ correspond to T_1^{-1} and T_2^{-1}; then we define the transformation $(T_1^{-1} T_2^{-1})$ by

$$\| \beta^1_{mn} \| \cdot \| \beta^2_{mn} \| = \| \bar{\beta}_{mn} \| \tag{4, 5}$$

Since $\bar{\beta}_{mn} = \sum_j \beta^1_{mj} \cdot \beta^2_{jn}$, properties (4, 1) – (4,3) also hold good for $\bar{\beta}_{mn}$.

Corresponding to (4, 4) we have

$$\bar{\beta}_{n, n+\rho} = \sum_{r=0}^{\rho} \beta^1_{n, n+r} \beta^2_{n+r, n+\rho}$$

$$\bar{\beta}_{n, n+\rho} \leqslant (\rho + 1) \tag{4, 5a}$$

If $\| \bar{\beta}_{mn} \|$ corresponds to the product of k such transformations

$$\text{then} \quad \bar{\beta}_{n, n+\rho} \leqslant (\rho + 1)^{k-1}. \tag{4, 5b}$$

Writing the system of equations (2, 4) at length, we have

$$\left.\begin{aligned}
\beta_{nn} &= 1 \\
\beta_{n, n+1} &= \alpha_{n, n+1} \beta_{nn} \\
\beta_{n, n+2} &= \beta_{nn} \alpha_{n, n+2} + \beta_{n, n+1} \alpha_{n+1, n+2} \\
&\cdots\cdots\cdots\cdots\cdots \\
\beta_{n, n+\rho} &= \beta_{nn} \alpha_{n, n+\rho} + \beta_{n, n+1} \alpha_{n+1, n+\rho} + \cdots + \beta_{n, n+\rho-1} \alpha_{n+\rho-1, n+\rho}
\end{aligned}\right\}$$

Adding we get

$$\beta_{nn} \left(1 - \sum_{r=1}^{\rho} \alpha_{n, n+r}\right) + \beta_{n, n+1} \left(1 - \sum_{r=2}^{\rho} \alpha_{n+1, n+r}\right) + \cdots$$
$$+ \beta_{n, n+\rho} (1) = 1. \tag{4, 6}$$

Let $\quad R_{n, n+\rho} = 1 - \sum_{r=1}^{\rho} \alpha_{n, n+r}$, (for $\rho > 1$) and $R_{n, n} = 1$.

Then (4, 6) can be written as

$$\sum_{r=0}^{\rho} \beta_{n,\,n+r}\, R_{n+r,\,n+\rho} = 1. \tag{4, 6a}$$

It is to be noted that $R_{n,\,n+\rho} \geqslant 0$ since by (*d*) of $2\cdot1 \sum_{1}^{\infty} a_{n,\,n+r} \leqslant 1$.

If in condition (*d*) of (2·1) we have $\sum_{\rho=1}^{\infty} a_{n,\,n+\rho} \leqslant k < 1$, for all n, then from (4, 6) we get

$$\sum_{r=0}^{\rho} \beta_{n,\,n+r}\, R_{n+r,\,n+\rho} = 1,$$

and $$R_{n,n+\rho} = 1 - \sum_{r=1}^{\rho} a_{n,\,n+r} \geqslant 1 - k\,;$$

hence $$(1 - k) \sum_{r=0}^{\rho} \beta_{n,\,n+r} \leqslant 1.$$

Hence $$\sum_{r=0}^{\infty} \beta_{n,\,n+r} \leqslant \frac{1}{1-k}. \tag{4, 7}$$

Again if in condition (*d*) of (2·1)

Let $\sum_{r=1}^{\infty} a_{n,\,n+r} = 1$, for all n, then it can be shown that

$$\sum_{r=0}^{\infty} \beta_{n,\,n+r} \text{ diverges.} \tag{4, 8}$$

Suppose that it converges; then choose k such that $\sum_{n=k}^{\infty} \beta_{n,\,n+\rho} \leqslant \epsilon$.

$$\sum_{n=0}^{\rho} \beta_{n,\,n+r}\, R_{n+r,\,n+\rho} = \sum_{r=0}^{k-1} + \sum_{r=k}^{\rho} = A + B\,;$$

$B < \sum_{k}^{\infty} \beta_{n,\,n+r}$, since $R_{n,\,n+\rho} \leqslant 1$ for all n and ρ; since by hypothesis $\sum_{0}^{\infty} a_{n,\,n+r} = 1$, $R_{n,\,n+\rho} \to 0$ as $p \to \infty$ (for every n). Now, $A < \sum_{r=0}^{k-1} R_{n+r,\,n+\rho}$. Choosing ρ sufficiently large

such that $R_{n+r,\,n+\rho} \leqslant \frac{\epsilon}{k}$, for $r = 0, 1, \cdots, k-1$.

$$A \leqslant \epsilon, \text{ for } \rho \geqslant \rho_0.$$

Hence, $\Sigma\, \beta_{n,\,n+r}\, R_{n+r,\,n+\rho} \leqslant 2\,\epsilon$ which contradicts 4·6

Hence, $$\sum_{\rho=0}^{\infty} \beta_{n,\,n+\rho} \text{ diverges.} \tag{4, 8}$$

§ 5. *Theorem of Unique Reciprocal Transformation for Null Sequences.*

We will now show that every transformation (T_0) has a unique reciprocal (T_0^{-1}) given by $\| \beta_{mn} \|$ [as defined in (4,1)], when applied to null sequences, *i.e.*,

THEOREM 4: *If* $\{x_n\}$ *be a null sequence, and* $y_n = T_0(x_n) = \sum_{\rho=0}^{\infty} a_{n,n+\rho}\, x_{n+\rho}$, *then* $x_n = T_0^{-1}(y_n) = \sum_{r=0}^{\infty} \beta_{n,n+r}\, y_{n+r}$, *and if the sequence* $\{y_n\}$ *be given, under the restriction that* $\{x_n\}$ *be a null sequence, the infinite set of equations* $\sum_{\rho=0}^{\infty} a_{n,n+\rho}\, x_{n+\rho} = y_n$, $(n = 0, 1, \cdots)$ *has utmost one solution.*

Proof:—Defining $a_{m,n}$ as in § 4,

$$T_0(x_n) = x_n - \sum_{1}^{\infty} a_{n,n+r}\, x_{n+r} = y_n. \tag{5, 1}$$

Let $\bar{x}_{n_0}$ be the upper bound of $|x_n|$ for all $n \geqslant n_0$, and let

$$\gamma_{n,n+\rho} = \sum_{r=\rho+1}^{\infty} a_{n,n+r}. \tag{5, 2}$$

Then,

$$\left.\begin{aligned} &T_0(x_n) = x_n - \sum_{1}^{\rho} a_{n,n+r}\, x_{n+r} + \theta, \gamma_{n,n+\rho}\, \bar{x}_{n+\rho} = y_n. \\ &\text{Similarly} \\ &T_0(x_{n+k}) = x_{n+k} - \sum_{k+1}^{\rho} a_{n,n+r}\, x_{n+r} + \theta_k \cdot \gamma_{n+k,n+\rho}\, \bar{x}_{n+\rho} = y_{n+k} \end{aligned}\right\}$$

for $k = 0, 1, \cdots \rho$ with $-1 \leqslant \theta_k \leqslant 1$.

$$\begin{aligned} \text{The sum } \sum_{r=0}^{\rho} \beta_{n,n+r}\, T_0(x_{n+r}) &= \Sigma\, \beta_{n,n+r} \Big(x_{n+r} - \sum_{k=r+1}^{\rho} a_{n,n+k}\, x_{n+k}\Big) \\ &\quad + (\Sigma\, \theta_k \cdot \beta_{n,n+k}\, \gamma_{n+k,n+\rho})\, \bar{x}_{n+\rho} \\ &= A + B\, \bar{x}_{n+\rho} = \sum_{r=0}^{\rho} \beta_{n,n+r}\, y_{n+r}. \end{aligned} \tag{5,3}$$

$$\text{Now,} \quad A = \beta_{nn}\, x_n + \sum_{r=1}^{\rho} x_{n+r} \Big(\beta_{n,n+r} - \sum_{k=0}^{r-1} \beta_{n,n+k}\, a_{n+k,n+r}\Big)$$

$$= x_n, \text{ by } (2, 4).$$

$$|B| < \sum_{k=0}^{\rho} \beta_{n,n+k}\, \gamma_{n+k,n+\rho} \tag{5, 4}$$

From condition (*d*) of 2·1 and 5·2 we have

$$\gamma_{n,n+\rho} = \sum_{r=1}^{\infty} a_{n,n+r} - \sum_{r=1}^{\rho} a_{n,n+r} \leqslant 1 - \sum_{r=1}^{\rho}.$$

Hence $\gamma_{n,\,n+p} \leqslant R_{n,\,n+p}$ for all n and $\rho > 0$. Therefore using (4, 6*a*) we derive from (5, 4) that

$$\sum_{k=0}^{\rho} \beta_{n,\,n+k}\,\gamma_{n+k,\,n+p} \leqslant 1 \text{ and } |\,B\,| < 1. \quad \text{Thus (5, 3) gives}$$

$$x_n + \theta\,\bar{x}_{n+p} = \sum_{0}^{\rho} \beta_{n,\,n+r}\,y_{n+r} \qquad (-1 \leqslant \theta \leqslant 1). \tag{5, 5}$$

Since $\{x_n\}$ is a null sequence, $x_{n+p} \to 0$ as $\rho \to \infty$; therefore

$$x_n = \sum_{r=0}^{\infty} \beta_{n,\,n+r}\,y_{n+r} \tag{5, 6}$$

$$i.e., \quad x_n = T_0^{-1}(y_n) = T_0^{-1}\,T_0\,(x_n).$$

To prove the uniqueness of the solution when it exists under the condition that $\{x_n\}$ be a null sequence, we proceed as follows :—

Let $\{x_n^1\}$ and $\{x_n^2\}$ be two null sequences which are solutions of the set of equations

$$x_n - \sum_{r=1}^{\infty} a_{n,n+r}\,x_{n+r} = y_n, \qquad (n = 0, 1, \cdots)$$

y_n being given. Let $Z_n = x_n^1 - x_n^2$, then by hypothesis, $\{z_n\}$ is a null sequence, and

$$z_n - \sum_{r=1}^{\infty} a_{n,n+r}\,z_{n+r} = 0.$$

Applying the result (5, 5), we have

$$z_n + \theta\,z_{n+p} = 0, \quad \text{where } -1 \leqslant \theta \leqslant 1.$$

Therefore $z_n \equiv 0$, and hence the uniqueness of the solution.

Note.—(1) Let us take a (T) for which

$$\sum_{r=1}^{\infty} a_{n,\,n+r} = 1; \quad (\text{for all } n).$$

Then if $\{x_n\}$ be a solution of $T\,(x_n) = y_n$, so is the sequence $\{L + x_n\}$ also a solution, for if L be a constant $T\,(L) = 0$. This shows the need for restricting $\{x_n\}$ to be a null sequence for the existence of a unique solution.

(2) For the same (T) as in note (1) it follows, by using (4, 8), that $\sum_{0}^{\infty} \beta_{n,\,n+r}$ diverges. We can find a null sequence $\{y_n\}$ such that $\Sigma\,\beta_{n_0,\,n_0+r}\,y_{n_0+r}$ diverges for one particular value n_0. Hence $T^{-1}(y_n)$ will, in general, have no meaning.

§ 6. *Complement to Theorem 4 and a Corollary.*

The question raised by the note in § 5 leads us to examine the necessary and sufficient conditions that the sequence $\{y_n\}$ must satisfy in order that the set of equations in x

$$\mathrm{T}(x_n) = y_n$$

has one solution $\{x_n\}$ which is a null sequence. We shall now prove

The sole condition (C) *is that the set of series* $\overset{\infty}{\Sigma}\, \beta_{mn}\, y_n$ *be* uniformly convergent *with regard to* m *for all* m $\geqslant 0$.

(*a*) *Proof that* (C) *is necessary:* Suppose that $\{x_n\}$ a null sequence is a solution of the set of equations $\mathrm{T}(x_n) = y_n$. Then from (5, 5) – (5, 6)

$$x_m + \theta\, \bar{x}_{\mathrm{A}} = \sum_{n=0}^{\mathrm{A}} \beta_{mn}\, y_n$$

$$= \overset{\infty}{\Sigma} - \sum_{n=\mathrm{A}+1}^{\infty} \qquad \text{for all } m \leqslant \mathrm{A}$$

$$= x_m - \sum_{n=\mathrm{A}+1}^{\infty}.$$

$$\therefore \quad \sum_{\mathrm{A}+1}^{\infty} \beta_{m,\,n}\, y_n = \theta\, \bar{x}_{\mathrm{A}} \qquad -1 \leqslant \theta \leqslant 1$$

(For all $m \geqslant \mathrm{A} + 1$).

$$\sum_{\mathrm{A}+1}^{\infty} \beta_{mn}\, y_n = \sum_{0}^{\infty} \beta_{mn}\, y_n = x_m, \text{ since } \beta_{mn} = 0 \text{ for } n < m.$$

Therefore,

$$\left| \sum_{n=\mathrm{A}+1}^{\infty} \beta_{mn}\, y_n \right| \leqslant \bar{x}_{\mathrm{A}}, \qquad \text{for all } m \geqslant 0$$

and

$$\left| \sum_{n=n_0}^{\infty} \beta_{mn}\, y_n \right| \leqslant x_{n_0} \leqslant \bar{x}_{\mathrm{A}}, \quad \text{for all } n_0 \geqslant \mathrm{A},$$

since $\bar{x}_{n_0}$, being the upper bound of x_n for $n \geqslant n_0$, is easily seen to be a monotonic sequence tending to zero.

Hence the condition (C) is necessary.

(*b*) *To prove that* (C) *is sufficient*: Let $\epsilon > 0$, and A_0 be so chosen that for all $\mathrm{A} \geqslant \mathrm{A}_0$, we have

$$\left| \sum_{\mathrm{A}+1}^{\infty} \beta_{mn}\, y_n \right| \leqslant \epsilon. \tag{6, 1}$$

Let x_n be defined by

$$x_n = \sum_{p=0}^{\infty} \beta_{np}\, y_p$$

when $n < A$,

$$x_n = \sum^{\rho=A} \beta_{n\rho}\, y_\rho + \sum_{A+1}^{\infty}$$

$$= \sum^{\rho=A} + \theta\epsilon$$

$$x_{n+r} = \sum^{\rho=A} \beta_{n+r,\,\rho}\, y_\rho + \theta_r \epsilon, \qquad n+r \leqslant A$$

and,

$$x_{A+k} = \sum^{\rho=A} \beta_{A+k,\,\rho}\, y_\rho + \sum_{A+1}^{\infty}$$

$$= \sum_{A+1}^{\infty} = \theta_{A+k} \cdot \epsilon, \quad \text{since } \beta_{mn} = 0 \quad (n<m).$$

Therefore,
$$x_n - \sum_{\rho=n+1}^{A} a_{n\rho}\, x_\rho = \sum_{\rho=n}^{A} \beta_{n\rho}\, y_\rho - \sum_{m=n+1}^{A} a_{nm} \cdot \Sigma\, \beta_{m\rho}\, y_\rho$$
$$+ \text{ a remainder (R).} \tag{6, 2}$$

Also,
$$|R| < \epsilon + \sum_{\rho=n+1}^{A} a_{n\rho}\, \epsilon \leqslant 2\epsilon. \quad \text{since } \sum_{n+1}^{\infty} a_{n,\rho} \leqslant 1.$$

The first two terms on the right-hand side of (6, 2)

$$= \beta_{nn}\, y_n + \sum_{\rho=n+1}^{A} (\beta_{n\rho} - \Sigma\, a_{nm}\, \beta_{m\rho})\, y_\rho$$

$$= y_n. \qquad \text{by } 2\cdot 6$$

Hence, $x_n - \sum_{n+1}^{A} a_{n\rho}\, x_\rho = y_n + 2\theta\epsilon$, and since

$$|x_{A+k}| \leqslant \epsilon, \text{ for all } k \geqslant 1$$

$$\left| \sum_{\rho=A+1}^{\infty} a_{n\rho}\, x_\rho \right| \leqslant \epsilon \sum_{A+1}^{\infty} a_{n\rho} \leqslant \epsilon. \quad \text{Therefore}$$

$$x_n - \sum_{\rho=n+1}^{\infty} a_{n\rho}\, x_\rho = y_n + 2\theta\epsilon - \sum_{\rho=A+1}^{\infty} a_{n\rho}\, x_\rho = y_n + 3\theta'\epsilon$$

i.e., $|T(x_n) - y_n| \leqslant 3\epsilon$.

Since ϵ is arbitrary, and also $|x_{A+k}| \leqslant \epsilon$, $x_n \to 0$ as $n \to \infty$, we have $T(x_n) = y_n$ which proves that (C) is sufficient. We may thus enunciate the

THEOREM: *Under Condition* (C)

$$T_1\,[T_1^{-1}(y_n)] = y_n. \tag{6, 3}$$

Cor. 1:—*If* $\{x_n\}$ *be a null sequence, and* $(T_1 T_2)(x_n) = y_n$ then

$$x_n = \Sigma\, \beta^2_{n,\,\rho} \cdot \Sigma\, \beta^1_{\rho,\,r} \cdot y_r$$

where $\| \beta_{mn}{}^1 \|$ *defines* T_1^{-1} *and* $\| \beta_{mn}{}^2 \|$ *defines* T_2^{-1} as in (4, 5).

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Proof:—Let $T_2(x_n) = z_n$, then $T_1(z_n) = y_n$. Since $\{x_n\}$ is a null sequence, it follows from Theorem 1 that $\{z_n\}$ is also a null sequence. Hence

$$z_n = \Sigma\ \beta^1{}_{np}\ y_p$$

and $$x_n = \Sigma\ \beta^2{}_{n,\ p}\ z_p = \Sigma\ \beta^2{}_{np} \cdot \Sigma\ \beta^1{}_{pr}\ y_r \text{ by Theorem IV}$$

or $$x_n = T_2{}^{-1}\,[T_1{}^{-1}\,(y_n)].$$

In general

$$\Sigma\ \beta_{np}{}^2 \cdot \Sigma\ \beta_{pr}{}^1\ y_r \neq \Sigma\ \bar{\beta}_{np}\ y_p \quad \text{with } \|\bar{\beta}_{mn}\| \text{ defined as in (4, 5),}$$

except under special conditions, *i.e.*, in general

$$T_1{}^{-1}\,(T_1{}^{-1}) \neq (T_2{}^{-1}\,T_1{}^{-1}).$$

We shall discuss in Part II the set of conditions under which the equality can be true, as well as other related questions as to the validity of several types of combinations of the transformations.