THEOREMS ON THE FUNCTIONAL LIMITS OF DERIVATIVES OF A FUNCTION AT INFINITY.

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§ 1. *Introduction.*

Hardy and Littlewood have proved the following theorem*:—

"If $f(x)$ is $O(\phi)$, and $f^n(x) = O(\psi)$ where ϕ and ψ are non-decreasing positive functions, then for $0 \leqslant r \leqslant n$

$$f^r(x) = O\left(\phi^{1-\frac{r}{n}} \cdot \psi^{\frac{r}{n}}\right).$$

In particular, if f and f^n are bounded so is f^r, for $0 \leqslant r \leqslant n$."

The object of this paper is, to go deeper into the question when f is bounded and f^n is bounded at least on one side, and examine the extent of the oscillation of $f^r(x)$ $(0 < r < n)$ at ∞. Let O_r be the oscillation of $f^r(x)$ at ∞. Then we prove the following theorems :—

(1) $$O_r \leqslant K_{r,n} \cdot O_0^{1-\frac{r}{n}} \cdot O_n^{\frac{r}{n}}$$

where $K_{r,n}$ are universal constants independent of $f, f', \cdots f^n$, but are functions of r and n only, and O_n = oscillation of $f^n(x)$ at ∞, and O_0 = oscillation of $f(x)$ at ∞, O_0 and O_n being finite $(0 < r < n)$.

(2) $$O_r \leqslant \overline{K}_{r,n}\, O_0^{1-\frac{r}{n}} \cdot (k_n)^{\frac{r}{n}} \qquad 0 < r < n,$$

where $\overline{K}_{r,n}$ are (as above) universal constants independent of $f, f', \cdots$ etc., and O_0 is as above, and $\underline{f_n(x)} = -k_n$ as $x \to \infty$ ($k_n > 0$ and finite).

A similar theorem when $\overline{f_n(x)} = k_n$ as $x \to \infty$ is given, follows easily ; also in the interesting case when $k_n = 0$ it can be easily seen that the above inequality will be still true.

(3) We deduce certain consequences like Theorems I (*a*) and II (*a*). II (*a*) is more general than a theorem of Hardy and Littlewood.†

(4) We shew that the constants $K_{r,n}$ occurring in (1) are the best possible when $n = 2$, and when $n = 3$.

(5) If $0 < \alpha < 1$, and $f^\alpha(x)$ be the α-th fractional derivative of f, then $$O_\alpha \leqslant K_\alpha \cdot O_0^{1-\alpha}\, O_1^{\alpha}$$

* *Proc. Lond. Math. Soc.*, 11 (New Series), 1913, p. 422.

† *Ibid.*, p. 426.

where O_0 = oscillation of $f(x)$ at ∞, O_1 = oscillation of $f'(x)$ at ∞ and K_α is a universal constant independent of f and f' and is a function of α only.

§ 2. *Theorem I.*

Let O_0 be the oscillation of $f(x)$ at ∞ and ($-k_2'$ and k_2) the functional limits of $f''(x)$ at ∞ ($k_2 > 0$, $k_2' > 0$). Let A_1 and $-B_1$ be the upper and lower functional limits of $f'(x)$ at ∞. We prove

$$\left(\frac{1}{k_2} + \frac{1}{k_2'}\right) \cdot \frac{A_1^2}{2} \leqslant O_0$$

$$\left(\frac{1}{k_2} + \frac{1}{k_2'}\right) \cdot \frac{B_1^2}{2} \leqslant O_0$$

and $\quad O_1 \leqslant \sqrt{2O_0 O_2}$.

Let X be a sufficiently large number such that for all $x \geqslant X$

$$(1) \quad -k_2' - \epsilon \leqslant f''(x) \leqslant k_2 + \epsilon, \quad (2) \quad |f(x_1) - f(x_2)| \leqslant O_0 + \epsilon$$

for $x_1, x_2 \geqslant X$.

Let x_0 be a number such that

$$f'(x_0) = A_1 + \epsilon' \quad (\epsilon' \text{ being small})$$

Then for $h > 0$ $\quad f'(x_0 + h) \geqslant f'(x_0) - (k_2' + \epsilon)\, h$

and $\quad f'(x_0 - h) \geqslant f'(x_0) - (k_2 + \epsilon)\, h.$

[*Note.*—We take x_0 great enough so that $x_0 - \dfrac{A_1 + \epsilon'}{k_2' + \epsilon} \geqslant X$]

Let

$$h_1 = \frac{A_1 + \epsilon'}{k_2' + \epsilon}, \quad h_2 = \frac{A_1 + \epsilon'}{k_2 + \epsilon}$$

then in the interval, $x_0 - h_2 \leqslant x \leqslant x_0 + h_1$, the curve $y = f'(x)$ will be above the broken line POP_1 indicated in Fig. 1.

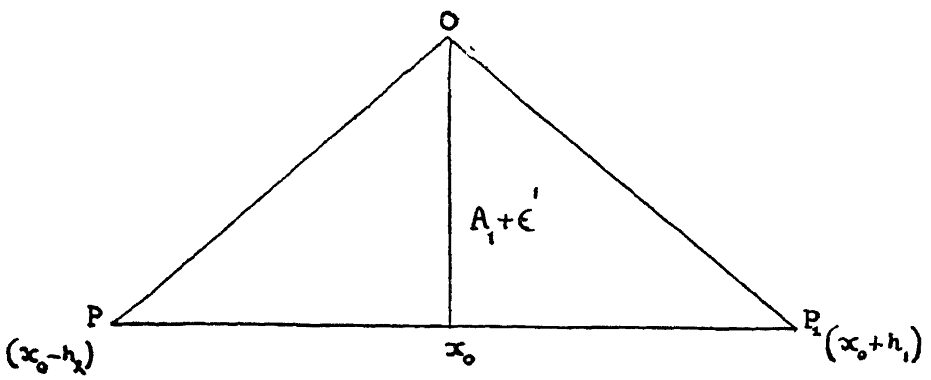

Fig. 1.

Therefore,

$$O_0 + \epsilon \geqslant f(x_0 + h_1) - f(x_0 - h_2) = \int_{-h_2}^{h_1} f'(x_0 + t)\, dt \geqslant \text{area of } \triangle \; POP_1$$

$$= \frac{(A_1 + \epsilon')^2}{2} \cdot \left\{\frac{1}{k_2 + \epsilon} + \frac{1}{k_2' + \epsilon}\right\}.$$

Since ϵ and ϵ' are arbitrarily small, we therefore have

$$\left.\begin{aligned} O_0 &\geqslant \tfrac{1}{2}\left(\frac{1}{k_2}+\frac{1}{k_2'}\right)A_1^2 \\ O_0 &\geqslant \tfrac{1}{2}\left(\frac{1}{k_2}+\frac{1}{k_2'}\right)B_1^2 \end{aligned}\right\} \tag{1}$$

Similarly,

Further,

$$2\cdot\sqrt{\frac{2\,k_2k_2'}{k_2+k_2'}\cdot O_0} \geqslant A_1+B_1 = O_1.$$

But

$$\frac{k_2+k'_2}{2} \geqslant \frac{2\,k_2k_2'}{k_2+k_2'}$$

Hence,

$$2\sqrt{O_0\,\frac{(k_2+k_2')}{2}} = \sqrt{2\cdot O_0O_2} \geqslant O_1. \tag{2}$$

We will now show by an example that (2) cannot be improved (*i.e.*, the value $\sqrt{2}$ for K_{12} cannot be replaced by anything smaller).

If non-existence of $f''(x)$ at an enumerable set be allowed then equality in (2) can actually occur.

Consider the function $\phi_0(t)$ defined as follows; $(A > 0,\ k > 0)$,

$$\begin{aligned} \phi_0(t) &= Kt \quad \text{in } \left(0, \frac{A}{K}\right) \\ &= A - Kt \quad \text{in } \left(\frac{A}{K}, \frac{3A}{K}\right) \\ &= -A + Kt \quad \text{in } \left(\frac{3A}{K}, \frac{4A}{K}\right) \end{aligned}$$

and $\phi_0\left(t+\dfrac{4A}{K}\right) = \phi_0(t)$ at all other points.

In order to remove the discontinuity of $\phi_0'(t)$ at $P_1\left(\dfrac{A}{K}\right)$ and $P_3\left(\dfrac{3A}{K}\right)$, and at corresponding points P_5, P_7, $\cdots$, etc., draw a circle to touch OP_1 and P_1P_2 just below the point P_1 (Fig. 2). We can draw it so that P_1P_1' and

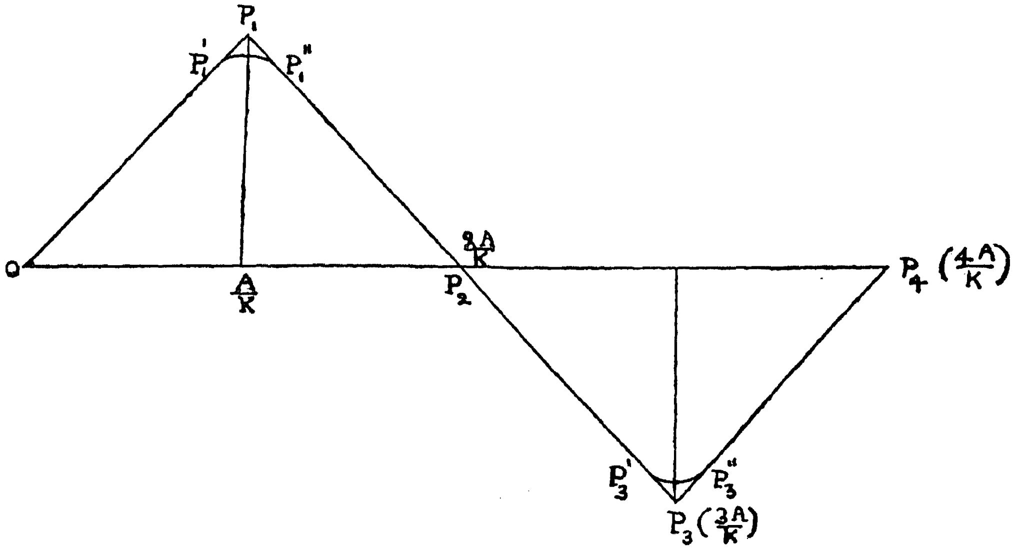

Fig. 2

P_1P_1'' are very small (${P_1}'$ and ${P_1}''$ being the points of contact of the circle with OP_1 and P_1P_2). Also draw equal circles near P_3, P_5, etc., to touch the lines P_3P_1 and P_3P_4 at corresponding points ${P_3}'{P_3}''$, etc. (*i.e.* $P_3{P_3}' = P_3{P_3}'' = P_1{P_1}' = P_1{P_1}''$, etc.).

Let the function corresponding to the curve $O{P_1}'{P_1}''{P_3}'{P_3}''{P_5}'{P_5}''$, etc. be $\phi(t)$. Then it is easy to see that (i) $\phi(t)$ is periodic in $\frac{4A}{K}$, and (ii) $\int_0^{\frac{4A}{K}} \phi(t)\, dt = 0$, (iii) since the derivative on arc ${P_1}'{P_1}''$ lies between K and $-$ K, $\phi'(t)$ lies between $-$ K and K.

Let $f(t) = \int_0^t \phi(t)\, dt$. Then $f(t)$ is periodic in $\frac{4A}{K}$ because of (ii) and maximum value of $f(t) = \int_0^{\frac{2A}{K}} \phi(t)\, dt$ = area of $\Delta\, OP_1P_2 - \epsilon$, where ϵ is very small.

$$= \frac{A^2}{K} - \epsilon \qquad (\text{in } 0 \leqslant x \leqslant \infty).$$

Minimum value of $f(t) = 0 \qquad (\text{in } 0 \leqslant x \leqslant \infty)$.

Therefore,

$$O_0 = \frac{A^2}{K} - \epsilon = \frac{A^2}{K}(1 - \epsilon_1).$$

It is easy to see that $O_1 = 2A - \epsilon' = 2A(1 - \epsilon_2)$

and $O_2 = 2\,K$.

Now,

$$\frac{O_1^2}{O_0 O_2} = \frac{4A^2(1-\epsilon_2)^2}{\frac{A^2}{K}(1-\epsilon_1).\, 2K} = 2(1 - \epsilon_3)$$

since the ϵ's are arbitrarily small we therefore see that the value $\sqrt{2}$ for K_{12} cannot be replaced by anything smaller.

Note.—If $f''(x)$ is bounded on one side only, *i.e.*, $\overline{\underset{x\to\infty}{f''(x)}} = k_2$, in our procedure for proving Theorem I we make use of the interval $(x_0 - h_2, x_0)$ only, and obtain

$$A_1 \leqslant \sqrt{2O_1k_2} \text{ and, similarly } B_1 \leqslant \sqrt{2O_1k_2}$$

and, $$A_1 + B_1 = O_1 \leqslant 2.\sqrt{2}\,\sqrt{O_1k_2}.$$

We can easily devise an example somewhat similar to the one above to show that the above result is the best possible, *i.e.*, $2^{\frac{3}{2}}$ is the best value for $\overline{K}_{12}$.

Cor. I.—Assuming Hardy and Littlewood's theorem that if f and f^n are bounded so is f^r for $0 \leqslant r \leqslant n$, we get by repeated use of (2)

$$\frac{O_0}{O_1} \geqslant \frac{1}{2}\cdot\frac{O_1}{O_2} \geqslant \frac{1}{2^2}\frac{O_2}{O_3} \cdots \geqslant \frac{1}{2^r}\frac{O_r}{O_{r+1}} \cdots \geqslant \frac{1}{2^{n-1}}\frac{O_{n-1}}{O_n}.$$

Let $d_r = \dfrac{O_r}{O_{r+1}}$; the above becomes $d_0 \geqslant \dfrac{d_1}{2} \cdots \geqslant \dfrac{d_r}{2^r} \cdots \geqslant \dfrac{d_{n-1}}{2^{n-1}}$.

Then $d_0\, d_1 \cdots d_{r-1} = \dfrac{O_0}{O_r}$ and $d_r\, d_{r+1} \cdots d_{n-1} = \dfrac{O_r}{O_n}$.

Then,

$$\frac{O_0}{O_r} \geqslant \frac{(d_{r-1})^r}{2^{\frac{(r-1)r}{2}}}$$

and

$$\frac{O_r}{O_n} \leqslant 2^{\frac{(n-r)(n-r+1)}{2}} \cdot (d_{r-1})^{n-r}$$

$$\frac{O_0^{\,n-r}\, O_n^{\,r}}{O_r^{\,n}} = \left(\frac{O_0}{O_r}\right)^{n-r} \Big/ \left(\frac{O_r}{O_n}\right)^{r} \geqslant \frac{1}{2^{\frac{r(n-r)\cdot n}{2}}}.$$

Hence, $$O_r \leqslant 2^{\frac{r(n-r)}{2}}\, O_0^{\,1-\frac{r}{n}} \cdot O_n^{\,\frac{r}{n}}.$$

The values for K_{rn} obtained here are of orders ranging between

$$e^{\alpha \cdot n} \text{ and } e^{\alpha' \cdot n^2}.$$

They are too high values for K_{rn}; even for $n = 3$ they give as will be obvious from what follows too high values.

We will presently prove two theorems in which we can obtain values for K_{rn}, much smaller than the ones given by this corollary.

§ 3. *A Theorem of Hardy on Cesàro Summability.*

We will here shew that a theorem in Cesàro summability due to Hardy can be easily deduced from Theorem I.

Let $\phi_0(x)$ be such that $\left| x \dfrac{d\phi_0}{ax} \right| \leqslant k.$

Let $\phi_r(x) = \dfrac{r}{x^r} \int_0^x \phi_0(t)\,(x-t)^{r-1}\,dt$ (r being integral positive)

and let us define D by $D = x\dfrac{d}{dx}\cdot$

Then $$D\phi_r = r(\phi_{r-1} - \phi_r)$$

and $$D^2\phi_r = r(D\phi_{r-1} - D\phi_r).$$

It is easy to prove

$$\text{(i)}\ D\phi_r = \frac{r}{x^r}\int_0^x D\phi_0\cdot(x-t)^{r-1}\,dt$$

$$\text{(ii)}\ r\,(D\phi_{r-1} - D\phi_r) = \frac{r\,(r-1)}{x^2}\cdot\int_0^x (D\phi_0 - D\phi_1)\,t\,(x-t)^{r-2}\,dt.$$

Since $D\phi_1 = \frac{1}{x}\int_0^x D\phi_0\,dx$ and since $|D\phi_0| \leqslant k$

$$|D\phi_0 - D\phi_1| \leqslant 2k.$$

By (ii) $\qquad |D^2\phi_r| \leqslant 2k. \qquad (r \geqslant 1).$

Let O_r be oscillation $\phi_r(x)$ at ∞. Then by (1) in Theorem I, we have

$$\text{upper limit of } |D\phi_r| \leqslant \sqrt{2kO_r}.$$

Hence if ϕ_r converges, $O_r = 0$

i.e., $\qquad |D\phi_r| \to 0$

i.e., $\qquad \phi_{r-1} - \phi_r \to 0 \qquad \text{as } x \to \infty$

i.e., ϕ_{r-1} converges to a finite limit.

It is easy now to deduce that $\phi_0(x)$ converges to a definite limit if ϕ_r converges to a definite limit.

§ 4. *Theorem II.*

Let oscillation of $f(x)$ at ∞ be O_0 (finite) and lower limit of $f^n(x) = -k_n$ at ∞ ($k_n > 0$) and (k_n finite).

If A_{n-1} and $-B_{n-1}$ are the upper and lower functional limits of $f^{n-1}(x)$ *at ∞ then*

$$A_{n-1} \leqslant \frac{n}{2^{\frac{1}{n}}}\cdot O_0^{\frac{1}{n}}\cdot k_n^{1-\frac{1}{n}}$$

$$B_{n-1} \leqslant \frac{n}{2^{\frac{1}{n}}}\,O_0^{\frac{1}{n}}\cdot k_n^{1-\frac{1}{n}}.$$

Let $\quad \Delta_{h_1} = f(x+h_1) - f(x)$ and $\Delta^2_{h_1,h_2} = \Delta_{h_2}\{f(x+h_1) - f(x)\}$, etc.

Then $\quad \Delta^{n-1}_{h_1h_2h_{n-1}} = \int_0^{h_1}\int^{h_2}\cdots\int_0^{h_{n-1}} f^{n-1}(x + u_1 + u_2 + \cdots + u_{n-1}).$

$$du_1,\,du_2\cdots du_{n-1}$$

As in Theorem I choose x_0 sufficiently large such that

$$f^{n-1}(x_0) = A_{n-1} + \epsilon' \quad \text{and } f^n\ (x) \geqslant -(k_n + \epsilon)$$

(assuming $\Sigma h_r \leqslant$ a constant ; all the $h_r < 0$).

Then $$f^{n-1}(x_0 + \Sigma u_r) \geqslant (A_{n-1} + \epsilon') - (k_n + \epsilon) \cdot (\Sigma u_r).$$

Hence, $$\Delta^{n-1}_{h_1 \cdots h_{n-1}} \geqslant \int_0^{h_1} \cdots \int_0^{h_{n-1}} [A_{n-1} + \epsilon' - (k_n + \epsilon) \cdot (\Sigma u_r)].\, du_1\, du_2 \cdots du_{n-1}.$$

The right-hand side

$$= h_1, h_2, h_{n-1} \left\{A_{n-1} + \epsilon' - \frac{(k_n + \epsilon)}{2} (\Sigma\, h_r)\right\} = F\,(h_1, h_2, h_{n-1}).$$

It is easy to prove that F is a maximum when

$$h_1 = h_2 = \cdots = h_{n-1} = \frac{2 \cdot (A_{n-1} + \epsilon')}{n \cdot (k_n + \epsilon)} = h.$$

and

$$F\,(h, \cdot h, \cdots) = \left(\frac{2}{k_n + \epsilon}\right)^{n-1} \cdot \left(\frac{A_{n-1} + \epsilon'}{n}\right)^n.$$

Now $\Delta_h^{n-1} f = \Delta_h^{n-2} \cdot \{f(x+h) - f(x)\} \leqslant 2^{n-2}(O_0 + \epsilon'')$ [since x_0 is very large].

Therefore

$$2^{n-2}(O_0 + \epsilon'') \geqslant \left(\frac{2}{k_n + \epsilon}\right)^{n-1} \left(\frac{A_{n-1} + \epsilon'}{n}\right)^n$$

since the ϵ's are arbitrarily small, we have

$$O_0 \geqslant \frac{2 \cdot A^n_{n-1}}{n^n \cdot k_n^{n-1}}$$

i.e.,

$$A_{n-1} \leqslant \left(\frac{n}{\frac{1}{2^n}}\right) \cdot O_0^{\frac{1}{n}} k_n^{1-\frac{1}{n}}. \tag{3}$$

a similar inequality of B_{n-1} can be established.

Therefore, $O_{n-1} \leqslant 2^{1-\frac{1}{n}} \cdot n \cdot O_0^{\frac{1}{n}} \cdot k_n^{1-\frac{1}{n}}$, and step by step calculation leads to the inqeuality for O_r, and the value for K_{rn} is given in the note below.

In case O_n is finite it can be easily deduced from above that

$$O_{n-1} \leqslant n \cdot O_0^{\frac{1}{n}} \cdot O_n^{1-\frac{1}{n}}.$$

§ 5. *Generalization of a Theorem of Hardy-Littlewood.*

(i) In case f is bounded and $f^n(x)$ is bounded on one side (at least) then all the intermediate derivatives $f^r(x)$ are bounded. This is at once obvious from Theorem II ($0 \leqslant r < n$).

(ii) In case $O_0 = 0$, *i.e.*, f is convergent and f^n is bounded on one side at least then all the intermediate derivatives $f^r(x)$ converge to zero as $x \to \infty$ $(0 \leqslant r < n)$. This is also obvious from Theorem II.

(iii) In case O_0 is finite and either lower functional limit of $f^n(x) = 0$ or upper functional limit of $f^n(x) = O$ at ∞
then also, all the intermediate derivatives $f^r(x)$ $(0 < r < n)$ tend to zero. This needs a slight argument to prove it.

Suppose the lower function limit of $f^n(x) = 0$ at ∞;
then for all large x, $f^n(x) \geqslant -\epsilon$ (ϵ being small) and, we have as in Theorem II,

$$A_{n-1} \leqslant 2^{1-\frac{1}{n}} \cdot n\, O_0^{\frac{1}{n}} \cdot \epsilon^{1-\frac{1}{n}}$$

$$\text{and} \quad B_{n-1} \leqslant 2^{1-\frac{1}{n}} \cdot n\, O_0^{\frac{1}{n}} \cdot \epsilon^{1-\frac{1}{n}}.$$

Since ϵ is arbitrarily small $A_{n-1} = B_{n-1} = 0$,

and now we can deduce easily $A_{n-2} = B_{n-2} = 0 \cdot \cdot \cdot$, etc.

Therefore the theorem follows. Similar argument when upper functional limit of $f^n(x) = 0$ is sufficient to prove the theorem.

This theorem is, as already noticed in the introduction, more general than one of Hardy and Littlewood (*ibid.*, p. 423) and is proved under more general conditions than there.

Note.—From (3), we can, by easy calculation, shew that for $0 < r < n$

$$\operatorname{Log} O_r - \left(1 - \frac{r}{n}\right)\operatorname{Log} O_0 - \frac{r}{n} \cdot \operatorname{Log} O_n \leqslant \operatorname{Log} K_{rn}$$

$$= r\left\{\frac{\operatorname{Log} r+1}{r} + \frac{\operatorname{Log} r+2}{r+1} + \cdot \cdot \cdot \frac{\operatorname{Log} n}{n-1}\right\}.$$

We obtained in Corollary I to Theorem I, for $\operatorname{Log} K_{rn}$ the value

$$\frac{r.(n-r)}{2} \cdot \operatorname{Log} 2.$$

$$\frac{\operatorname{Log} K_{rn} \text{ (in Theorem II)}}{r\dfrac{(n-r)}{2} \cdot \operatorname{Log} 2} = \text{K (a constant)}. \frac{\dfrac{\operatorname{Log} r+1}{r} + \cdots + \cdots + \dfrac{\operatorname{Log} n}{n-1}}{n-r}$$

$$= \text{utmost of order} \left(\frac{\operatorname{Log}^2 n}{n}\right)$$

Therefore (3) gives us much finer values for K_{rn} than Corollary I of Theorem I.

§ 6. *Theorem III.*

If $f^n(x)$ is bounded both ways we can, by the device of the following theorem, get much better values for $K_{r,n}$ than in Theorem II, if $n \geqslant 3$.

Theorem.—Let O_0 *and* O_n *be the oscillations of* $f(x)$ *and* $f^n(x)$ *at* ∞ *(both being finite). If* O_{n-2} *be the oscillation of* $f^{n-2}(x)$ *at* ∞, *then*

$$O_{n-2} \leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} O_n^{\frac{n-2}{n}}$$

and
$$O_{n-1} \leqslant \frac{n^{\frac{1}{2}}}{3^{\frac{n-2}{2n}}} \cdot O_0^{\frac{1}{n}} \cdot O_n^{1-\frac{1}{n}}.$$

Let $\Delta_{h_1} = f(x+h_1) - f(x-h_1)$ and $\Delta^2_{h_1,h_2} = \Delta_{h_1}\{\Delta_{h_2}\}$, etc.

Then
$$\Delta^{n-2}_{h_1 h_2 h_{n-2}} = \int_{-h_1}^{h_1}\int_{-h_2}^{h_2} \cdots \int_{-h_{n-2}}^{h_{n-2}} f^{n-2}(x+u_1+u_2+u_{n-2}) \cdot du_1 \cdots du_{n-2}.$$

Let A_{n-2} and $-B_{n-2}$ be the upper and lower functional limits of $f^{n-2}(x)$ at ∞, and let k_n and $-k_n'$ . . . . $f^n(x)$ at ∞. Then, there will be an infinity of values of x (as large as you like) for which $f^{n-2}(x) = A_{n-2} + \epsilon$ (where ϵ is arbitrarily small) and $f^{n-2}(x)$ will be a maximum at that point. (This will not necessarily happen when $A_{n-2} = 0 = - B_{n-2}$ in which case Theorem III is self-evident. So we may assume $O_{n-2} \neq 0$.) Let x_0 be such a value. Assuming Σh_r to be bounded, since for large values of x, $-(k_n'+\epsilon) \leqslant f^n(x) \leqslant (k_n+\epsilon)$, we have $f^{n-2}(x_0 + \Sigma u_r) \geqslant f^{n-2}(x_0) - \frac{(k_n'+\epsilon)}{\lfloor 2}(\Sigma u_r)^2 = (A_{n-2}+\epsilon') - \frac{(k'+\epsilon)}{\lfloor 2}(\Sigma u_r)^2$ (since x_0 is a maximum point, $f^{n-1}(x_0) = 0$).

Then
$$\Delta^{n-2}_{h_1 h_2 h_{n-2}} f(x_0) \geqslant \int_{-h_1}^{h_1} \cdots \int_{-h_{n-2}}^{h_{n-2}} \left[(A_{n-2}+\epsilon') - \frac{(k_n'+\epsilon)}{\lfloor 2}(\Sigma u_r)^2\right] \cdot du_1\, du_2 \cdots du_{n-2}.$$

The right-hand side $= F(h_1, h_2, \cdots h_{n-2})$

$$= 2^{n-2}\, h_1\, h_2 \cdot h_{n-2} \left\{(A_{n-2}+\epsilon') - \frac{(k_n'+\epsilon)}{\lfloor 3}(\Sigma h_r^2)\right\}.$$

Now F is a maximum when $h_1 = h_2 = h_{n-2} = \sqrt{\dfrac{6(A_{n-2}+\epsilon')}{n(k_n'+\epsilon)}} = h.$

Therefore, $\Delta_h^{n-2} \geqslant 2^{n-2} h^{n-2} \left\{(A_{n-2} + \epsilon') - \frac{(k_n' + \epsilon)}{\underline{|3}} \cdot (n-2)\, h^2\right\} =$

$$= 2^{n-2} \left\{\frac{6\,(A_{n-2} + \epsilon')}{n \cdot (k_n' + \epsilon)}\right\}^{\frac{n}{2}-1} \cdot 2 \cdot \left(\frac{A_{n-2} + \epsilon'}{n}\right)$$

$$= \frac{2^{\frac{3n}{2}-2} \cdot 3^{\frac{n}{2}-1}}{n^{\frac{n}{2}}} \cdot \frac{(A_{n-2} + \epsilon')^{\frac{n}{2}}}{(k_n' + \epsilon)^{\frac{n}{2}-1}}.$$

Now $\quad \Delta_h^{n-2} f(x_0) = \Delta_h^{n-3} \cdot \{f(x_0 + h) - f(x_0 - h)\} \leqslant 2^{n-3}\,(O_0 + \epsilon'').$

Hence, $\quad O_0 + \epsilon'' \geqslant \dfrac{2^{\frac{n}{2}+1} \cdot 3^{\frac{n}{2}-1}}{n^{\frac{n}{2}}} \cdot \dfrac{(A_{n-3} + \epsilon')^{\frac{n}{2}}}{(k_n' + \epsilon)^{\frac{n}{2}-1}}.$

Since the ϵ's are arbitrarily small we have

$$O_0 \geqslant \frac{2^{\frac{n}{2}+1} \cdot 3^{\frac{n}{2}-1}}{n^{\frac{n}{2}}} \cdot \frac{A_{n-2}^{\frac{n}{2}}}{(k_n')^{\frac{n}{2}-1}}$$

i.e.,
$$A_{n-2} \leqslant \frac{n}{2^{1+\frac{2}{n}} \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}}\, (k_n')^{1-\frac{2}{n}}. \qquad (4,\ a)$$

Similarly, we get

$$B_{n-2} \leqslant \frac{n}{2^{1+\frac{2}{n}} \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} \cdot (k_n)^{1-\frac{2}{n}}. \qquad (4,\ b)$$

Thus,

$$A_{n-2} + B_{n-2} = O_{n-2} \leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} \left\{\frac{k_n^{1-\frac{2}{n}} + k'^{\,1-\frac{2}{n}}_n}{2^{\frac{2}{n}}}\right\}$$

$$\leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}}\, (k_n + k_n')^{1-\frac{2}{n}}.$$

$$\leqslant \frac{n}{2 \cdot 3^{\frac{n-2}{n}}} \cdot O_0^{\frac{2}{n}} \cdot O_n^{1-\frac{2}{n}}. \qquad (4,\ c)$$

(Now $\dfrac{k_n^{1-\frac{2}{n}} + k_n'^{\,1-\frac{2}{n}}}{2} \leqslant \left(\dfrac{k_n + k_n'}{2}\right)^{1-\frac{2}{n}} = \dfrac{O_n^{1-\frac{2}{n}}}{2^{1-\frac{2}{n}}}$. Hence the inequality.)

Since by Theorem I, $O_{n-1} \leqslant \sqrt{2 O_{n-2} O_n}$, substituting for O_{n-1} in (4, *c*) we obtain

$$O_{n-1} \leqslant \left(n^{\frac{1}{2}} \Big/ 3^{\frac{n-2}{2n}}\right) \cdot O_0^{\frac{1}{n}} \cdot O_n^{1-\frac{1}{n}}. \tag{5}$$

§ 7. *Constants for* n = 3.

In case $n = 3$, we have from (4, *c*) and (5)

$$O_1 \leqslant \frac{3^{\frac{2}{3}}}{2} O_0^{\frac{2}{3}} \cdot O_3^{\frac{1}{3}}; \quad O_2 \leqslant 3^{\frac{1}{3}} O_0^{\frac{1}{3}} O_3^{\frac{2}{3}}.$$

We will give an example to shew that the constants K_{13}, K_{23}, given here are the best possible. If non-existence of $f'''(x)$ at an enumerable number of points be allowed then equality can actually occur in the above inequalities. Let $\phi(t)$ be defined as in the example under Theorem I and let

$$\phi_1(t) = \phi\left(t + \frac{A}{K}\right).$$

Then $\phi_1(t)$ will correspond to the graph (Fig. 3).

$Q_1 P_1' P_3' P_3'' P_5' Q_5$ in $0 \leqslant t \leqslant \dfrac{4A}{K}$.

The arcs Q_1P_1', $P_3'P_3''$, $P_5'Q_5$ being circular just as in the example under Theorem I.

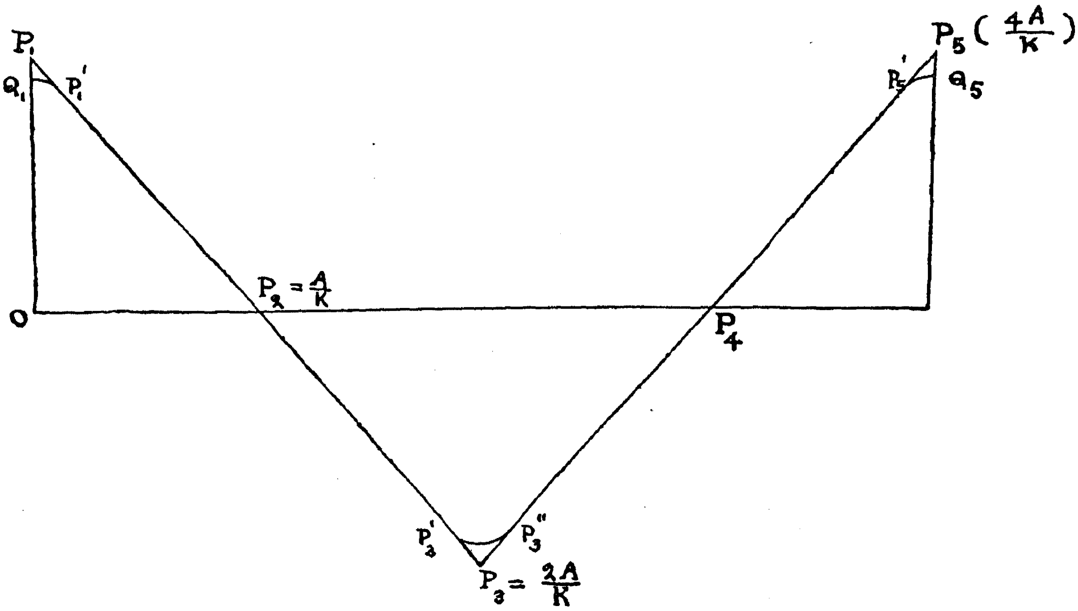

FIG. 3.

Then (i) $\phi_1(t)$ is periodic in $\frac{4A}{K}$ and (ii) $\int_0^{\frac{4A}{K}} \phi_1\, dt = 0$; (iii) oscillation of $\phi_1(t) = 2(A - \epsilon)$.

Let $\phi_2(t)$ be defined as $\int_0^t \phi_1 dt$. Then the graph of $\phi_2(t)$ will be slightly deformed parabolas as shown here (Fig. 4).

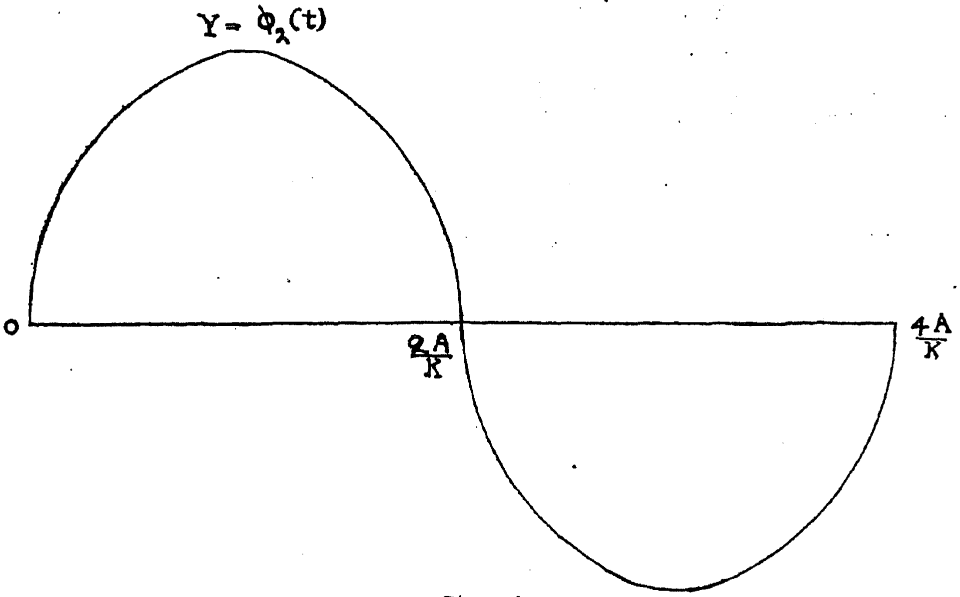

FIG. 4.

ϕ_2 is periodic in $\frac{4A}{K}$ because of (ii) and further as is obvious.

$$\phi_2\left(t + \frac{2A}{K}\right) = -\phi_2(t)\ ; \quad \text{therefore} \int_0^{\frac{4A}{K}} \phi_2\, dt = 0.$$

Therefore the function $f(t) = \int_0^t \phi_2(t)\, dt$ is periodic in $\frac{4A}{K}$

and maximum of $f(t) = \int_0^{\frac{2A}{K}} \phi_2\, dt = \frac{2}{3}\frac{A^3}{K^2} - \epsilon'' = \frac{2}{3}\frac{A^3}{K^2}(1 - \epsilon_3)$; minimum of $f(t) = 0$.

Similarly,

$$\text{maximum of } f'(t) = \int_0^{\frac{A}{K}} \phi_1(t)\, dt = \frac{A^2}{2K} - \epsilon' = \frac{A^2}{2K}(1 - \epsilon_2)$$

$$\text{minimum of } f'(t) = -\int_0^{\frac{A}{K}} \phi_1(t)\, dt = -\left(\frac{A^2}{2K} - \epsilon'\right) = -\frac{A^2}{2K}(1-\epsilon_2)$$

oscillation of $f''(t) = 2(A - \epsilon) = 2A(1 - \epsilon_1)$

oscillation of $f'''(t) = 2K$.

Therefore,

$$O_0 = \frac{2}{3}\frac{A^3}{K^2}(1-\epsilon_3)\,;\quad O_1 = \frac{A^2}{K}(1-\epsilon_2)\,;\quad O_2 = 2A(1-\epsilon_1) \text{ and } O_3 = 2K.$$

Now

$$\frac{O_1}{O_0^{\frac{2}{3}}\,O_0^{\frac{1}{3}}} = \frac{\frac{A^2}{K}(1-\epsilon_2)}{\left\{\frac{2}{3}\frac{A^3}{K^2}(1-\epsilon_3)\right\}^{\frac{2}{3}}(2K)^{\frac{1}{3}}} = \frac{3^{\frac{2}{3}}}{2}(1-\epsilon_4)$$

and,

$$\frac{O_2}{O_0^{\frac{1}{3}}\,O_2^{\frac{2}{3}}} = \frac{2A(1-\epsilon_1)}{\left\{\frac{2}{3}\frac{A^3}{K^2}(1-\epsilon_3)\right\}^{\frac{1}{3}}(2K)^{\frac{2}{3}}} = 3^{\frac{1}{3}}(1-\epsilon_5).$$

We therefore see that $K_{13} = \frac{3^{\frac{2}{3}}}{2}$ and $K_{23} = 3^{\frac{1}{3}}$ are the best possible values (when $n = 3$). We have shown the same to be true of K_{12} in § 2.

8. On Fractional Derivatives.

Let O_0 and O_1 be oscillations of $f(x)$ and $f'(x)$ at ∞

Let

$$f_0^{\alpha}(x) = \frac{1}{\overline{|1-\alpha}}\cdot\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{\alpha}}\,dt \qquad (0<\alpha<1).$$

Right-hand side =

$$\frac{1}{\overline{|1-\alpha}}\left\{\frac{f(0)}{x^{\alpha}} + \int_0^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt\right\}.$$

Since we are interested in the asymptotic behaviour of $f^{\alpha}(x)$ at ∞ the first term may be omitted. We will next shew that from our point of view, it is matter of indifference what origin we take, for

$$f_a^{\alpha}(x) = \frac{1}{\overline{|1-\alpha}}\,\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}\,dt = \frac{1}{\overline{|1-\alpha}}\left\{\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt\right\}.$$

As before the first term is of no importance and

$$\int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt - \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt = \int_0^a \cdots = O\left(\frac{1}{x^{\alpha}}\right).$$

Hence we will consider $\frac{1}{\overline{|1-\alpha}}\int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt$ as defining $f^{\alpha}(x)$.

Now

$$\overline{|1-\alpha}\cdot f_a^{\alpha}(x+a) \sim \int_a^{x+a} = \int_0^x \frac{f'(t+a)}{(x-t)^{\alpha}}\,dt = \int_0^x \frac{f'(a+x-t)}{t^{\alpha}}\,dt.$$

$$= \int_0^{x_1} + \int_{x_1}^{x} = \mathrm{I}_1 + \mathrm{I}_2 \qquad (0 < x_1 < x).$$

$$\mathrm{I}_1 = f'(a + x - \theta x_1)\,\frac{x_1^{1-\alpha}}{1-\alpha} \quad \text{and } \mathrm{I}_2 = \frac{1}{x_1^{\alpha}} \int_{x_1}^{\xi} f'(a + x - t)\,dt.$$

Let A_1 and $-\mathrm{B}_1$ be the upper and lower functional limits of $f'(x)$ at ∞. Taking 'a' fairly large we have

$$\mathrm{I}_1 + \mathrm{I}_2 \leqslant (\mathrm{A}_1 + \epsilon)\,\frac{x_1^{1-\alpha}}{1-\alpha} + \frac{\mathrm{O}_0 + \epsilon}{x_1^{\alpha}} = \mathrm{F}(x_1).$$

$\mathrm{F}(x)$ is a minimum when $x = \alpha \cdot \dfrac{\mathrm{O}_0 + \epsilon}{\mathrm{A}_1 + \epsilon}$.

Taking for x_1 the value $x_1 = \alpha \cdot \dfrac{\mathrm{O}_0 + \epsilon}{\mathrm{A}_1 + \epsilon}$.

We get, $|\overline{1-\alpha}\, f^{\alpha}(x) = \mathrm{I}_1 + \mathrm{I}_2 \leqslant$

$$\frac{(\mathrm{A}_1 + \epsilon)}{1-\alpha}\cdot\left(\alpha\cdot\frac{\mathrm{O}_0 + \epsilon}{\mathrm{A}_1 + \epsilon}\right)^{1-\alpha} + \frac{\mathrm{O}_0 + \epsilon}{\alpha^{\alpha}}\left(\frac{\mathrm{A}_1 + \epsilon}{\mathrm{O}_0 + \epsilon}\right)^{\alpha} \leqslant \frac{(\mathrm{A}_1 + \epsilon)^{\alpha}\cdot(\mathrm{O}_0 + \epsilon)^{1-\alpha}}{(1-\alpha)\cdot\alpha^{\alpha}}$$

Hence if A_α and $-\mathrm{B}_\alpha$ are the functional limits of $f^{\alpha}(x)$ at ∞ we have,

$$\mathrm{A}_\alpha \leqslant \frac{\mathrm{O}_0^{1-\alpha}\,\mathrm{A}_1^{\alpha}}{|\overline{2-\alpha}.\ \alpha^{\alpha}}.$$

$$\mathrm{B}_\alpha \leqslant \frac{\mathrm{O}_0^{1-\alpha}\,\mathrm{B}_1^{\alpha}}{|\overline{2-\alpha}\ \alpha^{\alpha}}.$$

Since

$$\mathrm{A}_1^{\alpha} + \mathrm{B}_1^{\alpha} \leqslant 2\cdot\left(\frac{\mathrm{A}_1 + \mathrm{B}_1}{2}\right)^{\alpha} = 2^{1-\alpha}\,(\mathrm{A}_1 + \mathrm{B}_1)^{\alpha}$$

we have

$$\mathrm{O}_\alpha = \mathrm{A}_\alpha + \mathrm{B}_\alpha \leqslant \frac{2^{1-\alpha}}{|\overline{2-\alpha}\cdot\alpha^{\alpha}}\cdot\ \mathrm{O}_0^{1-\alpha}\,\mathrm{O}_1^{\alpha}. \tag{6}$$

Since

$$\frac{2^{1-\alpha}}{|\overline{2-\alpha}\cdot\alpha^{\alpha}} \leqslant 2e^{\frac{1}{e}} \quad \text{we have} \quad \mathrm{O}_\alpha \leqslant 2\cdot e^{\frac{1}{e}}\cdot\mathrm{O}_0^{1-\alpha}\cdot\mathrm{O}_1^{\alpha}\cdot \qquad 0 < \alpha < 1.$$

An immediate corollary is that when either f or f' converges and the other remains bounded, all the intermediate derivatives $f^{\alpha}(x)$ tend to zero as $x \to \infty$.

Note.—We could equally well define by $f^{\alpha}(x)$ by $\dfrac{1}{|\overline{1-\alpha}} \displaystyle\int_x^{\infty} \frac{f'(t)}{(x-a)^{\alpha}}\,dt.$ (for large x), the latter integral being convergent since f is bounded and we get the same result as (6) for this definition of $f^{\alpha}(x)$.