ON LINEAR TRANSFORMATIONS OF BOUNDED SEQUENCES—II.

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PART II.

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Theorems II and III of Part I dealt completely with the way in which direct T’s combine, and the equivalence of combinations of T’s to other transformations defined by matrices. Theorem IV and its complement dealt with the simplest case of combinations of direct T’s and inverse T⁻¹’s. In this part we propose to prove some very general theorems on such combinations, in (8) and (9). (10) deals with a subclass of T designated T, whose inverses (T⁻¹) behave very much like any direct T; so that Theorems II and III of Part I characterize completely the way in which T⁻¹ or a product (T⁻¹₁ · T⁻¹₂ · · · T⁻¹ₖ) can combine with any T’s. In (8) and (9), we will notice that in all the theorems the sequence (xₙ), on which the transformations are applied will be at least a null sequence. In (11) two theorems applicable to bounded sequences, for special subclass of T’s are proved. (12) deals with the solution of an integral equation, an adaptation to the continuous variable of Theorem IV of Part I.

As in Part I the matrix defining a T will be denoted by \(a_{m,n}\) where \(a_{m,n}\) is characterized by the four conditions of (2.1), and the matrix defining T⁻¹ by \(\beta_{m,n}\). The numbering of theorems here is in continuation of the theorems of Part I.


Theorem V: Let S be a transformation defined by \(c_{m,n}\) such that \(c_{m,n} = 0\) \(n < m\), and R the product of a number of direct T’s, i.e., \(R = T₁T₂···Tₖ\),
defined by \( \| d_{m,n} \|, \| a^{1}_{m,n} \|, \| a^{2}_{m,n} \|, \ldots \| a^{k}_{m,n} \| \), where \( \| a^{k}_{m,n} \| \) defines \( T_{\rho} \) and let \( h_{m,n} \| \leq \| d_{m,n} \| \| c_{m,n} \| \). If the set of series \( \sum_{n=0}^{\infty} c_{m,n} y_{n} \) converge uniformly for all \( m \), then

(1) \( (R \cdot S)(y_{n}) \quad \quad (R \cdot S)(y_{n}) \); (2) \( \sum_{n=0}^{\infty} h_{m,n} y_{n} \) converge uniformly for all \( n \); (3) the sequence \( S(y_{n}) \) is a null one.

**Lemma 1.** \( \sum_{n=0}^{\infty} \| d_{n+1,n+\rho} \| \leq k \) for all \( n \).

**Proof:**

\[
\begin{align*}
\sum_{n=0}^{\infty} \| d_{n+1,n+\rho} \| & \leq \sum_{n=0}^{\infty} \sum_{\rho_{1}=n+\rho} d_{n+1,n+\rho_{1}} \cdot a^{1}_{n+\rho_{1},n+\rho_{1}} \cdot a^{2}_{n+\rho_{1},n+\rho_{2}} \cdot a^{3}_{n+\rho_{2},n+\rho_{3}} \cdot \ldots \cdot a^{k}_{n+\rho_{k},n+\rho} \\
& \leq \sum_{n=0}^{\infty} \sum_{\rho_{1}=n+\rho} d_{n+1,n+\rho_{1}} \cdot a^{1}_{n+\rho_{1},n+\rho_{1}} \cdot \sum_{\rho_{2}=n+\rho_{1}} a^{2}_{n+\rho_{2},n+\rho_{2}} \cdot \ldots \cdot a^{k}_{n+\rho_{k},n+\rho}.
\end{align*}
\]

Hence \( \sum_{n=0}^{\infty} \| d_{n+1,n+\rho} \| \leq k \).

Now by condition (2.1)

\[
\sum_{n=0}^{\infty} \| d_{n+1,n+\rho} \| \leq 2^{k} \cdot k \quad \quad [8.1]
\]

**Proof of Theorem V:** Let \( \epsilon > 0 \), \( N \) so chosen that for all \( A, N \) and for all \( m \geq N \)

\[
\left| \sum_{n=0}^{\infty} c_{m,n} y_{n} \right| < \epsilon.
\]

Let \( m \) be a fixed integer, and \( \Lambda_{0} \) another fixed integer such that \( \Lambda_{0} - N \) and \( \Lambda_{0} - m \).

Then

\[
\sum_{n=0}^{\infty} c_{m,n} y_{n} = \sum_{n=0}^{\infty} c_{m,\Lambda_{0}} y_{n} + \sum_{n=0}^{\infty} \sum_{n} a^{n}_{m,n} y_{n} \cdot \theta_{m,n} \epsilon.
\]

and

\[
\sum_{n=0}^{\infty} c_{m,\Lambda_{0}} y_{n} + \sum_{n=0}^{\infty} \sum_{n} a^{n}_{m,\Lambda_{0}} y_{n} \cdot \theta_{m,\Lambda_{0},n} \epsilon.
\]

and

\[
\sum_{n=0}^{\infty} c_{\Lambda_{0},n} y_{n} = \sum_{n=0}^{\infty} c_{\Lambda_{0},\Lambda_{0}+k} y_{n} \cdot \theta_{\Lambda_{0},\Lambda_{0}+k} \cdot \epsilon.
\]

where \( \cdot 1 < \theta_{\rho} < 1 \).

Hence \( |c_{\Lambda_{0},\Lambda_{0}+k}| \leq \epsilon \) for all \( k \geq 0 \) proving (3) of Theorem V.

[8.3]

Now

\[
R(\sum_{m=0}^{\infty} d_{m,n} \cdot z_{n} \cdot \sum_{\rho_{1}=n+\rho_{1}} a^{\rho_{1}}_{m,n} \cdot \sum_{n=0}^{\infty} a^{\rho_{1}+\rho_{2}} \cdot B_{1} + B_{2}.
\]

[8.4]

By (8.1) and (8.3) \( |B_{2}| \leq k \epsilon. \)
By (8.2) $B_1 = \frac{\rho = A_0}{\Sigma} \cdot d_{m, \rho} \cdot (\Sigma c_{\rho, r} y_r + \theta_{\rho} \epsilon) =$

$\frac{\rho = A_0}{\Sigma} \cdot d_{m, \rho} \cdot \frac{r = A_0}{\Sigma} c_{\rho, r} y_r + \epsilon \frac{\rho = A_0}{\Sigma} \theta_{\rho} \cdot d_{m, \rho}$

$= \frac{r = A_0}{\Sigma} (\Sigma d_{m, \rho} \cdot c_{\rho, r}) y_r + \theta' k \cdot \epsilon. \quad -1 \leq \theta' \leq 1.$

Hence $\Sigma d_{m, \rho} \cdot z_{\rho} = \frac{r = A_0}{\Sigma} h_{m, r} y_r + \theta'' 2k \cdot \epsilon \quad 1 \leq \theta'' \leq 1. \quad [8.5]$

Making $A_0 \rightarrow \infty$. We have $\Sigma d_{m, \rho} \cdot z_{\rho} = \frac{r = \infty}{\Sigma} h_{m, r} y_r \quad$ or $\quad [8.6]$

$R(z_n) = (R) [S(y_n)] = (RS) (y_n). \quad [8.7]$

From (8.5) and (8.6) we get $\left| \frac{r = \infty}{\Sigma} h_{m, r} z_r \right| \leq 2k \epsilon$ proving (2) of theorem V. \quad [8.8]

**Theorem VI**: Let $S$ be any transformation defined by $\|c_{m, n}\|$ such that $c_{m, n} = 0$ for $n < m$, and $R_1$ the product of a number of direct transformations, i.e., $R_1 = (T_{r_1} \cdot T_{r_2} \cdot T_{r_3})$ defined by $\|d_{m, n}\| = \|a_{i_m, n}\| \cdots \|a_{j_m, n}\|$, and $R_2$ the product of number of inverse transformations, i.e., $R_2 = (T_{r_1}^{-1} \cdot T_{r_2}^{-1} \cdot T_{r_3}^{-1})$ defined by $\|\beta_{m, n}\| = \|\beta'_{m, n}\| \cdots \|\beta''_{m, n}\|$. where $\|\beta'_{m, n}\|$ defines $(T_{r_1}^{-1})$. Let $b_{\rho}$ be the upper bound of $\beta_{m, n + \rho}$ for all $n$ and

and $\rho = \frac{n}{\sum} b_{\rho} = B_n$. If $\left| \frac{r = \infty}{\Sigma} c_{m, n} y_n \right| = 0 \left(\frac{1}{B_A}\right)$ for all $m$ and for all sufficiently large $A$, then

$(R_1 R_2) [S(y_n)] = (R_1 R_2 S) (y_n)$ when the latter exists.

**Proof**: Let $m_0$ be any positive integer, and $A_0$ sufficiently large and $A_0 > m_0$.

Let $z_n = S(y_n)$.

Then $z_{m_0} = \frac{r = \infty}{\sum} c_{m_0, n} y_n = \frac{r = A_0}{\sum} - 1 = \frac{r = A_0}{\sum} + \theta_{m_0} \cdot \epsilon_{A_0}$

$z_{m_0 + r} = \frac{r = A_0}{\sum} c_{m_0 + r, n} y_n + \theta_{m_0 + r} \cdot \epsilon_{A_0} \quad [8.9]$}

for $m_0 + r \leq A_0 - 1$ ; $-1 \leq \theta_{\rho} \leq 1$.

Let $\|y_{m_0}\| = \|d_{m, n}\| \cdot \|\beta_{m, n}\|$. 

Then

\[
\begin{align*}
\sum_{m=0}^{\infty} \gamma_{m,n} z_n &= \sum_{m=0}^{\infty} (\sum_{p=0}^{\infty} c_{m,p} y_p + \frac{\epsilon_{A_0}}{B_{A_0}}) \\
\gamma_{m,n} &= \sum_{m=0}^{\infty} (\sum_{p=0}^{\infty} c_{m,p} y_p) y_p + \frac{\epsilon_{A_0}}{B_{A_0}} \quad \text{for} \quad n = \gamma_{m,n} \theta_n = B_1 + B_2
\end{align*}
\]

Hence

\[
|B_2| < \frac{\epsilon_{A_0}}{B_{A_0}} \sum_{m=0}^{\infty} \gamma_{m,n}, \quad \text{and} \quad \gamma_{m,n} = \sum_{n_1=m_0}^{n} d_{m,n_1} \theta_{n_1,n}
\]

where

\[
\sum_{n=0}^{\infty} |d_{m,n}| = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lim_{n \to \infty} \beta_{m,n} = \frac{\epsilon_{A_0}}{B_{A_0}} \leq k \cdot B_{A_0} = 1 + \mu
\]

Hence

\[
|B_2| < k \cdot \frac{\epsilon_{A_0}}{B_{A_0}} \leq \epsilon_{A_0} \leq \epsilon_{A_0}
\]

where

\[
\|y_{m,n}\| = \|\gamma_{m,n}\|, \quad \|c_{m,n}\|
\]

If the series \(\sum_{n=0}^{\infty} \gamma_{m,n} y_p\) converges, i.e., \((R_1 R_2 S)(y_n)\) exists, then making \(A_0 \to \infty\) we have

\[
\sum_{n=0}^{\infty} \gamma_{m,n} z_n = \sum_{n=0}^{\infty} \gamma_{m,n} y_p \quad \text{[8.10]}
\]

i.e., \((R_1 R_2 [S(y_n)] = (R_1 R_2 S) (y_n)\) when the latter exists.

**Theorem VII:** Let \(R_1 = (T_1, T_2, T_3, T_4)\) defined by \(d_{m,n}\); \(R_2 = (T_1, T_2, T_3, T_4)\) defined by \(e_{m,n}\)

\[
\|d_{m,n}\| = \|a_{m,n}^{\sigma_1} \cdots a_{m,n}^{\sigma_{r}}\| \cdots \|a_{m,n}^{\sigma_1} \| = \|a_{m,n}^{\sigma_1} \cdots a_{m,n}^{\sigma_r}\|
\]

and \(I = (T_1^{-0} T_2^{-0} T_3^{-0})\) defined by \(\beta_{m,n}\)

Let \(S(R_1 I R_2)\) be defined by \(\gamma_{m,n}\)

\[
\|\beta_{m,n}\| = \|d_{m,n}\| \cdot \|e_{m,n}\|
\]

Let the upper bound of \(\beta_{m,n} y_p\) be \(b_{m,n}\), and \(\sum_{n=0}^{\infty} b_{m,n}\)

If \(y_n = 0 \left(\frac{1}{B_{m,n} + \delta}\right)\) in case \(B_n\) diverges, or \(y_n = 0 \ (1)\) when \(B_n\) converges, then \(\sum_{n=0}^{\infty} \gamma_{m,n} y_n\) converges uniformly for all \(m\);
(2) the terms of the product $S$ can be associated in any manner and the
resultants of the operations on the sequence $y_n$ are all equal to $S (y_n)$
for example \((T_{q_1} T_{q_2} T_{q_{r_1}}) (T_{q_{r_1} + 1} \cdot T_{q_r}) \cdot (T_{l}^{1-1} \cdot T_{l}^{-1}) (T_{l+1}^{1-1} \cdot T_{l}^{-1}) \cdot T_{r}^{1-1} \cdot T_{r}^{-1})

\((T_{s_{r_1}} \cdot T_{s_{r_2}} \cdot T_{r_{p+1}} \cdots T_{r_p}) (y_n) = S (y_n).\]

**Lemma I:** \( |\gamma_{m,m+p} | \leq \sum_r \sum_k |a_{m,m+r} | \cdot |e_{m+r+k,m+p} | \)

\( \leq \sum_k b_k \cdot \sum_r |a_{m,m+r} | \cdot |e_{m+r+k,m+p} | \) \[8.11\]

**Lemma II:** Let the terms of $R_1$ be associated in any manner.

i.e., Let $R_1 = (T_{q_1} \cdots T_{q_{r_1}}) (T_{q_{r_1} + 1} \cdots T_{q_{r_2}}) \cdots (T_{q_{r_p} + 1} \cdots T_{q_r})$

\( = R_1^1 \cdot R_3^1 \cdots R_p^1. \)

and let $R_1^1$ be defined by || $d^1_{m,n} ||$, $R_3^1$ by || $d^2_{m,n} ||$ and so on.

Let $f^1_{m,n}$ = absolute value of $d^1_{m,n}$, i.e. = $|d^1_{m,n}|$ etc.

If

\( || \bar{a}_{m,n} || = || f^1_{m,n} || \cdot || f^2_{m,n} || \cdot \cdots || f^p_{m,n} ||. \)

Then

\( \sum_{n=0}^{\infty} \bar{a}_{m,n} \leq 2^r. \) \[8.12\]

**Proof:**

Since by \[2.1\] \( \sum_{n=0}^{\infty} |a_{m,n} | \leq 2 \) for all $m$

\( \sum_{n=0}^{\infty} f_{m,n}^1 \leq 2^{r_1} \)

\( \cdots \)

\( \sum_{n=0}^{\infty} f_{m,n}^p \leq 2^{r_p - r_1} \cdot \cdots \cdot 2^{r - r_{p-1}} \)

and

\( \sum_{n=0}^{\infty} \bar{a}_{m,n} \leq 2^{r_1} \cdot 2^{r_2 - r_1} \cdots 2^{r - r_{p-1}} = 2^r. \) \[8.13\]

**Proof of (1):**

Let $A > 0$ be any fixed integer and let $B_n$ diverge; and

let $m$ be any positive integer. Let $|y_n | \leq \frac{K}{B_{n+\delta}}$ by hypothesis. Consider

now Case I when $m \leq A$, i.e., $m + q = A$.

\( \sum_{n=A}^{\infty} |\gamma_{m,n} | |y_n | = \sum_{p=q}^{\infty} |\gamma_{m,m+p} | |y_{m+p} | \leq \)

\( \sum_{p=q}^{\infty} \frac{b_k}{K} \sum_{k=0}^{\infty} |a_{m,m+r} | \cdot |e_{m+k+r,m+p} | \)

\( \frac{B^1 + 3}{B_{m+p}} \)
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\[ \sum_{k=0}^{\infty} b_k \cdot \sum_{r=0}^{\infty} |d_{m,m+r}| \cdot \sum_{\varrho=q}^{\infty} \frac{|e_{m+k+r,m+r+\varrho}|}{B_1^{1+\delta}} = K \cdot \sum_{k=0}^{\infty} b_k \cdot \varphi(k) = K \left( \sum_{k=1}^{A} + \sum_{k=A+1}^{\infty} \right) \]

\[ = K (B_1 + B_2). \]

Since by 8.1 \( \sum_{r=0}^{\infty} |e_{m,r}| \leq 2^p = K_3, \quad \sum_{\varrho=q}^{\infty} \frac{|e_{m+k+r,m+r+\varrho}|}{B_1^{1+\delta}} \leq \frac{K_3}{B_1^{1+\delta}} \]

Hence

\[ B_1 \leq K_2 \cdot \sum_{k=1}^{A} b_{A+k} \cdot \sum_{r=0}^{\infty} \frac{|d_{m,m+r}|}{B_1^{1+\delta}} \leq \frac{K_2 K_3}{B_1^{1+\delta}} \sum_{k=1}^{A} b_k \text{ since } \sum_{r=0}^{\infty} |d_{m,m+r}| \leq K_1 \text{ by (8.1)} \]

\[ = \frac{K_2 K_3}{B_1^{1+\delta}}. \]

\[ B_2 = \sum_{k=1}^{A} b_{A+k} \cdot \sum_{r=0}^{\infty} |d_{m,m+r}| \cdot \sum_{\varrho=q}^{\infty} \frac{|e_{m+r+A+k,m+r+\varrho}|}{B_1^{1+\delta}} \]

Now \( e_{m,n} = 0 \text{ and } m + q = A. \)

Hence

\[ \sum_{\varrho=q}^{\infty} \frac{|e_{m+r+A+k,m+\varrho}|}{B_1^{1+\delta}} \leq \frac{K_2}{B_1^{1+\delta}} \]

and

\[ \sum_{r=0}^{\infty} \frac{|d_{m,m+r}|}{B_1^{1+\delta}} \leq \frac{K_1}{B_1^{1+\delta}}. \]

Hence

\[ B_2 \leq \sum_{k=1}^{A} b_{A+k} \cdot \frac{K_1 K_2}{B_1^{1+\delta}} \leq K_1 K_2 \cdot \sum_{k=1}^{A} b_{A+k} \leq \frac{K_1 K_2}{B_1^{1+\delta}} \]

since

\[ \frac{1}{B_1^{1+\delta}} - \frac{1}{B_1^{n+\delta}} \geq \frac{\delta b_n}{B_1^{n+\delta}}. \]

Hence

\[ B_1 + B_2 = 0 \left( \frac{1}{B_1^{1+\delta}} \right). \]

Case II. \( m > A; \)

proceeding exactly as above we can prove for all \( m. \)

\[ \sum_{n=0}^{\infty} |\gamma_{mn} \psi_n| = \sum_{n=m}^{\infty} |\gamma_{mn} \psi_n| = 0 \left( \frac{1}{B_1^{n+\delta}} \right). \]

Hence when \( m > A \)

\[ \sum_{n=A}^{\infty} |\gamma_{mn} \psi_n| = \sum_{n=m}^{\infty} |\gamma_{mn} \psi_n| \text{ since } \gamma_{mn} = 0 \text{ for } n < m. \]
and
\[ \sum_{n=m}^{\infty} |\gamma_{mn} y_n| = 0 \left( \frac{1}{B^5_m} \right) = 0 \left( \frac{1}{B^5_A} \right) \text{ since } m > A. \]

Hence in all cases
\[ \sum_{n=A}^{\infty} |\gamma_{mn} y_n| = 0 \left( \frac{1}{B^5_A} \right). \quad \text{(8.17)} \]

Hence (1) is proved.

With regard to (2) of Theorem VII we shall prove a particular case of the
associations, the same proof being in all other cases.

To prove
\[ (T_{q_1} \cdot T_{q_1} \cdot T_{q_2} \cdot \cdots \cdot T_{q_r}) (T_{r_1+1} \cdot \cdots \cdot T_{r_t}) \cdot \cdots \cdot (T_{r_t+1} \cdot \cdots \cdot T_{r_p}) (y_n) = (T_{q_1} \cdot T_{q_r} \cdot T_{r_1+1} \cdot \cdots \cdot T_{r_t} \cdot \cdots \cdot T_{r_p}) (y_n) = S (y_n). \]

Let
\[ (T_{q_1} \cdot \cdots \cdot T_{q_r} \cdot T_{r_1+1} \cdot \cdots \cdot T_{r_p}) \]
and
\[ (T_{r_1+1} \cdot \cdots \cdot T_{r_p+1}) \]
and let
\[ d_{m,n} = \sum_{n_1}^{\infty} |d'_{m,n_1} | \cdot |d''_{n_1,n}| \]
and
\[ e_{m,n} = \sum_{n_1}^{\infty} |e'_{m,n_1} | \cdot |e''_{n_1,n}|. \]

Now
\[ \sum_{n_1}^{\infty} |d'_{m,n_1} | \cdot |d''_{n_1,n}| \cdot |e'_{n_1,n}| \cdot |e''_{n_1,n}| \cdot |f| \cdot |g| \]
\[ = \sum_{n_1}^{\infty} |d'_{m,n_1} | \cdot |d''_{n_1,n}| \cdot |e'_{n_1,n}| \cdot |e''_{n_1,n}| \cdot |f| \cdot |g| \]
\[ = \sum_{n_1}^{\infty} d_{m,n_1} \cdot \sum_{n_1}^{\infty} e_{n_1,n} \cdot |y_p| = \frac{\rho = \infty}{\rho} \sum_{n_1}^{\infty} d_{m,n_1} \cdot \sum_{n_1}^{\infty} e_{n_1,n} \cdot |y_p| \]

each as in the proof of (1) of this theorem, we prove
\[ \rho = \infty \quad \sum_{n_1}^{\infty} d_{m,n_1} \cdot \sum_{n_1}^{\infty} e_{n_1,n} \cdot |y_p| \]

Hence
\[ \sum_{n_1}^{\infty} d'_{m,n_1} \cdot \sum_{n_1}^{\infty} d''_{n_1,n} \cdot \sum_{n_1}^{\infty} e_{n_1,n} \cdot |y_p| \]
\[ = \rho = \infty \quad \sum_{n_1}^{\infty} d'_{m,n_1} \cdot \sum_{n_1}^{\infty} d''_{n_1,n} \cdot \sum_{n_1}^{\infty} e_{n_1,n} \cdot |y_p| \]
i.e., (8.18) is true.

We will state here some of the other associations that are equivalent to
S (y_n) as they will be needed later.
\begin{align*}
( T_{q_1}, T_{q_2}, \cdots, T_{q_r}, T_{r_1}^{-1} \cdots T_{r_l}^{-1} ) ( T_{l_1}^{-1} \cdots T_{l_k}^{-1} T_{t_1} \cdots T_{t_p} ) ( y_n ) & = \\
( T_{q_1}, T_{q_2}, T_{q_r}^{-1} ) ( T_{l_1}^{-1} \cdots T_{l_k}^{-1} T_{t_1} \cdots T_{t_p} ) ( y_n ) & = \\
( T_{q_1}, T_{q_r}^{-1} ) ( T_{t_1} \cdots T_{t_p} ) ( y_n ) & = \\
( T_{q_1}, T_{q_r}^{-1} ) ( T_{t_1} \cdots T_{t_p} ) ( y_n ) & = \\
( T_{q_1}, T_{q_2}, \cdots, T_{q_r}, T_{r_1}^{-1} \cdots T_{r_l}^{-1} ) ( T_{t_1} \cdots T_{t_p} ) ( y_n ) & = S ( y_n ). \tag{8.20}
\end{align*}

When \( B_n \) converges, then \( \rho = \infty \implies \Sigma \rho_{n, \rho} \leq \text{Lt. } B_\rho = K \) for all \( n \). \tag{8.21}

and since \( \rho = \infty \)
\[
\sum_{\rho = 0}^{\infty} \left| d_{n, \rho} \right| \leq K_1 \quad \text{and} \quad \sum_{\rho = 0}^{\infty} \left| e_{n, \rho} \right| \leq K_2
\]

\( \sum \left| \gamma_{m, n + \rho} \right| \leq k_1 k_2 k_3 = k_4 \). Let \( \bar{y}_n \) be an upper bound of \( |y_n| \) for \( n \geq A \);

then \( \sum_{\rho = A}^{\infty} \left| \gamma_{m, n + \rho} y_{\rho} \right| \leq k_4 \bar{y}_A \). Hence (1) of Theorem VII in this case is proved.

Because of (8.21) (2) of Theorem VII in this case is obvious.

**Theorem VIII:** Let \( \| \beta_m, n \| = \| \beta^1_{m, n} \| \cdots \| \beta^s_{m, n} \| \cdots \| \beta^k_{m, n} \| \) define \( ( T_{r_1}^{-1} \cdots T_{r_l}^{-1} ) \).

Let \( \delta_{\rho} \) be the upper bound of \( \beta_{n, n + \rho} \) for all \( n \), and \( B_n = \sum_{\rho = 0}^{\infty} \beta_{\rho} \) as in Theorem VII.

Let \( T_{\rho_1}, T_{\rho_2}, T_{\rho_3}, \cdots, T_{\rho_k} \) be direct \( T \)'s and \( \| d_{m, n} \| = \| a_{m, n} \| \cdots \| a_{r_{m, n}} \| \) define \( ( T_{\rho_1} \cdots T_{\rho_2} \cdots T_{\rho_k} ) \).

Let \( T_0 \) and \( T_0^{-1} \) be defined by \( \| a_{m, n} \| \) and \( \| \beta_{m, n} \| \).

Let \( S = ( T_{\rho_1} \cdots T_{\rho_2} \cdots T_{\rho_k} \cdots T_{\rho_l} ) \) be defined by \( \| d_{m, n} \| \cdots \| \beta_{m, n} \| = \| c_{m, n} \| \) and let \( S' = ( S T_0^{-1} ) \) be defined by \( \| c_{m, n} \| \cdots \| \beta_{m, n} \| = \| y_{m, n} \| \).

If \( y_m = 0 \left( \frac{1}{B_n} \right) \) \( \text{i.e., } ( B_n | y_n | ) \) is a null sequence, then

\[
\{ S' \left[ T_0 ( y_n ) \right] \} = \{ S' T_0 \} ( y_n ) = S ( y_n ), \text{ when the latter exists,}
\]

**Proof:** Let \( m \) and \( \rho \) be any positive integers and \( r_{n, n + \rho} \) be

\[
\sum_{r = \rho + 1}^{\infty} \left| a_{m, n + r} \right| \text{ as in (5.2)}.
\]

Let \( \bar{y}_n \) be upper bound of \( |y_n| \) for all \( n \geq n_0 \).

Let \( T_0 \) \( ( y_n ) = z_n \).

Then \( z_m = \sum_{r = 0}^{\infty} a_{m, m + r} y_{m + r} = \sum_{r = \rho + 1}^{\infty} \sum_{r = \rho + 1}^{\infty} \beta_{m, m + r} y_{m + r} + \sum_{r = \rho + 1}^{\infty} \theta_{m + k, m + r} y_{m + r} + \sum_{r = \rho + 1}^{\infty} \theta_{m + k, m + r} y_{m + r} \)

\[
\begin{align*}
&k \leq \rho, \quad z_{m + k} = \sum_{r = \rho + 1}^{\infty} a_{m + k, m + r} y_{m + r} + \theta_{m + k, m + r} y_{m + r}, \\
z_{m + r} = y_{m + r} + \theta_{m + r, m + r} y_{m + r}, \tag{8.22}
\end{align*}
\]
Consider now
\[ \sum_{r=0}^{\rho} \gamma_{m,m+r} z_{m+r} = A + B \]
where
\[ A = \sum_{r=0}^{s} \left( \sum_{r} \gamma_{m,m+r} \cdot \sum_{s} a_{r+s,m+s} \right) y_{m+s} + \gamma_{m,p} \sum_{r} \theta_{r} \cdot \gamma_{m,m+r} r_{m+r,m+p} \]
and
\[ B = \sum_{r=0}^{s} \gamma_{m,p} \cdot \sum_{r} \gamma_{m,m+r} \cdot r_{m+r,m+p} = \gamma_{m,p} \cdot B_{2} \]

Now
\[ \gamma_{m,m+r} = c_{m,m+k} \cdot \beta_{m+k,m+r} \]
\[ \therefore B_{2} \leq \sum_{r=0}^{\rho} \left( \sum_{s=0}^{k} c_{m,m+k} \cdot \beta_{m+k,m+r} \right) r_{m+r,m+p} \]
\[ = \sum_{r=0}^{\rho} \left| c_{m,m+k} \right| \cdot \sum_{s=0}^{k} \beta_{m+k,m+s} r_{m+r,m+p} \]

Just as in the course of proof of Theorem IV in (5.4) and (5.5), by (4.6) of Part I
we have
\[ \sum_{r=0}^{\rho} \beta_{m,m+r} r_{m+r,m+p} \leq 1 \text{ for all } m \text{ and all } \rho. \]

\[ \therefore B_{2} \leq \sum_{r=0}^{\rho} \left| c_{m,m+k} \right| \text{ and } c_{m,m+k} = \sum_{s=0}^{k} d_{m,m+s} \cdot \beta_{m+s,m+k}. \]

Hence
\[ \sum_{r=0}^{\rho} \left| c_{m,m+k} \right| \leq \sum_{s=0}^{k} \left| d_{m,m+s} \right| \cdot \beta_{m+s,m+k}. \]

Now \( \sum_{r=0}^{\rho} \beta_{m,m+r} \leq b_{r}. \) Hence
\[ \sum_{r=0}^{\rho} \left| d_{m,m+r} \right| \cdot b_{r} = \sum_{r=0}^{\rho} b_{n} \cdot \sum_{r=0}^{n} \left| d_{m,m+r} \right|. \]

Now by Lemma of (8.1) we have \( \sum_{r=0}^{\rho} \left| d_{m,m+r} \right| \leq K, \) a constant.

Hence \( B_{2} \leq K \cdot B_{k}. \)

Now \( B_{2} \leq \gamma_{m,p} \cdot B_{2} \leq K \cdot B_{k} \cdot \gamma_{m,p} = K \cdot B_{p} \cdot \gamma_{m,p} \leq K \cdot \epsilon_{m+p} \cdot \epsilon_{m+p} \).

since \( \gamma_{n} = 0 \left( \frac{1}{B_{n}} \right). \)

Hence
\[ \sum_{r=0}^{\rho} \gamma_{m,m+r} z_{m+r} = \sum_{s=0}^{\rho} c_{m,m+s} y_{m+s} + \theta K \cdot \epsilon_{m+p} \]

If the second series \( \sum_{s=0}^{\rho} c_{m,m+s} y_{m+s} \) converges, then
\[ \sum_{r=0}^{\rho} \gamma_{m,m+r} z_{m+r} \leq \sum_{s=0}^{\rho} c_{m,m+s} y_{m+s} \]
i.e., \( S' T_{0} (\gamma_{n}) = (S' T_{0} (\gamma_{n}) = S (\gamma_{n}) \) when the latter exists
\[ \left[ 8.27 \right] \]

\[ \left[ 8.28 \right] \]
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Theorem IX: Let $S$, $S'$ and $T_0$ be as in (8.21) and $R = (T_{q_1}, T_{q_2}, T_{q_r})$ defined by

$$
\| e_{m,n} \|.
$$

Then also $(S')[(T_0 R) (y_n)] = (S'T_0 R) (y_n)$

$$
= (SR) (y_n) \text{ if the latter exists.}
$$

Let $R (y_n) = z_n$ and $\gamma_{n_0}$ upper bound of $|y_n|$ for $n \geq n_0$

then $|z_n| = \sum_{\rho=m}^{\infty} c_{n,\rho} y_\rho \leq y_n \cdot \sum_{\rho=m}^{\infty} |c_{n,\rho}| < K y_n = 0 \left( \frac{1}{B_n} \right)$

Hence $z_n = 0 \left( \frac{1}{B_n} \right)$. \[8.29\]

Now $(T_0 R) (y_n) = T_0 (z_n)$ and

$S' [T_0 (z_n)] = (S'T_0) (z_n) = S (z_n)$

$= S [R (y_n)]$ (if the latter exists). \[8.30\]

Now the series $\sum_{\rho=\infty}^{\infty} |c_{n,\rho}| y_\rho \leq y_n \cdot K = 0 \left( \frac{1}{B_n} \right)$

since $\sum_{\rho=\infty}^{\infty} |c_{n,\rho}| < K$ by 8.1.

Hence by Theorem VI \( S \cdot R (y_n) = (SR) (y_n) \) when the latter exists. \[8.31\]

A particularly interesting case of Theorem IX is when $T_k \cdot T_k^{-1} = T_k^{-1} \cdots T_k^{-1} = 1$ unit transformation; then the theorem would be

Theorem X: If $S = (T_{q_1}, T_{q_2}, T_{q_r})$ and $y_n$ a null sequence and $R = (T_{q_1}, T_{q_2}, T_{q_r})$

then $(S' T_0 R) (y_n) = (SR) (y_n)$. \[8.32\]

§ 9. On Combinations of Inverse $(T^{-1})$ Transformations.

Most of the theorems in this section are either deductions from, or particular cases of theorems of previous section.

Theorem XI: Let $R_1 = (T_{q_1}, T_{q_2}, T_{q_r})$ defined by $\|d_{m,n}\| = \|a_{\ell_1 m,n}\| \cdots \|a_{\ell_r m,n}\|$

$$
R_2 = (T_{q_1}, T_{q_2}, \cdots, T_{q_r}) \cdots \|c_{m,n}\| = \|a_{\ell_1 m,n}\| \cdots \|a_{\ell_r m,n}\|.
$$

$T_k = (T_{k_1}^{-1}, T_{k_2}^{-1}, \cdots, T_{k_r}^{-1}) \cdots \|\beta_{m,n}\| = \|\beta_{k_1 m,n}\| \cdots \|\beta_{k_r m,n}\|$

and

$S = (IR_2)$ defined by $\|e_{n,n}\| = \|\beta_{m,n}\| \cdot |c_{n,n}|$.

If $\sum_{n=1}^{\infty} c_{n,n} y_n$ converges uniformly for all $m$, then

$$(R_1) [S (y_n)] = (R_1 S) (y_n).$$

\[9.1\]

This is a particular case of Theorem V and result follows from (8.7).
THEOREM XII: Let \( S = (T_1^{-1} T_{k}^{-1}) \) defined by \( \| \beta_{m, n} \| \).

If \( \sum_{n=0}^{\infty} \beta_{m, n} y_n \) converges uniformly for all \( m \), then

1. \( (T_{l} T_{l-1} \cdots T_{1}) (T_{l}^{-1} \cdots T_{k}^{-1}) (y_n) = (T_{l+1}^{-1} \cdots T_{k}^{-1}) (y_n) \) \( 0 \leq l \leq k \).

2. \( (T_{k} T_{k-1} \cdots T_{1}) (T_{k}^{-1} \cdots T_{k}^{-1}) (y_n) = y_n \).

3. \( (T_{l}^{-1} T_{l-1} \cdots T_{k}^{-1}) (y_n) = (T_{l}^{-1}) (T_{l-1}^{-1}) (T_{l+1}^{-1} \cdots T_{k}^{-1}) (y_n) \) \( = (T_{l}^{-1}) (T_{l-1}^{-1}) \cdots (T_{k}^{-1}) (y_n) \).

Proof:

1. is a particular case of (8.33) \([9.2]\)

2. is a particular case of (1) when \( l = k \). \([9.3]\)

3. Let \( z_m = \sum_{n=0}^{\infty} \beta_{mn} y_n \) then by (1) \( (T_{l} T_{l-1} \cdots T_{1}) (z_m) = (T_{l+1}^{-1} \cdots T_{k}^{-1}) (y_n) = z_m \).

Since \( \sum \beta_{m, n} y_n \) is uniformly convergent by (3) of theorem V it is a null sequence.

By repeated application of Theorem IV of Part I as in corollary to it

\[
\begin{align*}
\sum_{n=0}^{\infty} \beta_{m, m+r_1} y_{r_1} + \sum_{n=0}^{\infty} \beta_{m+r_1, m+r_2} y_{r_2} + \cdots + \sum_{n=0}^{\infty} \beta_{m+r_{l-1}, m} y_{r_{l-1}} + \beta \cdot y_{r_{l}} & = \\
(T_{l}^{-1} T_{l-1} \cdots T_{k}^{-1} (y_n)) & = (T_{l}^{-1} T_{l-1}^{-1}) (y_n) \end{align*}
\]

and in particular when \( l = k \) \( z_m = (T_{l}^{-1} T_{l-1}^{-1} \cdots T_{k}^{-1}) (y_n) \)

\[
\begin{align*}
\sum_{n=0}^{\infty} \beta_{m, m+r_1} y_{r_1} + \sum_{n=0}^{\infty} \beta_{m+r_1, m+r_2} y_{r_2} + \cdots + \sum_{n=0}^{\infty} \beta_{m+r_{k-1}, m} y_{r_{k-1}} + \beta \cdot y_{r_{k}} & = \\
\sum_{n=0}^{\infty} \beta_{m, m+r_{l}} y_{r_{l}} & = (T_{l}^{-1} T_{l-1}^{-1} \cdots T_{k}^{-1}) (y_n) \end{align*}
\]

A particularly interesting case of Theorem VI is the following:

THEOREM XIII: Let \( R_1 = (T_{s_1} \cdots T_{s_2} \cdots T_{s_r} \cdots T_{s_p}) \), be defined by \( \| d_{mn} \| \)

and \( R_2 = (T_{l}^{-1} \cdots T_{k}^{-1}) \) and \( S = (T_{l}^{-1} \cdots T_{l}^{-1} T_{s_1} \cdots T_{s_r}) \) defined by \( \| c_{m, n} \| \).

If \( \sum_{n=0}^{\infty} c_{m, n} y_n \) = 0 \( \left( \frac{1}{B_m} \right) \) for all \( m \)

then \( (R_1 R_2) (S (y_n)) \) \( (R_1 R_2 S (y_n)) \) when the latter exists. \([9.6]\)

THEOREM XIV: Let \( (T_{l}^{-1} T_{l}^{-1} \cdots T_{k}^{-1}) \) be defined by \( \| \beta_{m, n} \| \) then if \( \sum \beta_{m, n} \| y_n \| \) converges for all \( m \), then the terms of the product \( (T_{l}^{-1} \cdots T_{k}^{-1}) \) can be associated in any arbitrary manner, and the resultant of all the associations on \( y_n \) is equal to \( (T_{l}^{-1} T_{l}^{-1} \cdots T_{k}^{-1}) (y_n) \).

We shall take a particular association and prove it, the proof for all other associations being the same.
Let \((T_1^{-1}, T_2^{-1}, T_k^{-1})\) be defined by \(\|\beta'_{m,n}\|\).
\[
\begin{align*}
(T_{i_1 + 1}^{-1} \cdots T_{i_2}^{-1}) & \quad \cdots \quad (T_{i_2 + 1}^{-1} T_k^{-1}) \quad \cdots \quad \|\beta''_{m,n}\|.
\end{align*}
\]
To prove \(\sum \beta'_{m,n_1} \cdot \sum \beta''_{n_2,n_2} \cdot \sum \beta'''_{n_3,n_3} \cdot y_\rho = \sum \beta_m \cdot y_\rho\).

Now all the \(\beta_{m,n} \geq 0\), and \(\|\beta'_{m,n}\| \cdot \|\beta''_{m,n}\| \cdot \|\beta'''_{m,n}\| = \|\beta_{m,n}\|\).

Since \(\sum \beta_m y_\rho \) converges
\[
\begin{align*}
\rho &= \infty \sum \beta_m y_\rho = \sum \left(\sum \beta'_m \beta''_{m,p_1} \beta'''_{p_2,p_2} \cdot y_\rho\right)
\end{align*}
\]
\[
= \sum \beta'_m \beta''_{m,p_1} \beta'''_{p_2,p_2} \cdot y_\rho
\]

where \(\sum \beta'_m \beta''_{m,p_1} \beta'''_{p_2,p_2} \cdot y_\rho = \sum \beta_\rho \).

i.e., \((T_1^{-1} T_2^{-1} T_k^{-1}) (T_{i_1 + 1}^{-1} \cdots T_{i_2}^{-1} (T_{i_2 + 1}^{-1} T_k^{-1}) (y_n)
\]

\[
= (T_1^{-1} \cdots T_k^{-1}) (y_n). \quad [9.7]
\]

**Theorem XV**: \(\|\beta_{m,n}\|\) defines \((T_1^{-1} \cdots T_k^{-1})\). If \(\sum \beta_{m,n} | y_n |\) converges uniformly for all \(m\) then (1), (2), (3) of Theorem XII and the conclusion of XIV are all true in this case.

This is obvious. An interesting case of this is when \(y_n = 0 \left(\frac{1}{B_n^{1+\delta}}\right)\)

for by Theorem VII \(\sum \beta_{m,n} | y_n |\) converges uniformly if \(y_n = 0 \left(\frac{1}{B_n^{1+\delta}}\right)\).

\[9.8\]

§ 10. **On the Properties of a Particular Subclass of \((T)\).**

We notice that in the theorems proved till now, in order that a combination of inverse and direct transformations on a sequence \(\{y_n\}\) may be valid, there must be some strict restriction on \(y_n\) (for e.g., in Theorem IV, \(y_n \rightarrow 0\) as \(n \rightarrow \infty\)). In general, when a combination involves an inverse, the restriction on \(y_n\) is greater than when the combination is purely of direct \(T\)'s in which latter case the only restriction being \(y_n\) be bounded. In § 10 we deal with a subclass of \((T)\) designated by \(T_0\). The algebra of combinations involving any number of \(T\)'s and \((T)^{-1}\) is very simple and exactly like that of direct \((T)\) transformation combinations.

The matrix \(\|\beta_{m,n}\|\) of a \(T\) is characterised by the following:

\((a)\) \(\beta_{m,n} = 1\) \(\beta_{m,n} = 0, \quad n < m \) \(\beta_{m,n} < 0, \quad n > m\) and

\((d)\) \(-\sum_{\rho=1}^{\infty} \beta_{m,n+\rho} \leq k < 1\) for all \(m\); where the first three conditions are the same as in (2·1) for a \(T\) and \((d)\) is a greater restriction than \((d)\) of (2·1)

\[10.1\]
From (4.7) if \( \| \beta_{mn} \| \) defines a \((\bar{T})^{-1}\) then \( \sum_{\rho=0}^{\infty} \bar{\beta}_{m,m+\rho} \leq \frac{1}{1 - k} \). \[10.2\]

Let us designate the matrix of any \( T, \bar{T} \) or \((\bar{T})^{-1}\) by \( \| b_{mn} \| \). Then from (10.1), and (10.2) and condition (d) of (2.1) we have \( \sum_{\rho=0}^{\infty} | b_{m,m+\rho} | \leq \mathcal{K} \) (a constant) for all \( m \).

If \( V \) stands for any \( T, \bar{T} \), or \((\bar{T})^{-1}\) the most general theorem involving \( T, \bar{T} \) and \((\bar{T})^{-1}\) will be the following:

**Theorem XVI:** If \( \{x_n\} \) be \( \{\text{bounded}\} \) sequence so is \( \{V_1, V_2 \cdots V_k\} \) \( \{x_n\} \), and

\[
(V_1 V_2 V_k)(x_n) = (V_1 V_2 \cdots V_\rho) \cdots (V_{\rho+1} \cdots V_\ell) \cdots (V_{\ell+1} V_k)(x_n).
\]

Proof is immediate because it means that all \((\bar{T})^{-1}\) behave exactly like the direct transformation \( T \).

In particular it is interesting to notice the form which Theorem IV and its complement take in the case of \( \bar{T} \) and \( T^{-1} \).

**Theorem XVII:** \( \{x_n\} \) being a bounded sequence, if \( T(x_n) = y_n \), then

1. \( x_n = (T)^{-1}(y_n) \);
2. Given \( \{y_n\} \) the infinite set of equations \( T(x_n) = \sum \bar{a}_{m,\rho} x_\rho = y_m \), \( m = 0, 1, 2, \cdots n \),—under the restriction \( x_n \) be bounded, has utmost one solution.
3. The necessary and sufficient condition that the infinite set of equations \( \sum \bar{a}_{m,\rho} x_\rho = y_m \) has one solution \( \{x_n\} \) under the restriction that \( x_n \) be bounded is that \( y_n \) be bounded.

The proofs of (1), (2) and (3) follow very simply from Theorem XI. \[10.5\]

§ 11. **Two theorems applicable for a special subclass of (T).**

Let \( T_0 \) be defined by \( \| a^{0}_{m,n} \| \) and \( T_{0}^{-1} \) by \( \| \beta^{0}_{m,n} \| \).

Let \( S \) be any transformation denoted by \( \| s_{m,n} \| \) where \( s_{m,n} = 0, n < m \);

**Theorem XVIII:** If (1) \( \{x_n\} \) be a bounded sequence and (2) \( \frac{s_{m,m+\rho}}{B^{0}_{m,m+\rho}} \rightarrow 0 \) as \( \rho \rightarrow \infty \) for every \( m \) and (3) \( | s_{m,n} | \leq \mathcal{K} \) for all \( m, n \), then \( S[T_0(x_n)] = (S T_0)(x_n) \), when the latter exists.

**Proof:**

Let \( |x_n| < \mathcal{K} \). Let \( \sum_{\rho=\rho+1}^{\infty} |a^{0}_{m,n+\rho}| = - \sum_{\rho=\rho+1}^{\infty} a^{0}_{m,n+\rho} = r_{n,n+\rho} \), \( \rho \geq 1 \) and let \( T_0(x_n) = y_n \).
Then \( y_n = \sum_{\rho = 0}^{\infty} \alpha_{n, n + \rho} x_{n + \rho} = \sum_{\rho = 0}^{\infty} \beta_{n, n + \rho} + \theta K \cdot r_{n, n + \rho} \).

Similarly \( y_{n + k} = \sum_{\rho = 0}^{k} \alpha_{n, n + \rho} x_{n + \rho} + \theta_k \cdot K \cdot r_{n + k, n + \rho} \)
for \( k = 0, 1, 2, \ldots \text{ and } -1 \leq \theta_k \leq 1. \)

Then \( \sum_{\rho = 0}^{k} s_{n, n + \rho} y_{n + \rho} = \sum_{\rho = 0}^{k} s_{n, n + \rho} \cdot \sum_{\rho = 0}^{k} \alpha_{n, n + \rho} x_{n + \rho} = \sum_{\rho = 0}^{k} S_{n, n + \rho} \cdot r_{n, n + \rho} \)

\[ = A + B. \]

\[ A = \sum_{\rho = 0}^{k} \beta_{n, n + \rho} \cdot r_{n, n + \rho} \]

\[ \leq K \cdot \sum_{\rho = 0}^{k} |s_{n, n + \rho}| \leq K (\sum_{\rho = 0}^{k} l + \sum_{\rho = 0}^{k} l) \]

Choose \( k \) sufficiently large such that \( \frac{s_{n, n + \rho}}{\beta_{n, n + \rho}} \leq \epsilon \)

\[ B_2 \leq \epsilon \cdot \sum_{\rho = k}^{\infty} \beta_{n, n + \rho} \cdot r_{n, n + \rho} \leq \epsilon \cdot \sum_{\rho = 0}^{\infty} \beta_{n, n + \rho} \cdot r_{n, n + \rho} \]

Now \( r_{n, n + \rho} = \sum_{\rho = 0}^{\infty} -a_{n, n + \rho} = \sum_{\rho = 1}^{\infty} \beta_{n, n + \rho} \cdot r_{n, n + \rho} \leq R_{n, n + \rho} \) of (4.6 a)
and by (4.6a) \( \sum_{\rho = 0}^{k} \beta_{n, n + \rho} R_{n, n + \rho} = 1 \); Hence \( B_2 \leq \epsilon. \)

Now \( B_1 \leq K \cdot \sum_{\rho = 0}^{\infty} r_{n, n + \rho} \), by hypothesis (3).

Now each of the sequences \( r_{n, n + \rho}, r_{n + 1, n + \rho} \cdots r_{n + k - 1, n + \rho} \) tends to zero by condition (d) of 2.1; choosing \( \rho \) sufficiently large, \( B_1 \leq \epsilon. \)

Hence for sufficiently large \( \rho \) \(|B| \leq \epsilon (\rho)\) and we have

\[ \sum_{\rho = 0}^{k} s_{n, n + \rho} \cdot y_{n + \rho} = \sum_{\rho = 0}^{k} (S_{n, n + \rho} + \rho \cdot \rho_1 \alpha_{n, n + \rho}) x_{n + \rho} + \theta \epsilon (\rho). \]

\[ = \sum_{\rho = 0}^{k} d_{n, n + \rho} x_{n + \rho} + \theta \epsilon (\rho). \]

where \( ||d_{m, n}|| = ||s_{m, n}|| \cdot ||\alpha_{m, n}||. \)

If the latter series \( \sum_{\rho = 0}^{k} d_{n, n + \rho} x_{n + \rho} \) converges then in the limit making \( \rho \to \infty \)

\[ \sum_{\rho = 0}^{\infty} s_{n, n + \rho} \cdot y_{n + \rho} = \sum_{\rho = 0}^{\infty} d_{n, n + \rho} x_{n + \rho} \] or

\[ S \left( T_0 \left( x_n \right) \right) = \left( S \circ T_0 \right) \left( x_n \right) \text{ when the latter exists.} \]
THEOREM XIX. Let $T_1$ and $T_2$ and $(T_1 T_2)$ be defined by $\| a_{m,n} \|$, $\| b_{m,n} \|$ and $\| c_{m,n} \| = \| a_{m,n} \| \cdot \| b_{m,n} \|$. Let

\[
(1) \quad \sum_{\rho = \rho + 1}^{\infty} |a_{m,m+\rho}| = r_{a_{m},m+\rho}^\rho
\]

\[
(2) \quad \sum_{\rho = \rho + 1}^{\infty} |c_{m,m+\rho}| = r_{c_{m},m+\rho}^\rho
\]

and $T_1^{-1}$ be defined by $\| \beta_{m,n} \|$.

Then if $r_{a_{m},m+\rho}^\rho \to 0$ as $\rho \to 0$ uniformly for all $m$ (2) $\beta_{n,\rho} = 0$ as $\rho \to \infty$.

and (3) $|x_n| \leq K$, then

\[
(T_1^{-1} \cdot ((T_1 T_2) (x_n))) = T_2 (x_n). \tag{11.7}
\]

Proof: Let $(T_1 T_2) (x_n) = y_n$ and $\rho$ a positive integer

and $y_n = \sum_{\rho = \rho + 1}^{\infty} c_{n,n+\rho} x_{n+\rho} = \sum_{\rho = \rho + 1}^{\infty} c_{n,n+\rho} x_{n+\rho} + K \theta \cdot \sum_{\rho = \rho + 1}^{\infty} c_{n+\rho,n+\rho} x_{n+\rho} + K \theta \cdot \sum_{\rho = \rho + 1}^{\infty} c_{n+\rho,n+\rho} x_{n+\rho}$

Similarly $y_n = \sum_{\rho = \rho + 1}^{\infty} c_{n+k,n+\rho} x_{n+\rho} + K \theta \cdot \sum_{\rho = \rho + 1}^{\infty} c_{n+k,n+\rho} x_{n+\rho}$

for $k = 0, 1, \rho$ and $-1 \leq \theta \leq 1$.

\[
\sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} y_{n+\rho} = \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} \sum_{\rho = 0}^{\rho} c_{n+\rho,n+\rho} x_{n+\rho} + K \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} \sum_{\rho = 0}^{\rho} c_{n+\rho,n+\rho} x_{n+\rho} + K \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} \sum_{\rho = 0}^{\rho} c_{n+\rho,n+\rho} x_{n+\rho}
\]

\[
A = \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} \cdot \sum_{\rho = 0}^{\rho} c_{n+\rho,n+\rho} \cdot x_{n+\rho} + K \sum_{\rho = 0}^{\rho} b_{n,n+\rho} \cdot x_{n+\rho}
\]

\[
|B| \leq K \cdot \left( \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} r_{n,n+\rho} + \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} r_{n,n+\rho} \right) = K (B_1 + B_2) \tag{11.9}
\]

Choosing $k$ large enough such that $r_{n,n+\rho} \leq \epsilon$ for all $n$ and $\rho \geq k$

\[
B_1 \leq \epsilon \cdot \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} r_{n,n+\rho} \leq \epsilon \text{ just as in (12.3).} \tag{11.10}
\]

and $B_2 = \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho} r_{n,n+\rho} \leq K_1 \sum_{\rho = 0}^{\rho} \beta_{n,n+\rho}$

since by condition (d) of (2.1) for $T_1$ and $T_2, r_{n,n} \leq K_1$ for all $n$ and $\rho$. 

Choosing \( \rho \) sufficiently large, by condition (2) of Theorem XIX, we have

\[
\sum_{\rho = \rho - k + 1}^{\rho = \rho} \beta_{n, \rho} \leq \epsilon, \text{ i.e., } B_2 \leq \epsilon' (\rho)
\]

Hence

\[
| B | \leq \epsilon'' (\rho) \quad [11.11]
\]

and

\[
\sum_{\rho = \rho}^{\rho = \rho} \beta_{n, \rho} y_{n+\rho} = \sum_{\rho}^{\rho} b_{n, \rho} x_{n+\rho} + \theta \epsilon'' (\rho)
\]

and in the limit

\[
\sum_{\rho}^{\rho = \infty} \beta_{n, \rho} y_{n+\rho} = \sum_{\rho}^{\rho} b_{n, \rho} x_{n+\rho} \quad \text{or}
\]

\[
(T_1^{-1}) [ (T_1 T_2) (x_n) ] = T_2 (x_n). \quad [11.12]
\]

\section*{§ 12. On an Integral Equation.}

Corresponding to a \( T \) for a sequence, we define the following transformation for the continuous variable.

Let

1. \( K (x, t) > 0 \) for all \( x > 0, t \geq 0 \);
2. \( K (x, t) = 0, t < x \);
3. \( K (x, t) \) being continuous in \( (x, t) \), for all \( x > 0, t \geq 0 \);
4. \( \int_0^\infty K (x, t) \, dt \leq 1 \) for all \( x \). \quad [12.1]

Let \( u (t) \) be a given continuous function defined for \( t \geq 0 \), then the transformation \( T \) will be \( T (u) = u (t) - \int_0^t K (x, t) \, u (r) \, dr = v (t) \). \quad [12.2]

We discuss here only the existence of the inverse of this transform and show that under certain conditions \( u (t) = T^{-1} (v (t)) \). The main point of interest in the solution of the integral equation in \( u \) of (12.2) is that the range is infinite. We shall show that the solution can be given in terms of Volterra's reciprocal of \( K (x, t) \) just as in the ordinary case of finite range when \( v (t) \) satisfies a certain condition.

**Theorem XX:** (1) If \( u (x) \to 0 \) as \( x \to \infty \) and

\[
T (u) = u (x) - \int_0^\infty K (x, t) \, u (t) \, dt = v (x),
\]

then

\[
u = T^{-1} [(v (x)) = v (x) + \int_0^\infty K (x, t) \, v (t) \, dt.
\]

where

\[
[K (x, t)] = K (x, t) + K_1 (x, t) + \cdots K_n (x, t) + \cdots
\]

where

\[
[K_n (x, t)] = \int_0^t K_{n-1} (x, u) \, K (u, t) \, dt.
\]

(2) Given \( v (x) \), the equation \( T (u) = v \) will have utmost one solution under the restriction \( u (x) \to 0 \) as \( x \to \infty \).
(3) Given \( v(x) \) the necessary and sufficient condition that the equation 
\( T(u) = v \) has one solution under the restriction \( u(x) \to 0 \) as \( x \to \infty \) is that
\[
\int_0^\infty K(x, t) v(t) \, dt \text{ converges uniformly for all } x > 0.
\]

**Lemmas:**

(1) \( K_n(x,t) = 0 \) \( t < x \) (proof by mathematical induction and (2) \( o \) \[12.1\])

(2) \( K_n(x,t) \geq 0 \) for \( x \) and \( t \), obvious from definition, \[12.4\]

(3) By definition \( K_n(x,t) = \int_0^t K_{n-1}(x,u) \cdot K(u,t) \, du \). To prove
\[
K_n(x,t) = \int_0^t K_{n-1}(u,t) \cdot K(x,u) \, du,
\] \[12.5\]
suppose this is true for \( 1, 2, \cdots n-1 \), then we will prove it is true for \( n \) and hence for all \( n \geq 1 \) since it is true for \( n > 1 \).

By definition \( K_n(x,t) = \int_0^t K(u,t) \cdot K_{n-1}(x,u) \, du \)

and by hypothesis \( K_{n-1}(xu) = \int_0^t K_{n-2}(u,v) \cdot K(xu) \, du \).

Hence \( K_n(x,t) = \int_0^t K(u,t) \cdot du \cdot \int_0^v K(xu) \cdot K_{n-2}(v,u) \, dv \)

\[
= \int_0^t K(x,v) \, dv \int_0^t K(u,t) \cdot K_{n-2}(v,u) \, du
\]

\[
= \int_0^t K(x,v) \, dv \int_0^t K(u,t) \cdot K_{n-2}(v,u) \, du
\]

since \( K_n(v,u) = 0 \) \( u < v \)

\[
= \int_0^t K(x,v) \cdot K_{n-1}(vt) \, dt
\] \[12.5\]

for \( n = 1 \). \(12.5\) is obvious since \( K_n(x,t) \equiv K(x,t) \). Hence \(12.5\) is true for \( n \geq 1 \).

(4) \( \sum_0^\infty K_n(n,t) \) converges. This follows from continuity of \( K(x,t) \) i: \( x \geq 0, t > 0 \) \[12.6\]

(5) \( K(x,t) \) is continuous; this follows from uniform convergence of \( \sum K \) and continuity of \( K_n \). \[12.7\]
(6) $\bar{K}(x, t) \geq 0$ for all $x, t$; (2) $\bar{K}(x, t) = 0$ if $t < x$. (1) is obvious, (2) follows from (12.3).

(7) The reciprocal function $\bar{K}(x, t)$ satisfies two integral equations.

$$K(x, t) - K(x, t) = \int_0^t K(x, u) K(u, t) \, dt = \int_0^t \bar{K}(u, t) \cdot K(x, u) \, du.$$  

This follows immediately from definition of $K_n$ and (12.5), and term by term integration of $\Sigma K_n(x)$.

[12.8]

(8) Let $R(x, A) = 1 - \int_0^A K(x, t) \, dt$. To prove

$$\int_0^A \bar{K}(x, u) \cdot R(u, A) \, du = 1 - R(x, A).$$  

[12.9]

Proof: \[
\int_0^A \bar{K}(x, u) \left\{1 - \int_0^A K(u, t) \, dt\right\} \, du = \int_0^A \bar{K}(x, u) \, du - \int_0^A \bar{K}(x, u) \int_0^u K(u, t) \, dt \, du = a - \beta.
\]

$$\beta = \int_0^A \bar{K}(x, u) \, du \int_0^u K(u, t) \, dt, \text{ since } K(u, t) = 0 \quad t < u$$

$$= \int_0^A dt \cdot \int_0^t \bar{K}(x, u) K(u, t) \, du = \int_0^A \{\bar{K}(x, t) - K(x, t)\} \, dt \text{ by (12.8)}$$

Hence $a - \beta = \int_0^A K(x, t) \, dt = 1 - R(x, A)$

$$= \int_0^A \bar{K}(x, u) R(u, A)$$  

[12.9]

(12.9) corresponds to formula (4.7) in Part I.

Proof (1) of Theorem XX:

$$\int_0^A \bar{K}(xt) \, dt = \int_0^A \bar{K}(x, t) u(t) \, dt - \int_0^A \bar{K}(x, t) \, dt \int_0^\infty K(t, z) u(z) \, dz \text{ by (12.2)}$$

$$= B_1 - B_2.$$  

[12.10]

Let $\bar{u}(z_0)$ be the upper bound of $|u(z)|$ for $z > z_0$

$$\int_0^\infty \bar{K}(t, z) u(z) \, dz = \int_0^\infty + \int_\infty^\infty = \int_0^\infty + \theta \cdot r(t, A) \bar{u}(A_0),$$  

[12.11]

where $r(t, A) = \int_0^\infty \bar{K}(t, z) \, dz$ and $-1 \leq \theta \leq 1.$
Hence \( B_2 = \int_0^A K(x, t) \, dt \int_0^A K(tz) \, dz + \ddot{u} \cdot \theta \cdot K(x, t) \cdot r(t, A) \, dt \)
\[= C_1 + C_2. \quad [12\cdot12] \]

Now \( r(t, A) = \int_0^\infty K(t, z) \, dz - \int_0^A K(tz) \, dz \leq 1 - \int_0^A = R(t, A). \)

Hence \( |c_2| \leq \ddot{u}(A) \cdot \int_0^A K(x, t) \cdot R(t, A) \, dt = \ddot{u}(A) \{1 - R(x, A)\} \) by (12\cdot9)

and \( c_1 = \int_0^A K(x, t) \, dt \int_t^A K(tz) \cdot u(z) \, dz \) since \( K(tz) = 0, \ z < t \)
\[= \int_0^A u(z) \, dz \cdot \int_0^z K(x, t) \cdot K(t, z) \, dt \]
\[= \int_0^A (K(x, z) - K(x, z) \cdot u(z) \, dz. \quad [12\cdot13] \]

Hence \( B_1 - B_2 = \int_0^A K(x, z) \cdot u(z) + \theta' \cdot u(A) \{1 - R(x, A)\} \)
\[= \int_0^\infty K(x, z) \ddot{u}(z) + \theta'' \cdot R(x, A) \cdot u(A) + \theta' \cdot \{1 - R(x, A)\} \ddot{u}(A) \]
\[= \int_0^A K(x, z) \ddot{u}(z) \, dz + \theta''' \ddot{u}(A) \quad [12\cdot15] \]

Hence \( \int_0^A K(x, t) \cdot v(t) \, dt = \int_0^\infty K(x, z) \cdot u(z) \, dz + \theta''' \ddot{u}(A) \)
\[= u(x) - v(x) + \theta''' \ddot{u}(A). \quad [12\cdot16] \]

Since \( \ddot{u}(A) \to 0, \ A \to \infty \) by hypothesis, we have
\[u(x) = v(x) + \int_0^\infty K(x, t) \cdot v(t) \, dt = T^{-1} \{v(x)\}. \quad [12\cdot17]\]

Proofs of (2) and (3) of Theorem XX follow almost exactly by the same way as the corresponding portion of the Theorem IV of Part I and complement of Theorem IV of Part I.