ON LINEAR TRANSFORMATIONS OF BOUNDED SEQUENCES—II.

By K. S. K. Iyengar.

(From the Department of Mathematics, University of Mysore, Bangalore.)

Received July 7, 1938.

Part II.

§ 7. *General Remarks.*

Theorems II and III of Part I dealt completely with the way in which direct T's combine, and the equivalence of combinations of T's to other transformations defined by matrices. Theorem IV and its complement dealt with the simplest case of combinations of direct T's and inverse T^{-1}'s. In this part we propose to prove some very general theorems on such combinations, in (8) and (9). (10) deals with a subclass of T designated $\overline{T}$, whose inverses ($\overline{T}^{-1}$) behave very much like any direct T ; so that Theorems II and III of Part I characterize completely the way in which $\overline{T}^{-1}$ or a product $(\overline{T}_1^{-1} \cdot \overline{T}_2^{-1} \cdot T_k^{-1})$ can combine with any T's. In (8) and (9), we will notice that in all the theorems the sequence (x_n), on which the transformations are applied will be at least a null sequence. In (11) two theorems applicable to bounded sequences, for special subclass of T's are proved. (12) deals with the solution of an integral equation, an adaptation to the continuous variable of Theorem IV of Part I.

As in Part I the matrix defining a T will be denoted by $\| a_{m,n} \|$ where $a_{m,n}$ is characterized by the four conditions of (2·1), and the matrix defining T^{-1} by $\| \beta_{m,n} \|$. The numbering of theorems here is in continuation of the theorems of Part I.

§ 8. *General Theorems.*

Theorem V : Let S be a transformation defined by $\|c_{m,n}\|$ such that $c_{m,n} = 0$ $n < m$, and R the product of a number of direct T's, *i.e.*, $R = (T_1 T_2 \cdot \cdot T_k)$,

[redacted]

defined by $\| d_{m,n} \| = \| a^1_{m,n} \| \cdot \| a^2_{m,n} \| \cdots \| a^k_{m,n} \|$, where $\| a^k_{m,n} \|$ defines T_p and let $\| h_{m,n} \| = \| d_{m,n} \| \cdot \| c_{m,n} \|$. If the set of series $\sum_n^\infty c_{m,n} y_n$ converge uniformly for all m, then

(1) $(R) \cdot [S(y_n)] = (R \cdot S)(y_n)$; (2) $\sum_n^\infty h_{m,n} y_n$ converge uniformly for all m : (3) the sequence $S(y_n)$ is a null one.

Lemma 1. $$\sum_p^\infty | d_{n,n+p} | < k \quad \text{for all } n.$$

Proof : $$d_{n,n+p} = \sum \cdot \sum\sum a^1_{n,n+p_1} \cdot a^2_{n+p_1,n+p_2} \cdots a^k_{n+p_{k-1},n+p}.$$

Hence $$\sum_p^\infty | d_{n,n+p} | \leq \sum_p^\infty \left(\sum \cdots \sum | a^1_{n,n+p_1} | \cdots | a^k_{n+p_{k-1},n+p} | \right)$$
$$= \sum_{p_1}^\infty | a^1_{n,n+p_1} | \cdots \sum_{p_{k-1}}^\infty | a^{k-1}_{n+p_{k-2},n+p_{k-1}} | \cdot \sum_{p}^{\infty} | a^k_{n+p_{k-1},n+p} |.$$

Now by condition (d) of 2·1 $$\sum_{p=0}^\infty | a^r_{n,n+p} | < 2.$$

Hence $$\sum_{p=0}^\infty | d_{n,n+p} | < 2^k = k. \qquad [8\cdot 1]$$

Proof of Theorem V : Let $\epsilon > 0$, N_0 so chosen that for all $A > N_0$ and for all $m > 0$

$$\left| \sum_{n=A}^\infty c_{m,n} y_n \right| < \epsilon.$$ Let m be a fixed integer, and A_0 another fixed integer such that $A_0 > N_0$ and $A_0 > m$.

Let $z_n = \sum^\infty c_{n,p} \cdot y_p = S(y_n)$.

Then

$$z_m = \sum_n^\infty c_{m,n} y_n = \sum_n^{A_0} + \sum_{n=A_0+1}^\infty = \sum_n^{A_0} + \theta_m \epsilon.$$

and $$z_{m+r} = \sum_n^{A_0} c_{m+r,n} y_n + \theta_{m+r} \cdot \epsilon$$
for $m + r < A_0$;

and $$z_{A_0+k} = \sum_{n=A_0+k}^\infty c_{A_0+k,n} \cdot y_n = \theta_{A_0+k} \cdot \epsilon, \qquad [8\cdot 2]$$

where $-1 \leqslant \theta_p \leqslant 1$.

Hence $| z_{A_0+k} | \leqslant \epsilon$ for all $k \geqslant 0$ proving (3) of Theorem V. [8·3]

Now $$R(z_m) = \sum_p^\infty d_{m,p} z_p = \sum_p^{A_0} + \sum_{p=A_0+1}^\infty = B_1 + B_2. \qquad [8\cdot 4]$$

By (8·1) and (8·3) $| B_2 | \leqslant k\epsilon$.

By (8·2) $B_1 = \sum^{p=A_0} \cdot d_{m,p} \cdot \left(\sum^{r=A_0} c_{p,r}\, y_r + \theta_p\, \epsilon \right) =$

$$\sum^{p=A_0} d_{m,p} \cdot \sum^{r=A_0} c_{p,r} \cdot y_r + \epsilon \sum^{p=A_0} \theta_p \cdot d_{m,p}$$

$$= \sum^{r=A_0} (\Sigma d_{m,p} \cdot c_{p,r})\, y_r + \theta' \, k \cdot \epsilon. \qquad -1 \leqslant \theta' \leqslant 1.$$

$$= \sum^{r=A_0} h_{m,r} \cdot y_r + \theta' \, k\epsilon.$$

Hence $\sum^{p=\infty} d_{m,p} \cdot z_p = \sum^{r=A_0} h_{m,r}\, y_r + \theta'' \cdot 2k \cdot \epsilon \quad -1 \leqslant \theta'' \leqslant 1.$ [8·5]

Making $A_0 \rightarrow \infty$. We have

$$\sum^{p=\infty} d_{m,p}\, z_p = \sum^{r=\infty} h_{m,r}\, y_r \quad \text{or} \qquad [8\cdot 6]$$

$$R\,(z_n) = (R)\,[S\,(y_n)] = (RS)\,(y_n). \qquad [8\cdot 7]$$

From (8·5) and (8·6) we get $\left| \sum_{r=A_0+1}^{\infty} h_{m,r}\, z_r \right| \leqslant 2k\epsilon$ proving (2) of theorem V. [8·8]

Theorem VI: Let S be any transformation defined by $\|c_{m,n}\|$ such that $c_{m,n} = 0$ for $n < m$, and R_1 the product of a number of direct transformations, *i.e.*, $R_1 = (T_{s_1}\ T_{s_2} \cdot T_{s_p})$ defined by $\| d_{m,n} \| = \| a^{s_1}{}_{m,n} \| \cdots \| a^{s_p}{}_{m,n} \|$, and R_2 the product of number of inverse transformations, *i.e.*, $R_2 = (T_1^{-1}\, T_2^{-1}\, T_k^{-1})$ defined by $\| \bar\beta_{mn} \| = \| \beta^1{}_{m,n} \| \cdots \| \beta^k{}_{m,n} \|$. where $\| \beta^s{}_{m,n} \|$ defines (T_s^{-1}). Let b_p be the upper bound of $\bar\beta_{n,n+p}$ for all n and $\sum_0^{p=n} b_p = B_n$. If $\left| \sum_{n=A}^{\infty} c_{m,n}\, y_n \right| = 0 \left(\frac{1}{B_A} \right)$ for all m and for all sufficiently large A, then

$(R_1\, R_2)\,[S\,(y_n)] = (R_1\, R_2\, S)\,(y_n)$ when the latter exists.

Proof:—Let m_0 be any positive integer, and A_0 sufficiently large and $A_0 > m_0$.

Let $z_n = S\,(y_n)$.

Then $z_{m_0} = \sum^{n=\infty} c_{m_0,n} \cdot y_n = \sum^{n=A_0-1} + \sum_{A_0}^{\infty} = \sum^{n=A_0-1} + \theta_{m_0} \cdot \frac{\epsilon_{A_0}}{B_{A_0}}$

$$z_{m_0+r} = \sum^{n=A_0-1} c_{m_0+r,n}\, y_n + \theta_{m_0+r} \cdot \frac{\epsilon_{A_0}}{B_{A_0}} \qquad [8\cdot 9]$$

for $m_0 + r \leqslant A_0 - 1 \quad ; \quad -1 \leqslant \theta_p \leqslant 1.$

Let $\| \gamma_{mn} \| = \| d_{m,n} \| \cdot \| \bar\beta_{m,n} \|$.

Then $$\sum_{n=m_0}^{n=A_0-1} \gamma_{m_0,n} z_n = \sum^{n=A_0-1} \gamma_{m_0 n}\cdot\left(\sum^{p=A_0-1} c_{n,p} y_p + \theta_n \frac{\epsilon_{A_0}}{B_{A_0}}\right)$$

$$= \sum^{p=A_0-1} (\Sigma \gamma_{m_0,n} c_{n,p}) y_p + \frac{\epsilon_{A_0}}{B_{A_0}} \sum^{n=A_0-1} \gamma_{m_0 n}\,\theta_n = B_1 + B_2$$

$$|B_2| \leqslant \frac{\epsilon_{A_0}}{B_{A_0}}\cdot \sum^{n=A_0-1} |\gamma_{m_0 n}|, \text{ and } \gamma_{m_0 n} = \sum_{n_1=m_0}^{n_1=n} d_{m_0 n_1} \bar{\beta}_{n_1,n}$$

Hence $|\gamma_{m_0 n}| \leqslant \sum_{n_1=m}^{n_1=n} |d_{m\,n_1}|\, b_{n-n_1}$ and $\sum_{n=m_0}^{n=A_0-1} |\gamma_{m_0 n}| = \sum_{n=m_0}^{n=A_0-1} \sum_{n_1=m_0}^{n_1=n} |d_{m_0 n_1}|\, b_{n-n_1}$

$$= \sum_{k=0}^{A_0-1-m_0} b_k \sum_{n=m_0}^{A_0-1-\mu} |d_{m_0 n}| \leqslant k\cdot B_{A_0-1-m_0} \text{ where } \sum_{n=m_0}^{\infty} |d_{m_0 n}| \leqslant k \text{ by}(8\cdot 1)$$

Hence $$|B_2| \leqslant k.\frac{B_{A_0-1-m_0}}{B_{A_0}}\cdot\ \epsilon_{A_0} \leqslant k\cdot\ \epsilon_{A_0}.$$

Hence $$\sum^{n=A_0-1} \gamma_{m_0 n} z_n = \sum^{p=A_0-1} (\Sigma\, \gamma_{m_0 n}\cdot c_{n,p})\, y_p + \theta k \epsilon_{A_0}$$

$$= \sum^{p=A_0-1} \gamma^1_{m_0 p}\, y_p + \theta k \epsilon_{A_0},$$

where $\| \gamma^1{}_{m\,n} \| = \|\gamma_{m,n}\|\cdot \| c_{m,n} \|$.

If the series $\sum^{\infty} \gamma^1{}_{m_0 p}\, y_p$ converges, *i.e.*, $(R_1 R_2 S)\ (y_n)$ exists, then making $A_0 \to \infty$ we have

$$\sum^{n=\infty} \gamma_{m_0 n} z_n = \sum^{p=\infty} \gamma_{m_0 p}\, y_p \qquad [8\cdot 10]$$

i.e., $(R_1 R_2)\,[S\,(y_n)] = (R_1 R_2 S)\,(y_n)$ when the latter exists.

THEOREM VII: Let $R_1 = (T_{q_1} T_{q_2} T_{q_r})$ defined by $\| d_{m,n} \|$; $R_2 = (T_{s_1} T_{s_2} T_{s_p})$ defined by $\| e_{m,n} \|$

$$[\| d_{mn} \| = \| a^{a_1}_{m,n} \| \cdots \| a^{a_r}_{m,n} \|;\ \| e_{mn} \| = \| a^{s_1}_{m,n} \| \cdots \| a^{s_p}_{m,n} \|.]$$

and $I = (T_1{}^{-1} T_2{}^{-1} T_k{}^{-1})$ defined by $\| \bar{\beta}_{mn} \| = \| \beta^1{}_{m,n} \| \cdots \| \beta^k{}_{m,n} \|$.

Let $S\,(R_1\, I\, R_2)$ be defined by $\| \gamma_{mn} \| = \| d_{m,n} \| \cdot \| \bar{\beta}_{m,n} \| \cdot \| e_{m,n} \|$.

Let the upper bound of $\bar{\beta}_{n,n+p}$ be b_p, and $\sum_{p=0}^{n} b_p = B_n$.

If $y_n = 0\left(\frac{1}{B_n{}^{1+\delta}}\right)$ in case B_n diverges, or $y_n = 0\ (1)$ when B_n converges, then (1) $\sum^{n=\infty} |\gamma_{m,n} y_n|$ converges uniformly for all m;

(2) the terms of the product S can be associated in any manner and the resultants of the operations on the sequence y_n are all equal to S (y_n) [for example $(T_{q_1} T_{q_2} T_{q_{r_1}}) (T_{q_{r_1+1}} \cdot\cdot T_{q_r}) \cdot (\cdot T_1^{-1} \cdot\cdot T_l^{-1}) (T_{l+1}^{-1} \cdot\cdot T_k^{-1})$

$$(T_{s_1} \cdot T_{s_p{}^1}) (T_{s_p{}^1+1} \cdot\cdot T_{s_p}) (y_n) = S (y_n).]$$

Lemma I: $|\gamma_{m, m+p}| \leqslant \sum_r \sum_k | d_{m, m+r} | \cdot \bar{\beta}_{m+r, m+r+k} \cdot | e_{m+r+k, m+p} |$

$$\leqslant \sum_{k=0}^{k=p} b_k \cdot \sum_{r=0}^{r=p-k} | d_{m, m+r} | \cdot | e_{m+r+k, m+p} | \qquad [8 \cdot 11]$$

Lemma II : Let the terms of R_1 be associated in any manner.

i.e., Let $R_1 = (T_{q_1} \cdot\cdot T_{q_{r_1}}) \cdot (T_{q_{r_1+1}} \cdot\cdot T_{q_{r_2}}) \cdots (T_{q_{r_{p-1}+1}} \cdots T_{q_r})$

$$= R_1{}^1 \cdot R_2{}^1 \cdots R_p{}^1.$$

and let $R_1{}^1$ be defined by $\| d^1{}_{m, n} \|$, $R_2{}^1$ by $\| d^2{}_{m, n} \|$ and so on.

Let $f^1{}_{mn}$ = absolute value of $d^1{}_{mn}$, *i.e.* $= |d^1{}_{m, n}|$ etc.

If $\| \bar{d}_{mn} \| = \| f^1{}_{m, n} \| \cdot \| f^2{}_{m, n} \| \cdot\cdot \| f^p{}_{m, n} \|$.

Then $$\sum_{n=0}^{\infty} \bar{d}_{m, n} \leqslant 2^r. \qquad [8 \cdot 12)$$

Proof :— Since by $2 \cdot 1$ $\sum_{n=0}^{n=\infty} | a_{m, n} | \leqslant 2$ for all m

$$\sum_{n=0}^{n=\infty} f^1{}_{m, n} \leqslant 2^{r_1}$$

$$\sum^{n=\infty} f^2{}_{m, n} \leqslant 2^{r_2 - r_1} \cdot\cdot$$

and $$\sum^{n=\infty} f^p{}_{m, n} \leqslant 2^{r - r_{p-1}} \qquad \text{by } 8 \cdot 1.$$

Hence $$\sum_{n=0}^{\infty} \bar{d}_{m, n} \leqslant 2^{r_1} \cdot 2^{r_2 - r_1} \cdots 2^{r - r_{p-1}} = 2^r. \qquad [8 \cdot 13]$$

Proof of (1) : Let $A > 0$ be any fixed integer and let B_n diverge ; and let m be any positive integer. Let $|y_n| \leqslant \frac{K}{B_n^{1+\delta}}$ by hypothesis. Consider now Case I when $m \leqslant A$, *i.e.*, $m + q = A$.

$$\sum_{n=A}^{\infty} |\gamma_{m, n} | y_n | = \sum_{p=q}^{\infty} | \gamma_{m, m+p} | | y_{m+p} | \leqslant$$

$$K. \sum_{p=q}^{p=\infty} \frac{\sum_{k=0}^{p} b_k \cdot \Sigma | d_{m, m+r} | \cdot | e_{m+k+r, m+p} |}{B^{1+\delta}_{m+p}}$$

$$= K \sum_{k=0}^{\infty} b_k \cdot \sum_{r=0}^{\infty} |d_{m,m+r}| \cdot \sum_{p=q}^{\infty} \frac{|e_{m+k+r,m+p}|}{B^{1+\delta}_{m+p}} = K \cdot \sum_{k=0}^{\infty} b_k \cdot \phi(k) = K\left(\sum^{k=A} + \sum_{A+1}^{\infty}\right)$$

$$= K(B_1 + B_2). \qquad [8 \cdot 14]$$

Since by 8·1 $\sum_{r=0}^{r=\infty} |e_{m,r}| \leqslant 2^{p} = K_2$, $\sum_{p=q}^{\infty} \frac{|e_{m+k+r,m+p}|}{B^{1+\delta}_{m+p}} \leqslant \frac{K_2}{B^{1+\delta}_{m+q}} = \frac{K_2}{B^{1+\delta}_{A}}$

Hence

$$B_1 \leqslant K_2 \cdot \sum^{k=A} b_k \sum_{r=0}^{\infty} \frac{|d_{m,m+r}|}{B^{1+\delta}_{A}} \leqslant \frac{K_2 K_1}{B^{1+\delta}_{A}} \sum^{k=A} b_k \text{ since } \Sigma |d_{m,m+r}| \leqslant K_1 \text{ by } (8 \cdot 1)$$

$$= \frac{K_2 K_1}{B^{\delta}_{A}}. \qquad [8 \cdot 15]$$

$$B_2 = \sum_{k=1}^{\infty} b_{A+k} \cdot \sum_{r=0}^{\infty} |d_{m,m+r}| \cdot \sum_{p=q}^{\infty} \frac{|e_{m+r+A+k,m+p}|}{B^{1+\delta}_{m+p}}$$

Now $e_{m,n} = 0 \quad n < m$ and $m + q = A$.

Hence $\sum_{p=q}^{\infty} \frac{|e_{m+r+A+k,m+p}|}{B^{1+\delta}_{m+p}} = \sum_{p=r+A+k}^{\infty} \frac{|e_{m+r+A+k,m+p}|}{B^{1+\delta}_{m+p}} \leqslant \frac{K_2}{B^{1+\delta}_{m+r+A+k}}$

and $\sum_{r=0}^{\infty} \frac{|d_{m,m+r}|}{B^{1+\delta}_{m+r+A+k}} \leqslant \frac{K_1}{B^{1+\delta}_{m+A+k}}$.

Hence $B_2 \leqslant \sum_{k=1}^{\infty} b_{A+k} \cdot \frac{K_1 K_2}{B^{1+\delta}_{m+A+k}} \leqslant K_1 K_2 \cdot \sum_{k=1}^{\infty} \frac{b_{A+k}}{B^{1+\delta}_{A+k}} \leqslant \frac{K_1 K_2}{\delta B^{\delta}_{A}}$

since $\frac{1}{B^{\delta}_{n-1}} - \frac{1}{B^{\delta}_{n}} \geqslant \frac{\delta b_n}{B^{1+\delta}_{n}}$.

Hence $B_1 + B_2 = 0\left(\frac{1}{B^{\delta}_{A}}\right)$ [8·16]

Case II. $m > A$;

proceeding exactly as above we can prove for all m.

$$\sum_{n=0}^{\infty} |\gamma_{mn} y_n| = \sum_{n=m}^{\infty} |\gamma_{mn} y_n| = 0\left(\frac{1}{B^{\delta}_{m}}\right).$$

Hence when $m > A$

$$\sum_{n=A}^{\infty} |\gamma_{mn} y_n| = \sum_{n=m}^{\infty} |\gamma_{mn} y_n| \quad \text{since } \gamma_{mn} = 0 \quad n < m$$

and $\sum\limits_{n=m}^{\infty} |\gamma_{mn}\, y_n| = 0\left(\frac{1}{B_m^{\delta}}\right) = 0\left(\frac{1}{B_A^{\delta}}\right)$ since $m > A$.

Hence in all cases $\sum\limits_{n=A}^{\infty} |\gamma_{m,n}\, y_n| = 0\left(\frac{1}{B_A^{\delta}}\right)$. (8·17)

Hence (1) is proved.

With regard to (2) of Theorem VII we shall prove a particular case of the associations, the proof being the same in all other cases.

To prove $(T_{q_1}T_{q_2}T_{q_{r_1}})\,(T_{q_{r_1+1}} \cdot\cdot\, T_{q_r})\,(T_1^{-1} \cdot\cdot\, T_l^{-1})\,(T_{l+1}^{-1} \cdot\cdot\, T_k^{-1})$ [illegible]

$\times (T_{s_1} \cdot\cdot T_{s_{p^1}})\,(T_{s_{p^1+1}} \cdots T_{s_p})\,(y_n) = (T_{q_1} \cdot\cdot T_{q_r} \cdot T_1^{-1} \cdot\cdot T_k^{-1} \cdot T_{s_1} \cdot\cdot T_{s_p})\,(y_n)$

$= S\,(y_n)$ (8·18)

Let $(T_{q_1} \cdot\cdot\, T_{q_{r_1}})$ be defined by $\| d'_{m,n} \|$, $(T_{q_{r_1+1}} \cdot\cdot\, T_{q_r})$ by $\| d''_{m,n} \|$

$(T_1^{-1} \cdot\cdot\, T_l^{-1})$ by $\| \bar{\beta}'_{m,n} \|$, $(T_{l+1}^{-1} \cdot\cdot\, T_k^{-1})$ by $\| \bar{\beta}''_{m,n} \|$

$(T_{s_1} \cdot\cdot\, T_{s_{p^1}})$ by $\| e'_{m,n} \|$, $(T_{s_{p^1+1}} \cdot\cdot\, T_{s_p})$ by $\| e''_{m,n} \|$

and let $\bar{d}_{m,n} = \sum\limits_{n_1} | d'_{m,n_1} | \cdot | d''_{n_1,n} |$

and $\bar{e}_{m,n} = \sum | e'_{m,n} | \cdot | e''_{m,n} |$.

Now $\sum\limits_{n_1} | d'_{m,n_1} | \cdot \sum\limits_{n_2} | d''_{n_1,n_2} | \cdot \sum\limits_{n_3} \bar{\beta}'_{n_2,n_3} \cdot \sum\limits_{n_4} \bar{\beta}''_{n_3,n_4} \cdot \sum\limits_{n_5} | e'_{n_4,n_5} | \cdot \sum\limits_{p} | e''_{n_5,p} |\,| y_p |$

$= \sum \big(\sum\limits_{n_1} | d'_{m,n_1} |\, | d''_{n_1 n_2} |\big) \cdot \sum \bar{\beta}_{n_2,n_4} \cdot \sum \big(\sum\limits_{n_5} | e'_{n_4,n_5} | \cdot | e''_{n_5,p} |\big)\, | y_p |$

$= \sum \bar{d}_{m,n_1} \cdot \sum \bar{\beta}_{n_1,n_2} \sum \bar{e}_{n_2 p}\, | y_\rho | = \sum\limits_{\rho}^{\rho=\infty} (\sum\sum \bar{d}_{m,n_1} \cdot \bar{\beta}_{n_1,n_2} \cdot \bar{e}_{n_2,p})$ [illegible]

Since by (8·13), $\sum\limits_{p=0}^{\infty} \bar{d}_{n,p} \leqslant K$ and $\sum \bar{e}_{n,p} \leqslant K^1$

exactly as in the proof of (1) of this theorem, we prove $\sum\limits^{p=\infty} (\sum\sum \bar{d}_{m,n_1} \cdot \bar{\beta}_{n_1,n_2} \cdot \bar{e}_{n_2 p})\, | y_\rho |$ converges.

Hence

$\sum\limits^{n_1=\infty} d'_{m,n_1} \sum\limits^{n_2=\infty} d''_{n_1 n_2} \cdot \sum\limits^{n_3=\infty} \bar{\beta}'_{n_2,n_3} \cdot \sum\limits^{n_4=\infty} \bar{\beta}''_{n_3,n_4} \cdot \sum\limits^{n_5=\infty} e'_{n_4,n_5} \cdot \sum\limits^{p} e''_{n_5,p}\, y_p$

$= \sum\limits^{p=\infty} (\sum\sum \cdot\cdot \sum \cdot\cdot \sum\sum d'_{m,n_1} \cdot d''_{n_1,n_2}\, \bar{\beta}'_{n_2,n_3}\, \bar{\beta}''_{n_3,n_4}\, e'_{n_4,n_5}\, e''_{n_5,p})\, y_p$ [illegible]

i.e., (8·18) is true. (8·19)

We will state here some of the other associations that are equivalent to $S\,(y_n)$ as they will be needed later.

$$(T_{q_1} T_{q_2} \cdots T_{q_r} T_1^{-1} \cdot\cdot T_l^{-1}) (T_{l+1}^{-1} \cdot\cdot T_k^{-1} T_{s_1} \cdot\cdot T_{s_p}) (y_n) =$$
$$(T_{q_1} T_{q_2} \cdot T_{q_r}) \cdot (T_1^{-1} \cdot\cdot T_k^{-1} T_{s_1} \cdot\cdot T_{s_p}) (y_n) =$$
$$(T_{q_1} \cdot\cdot T_{q_r} \; T_1^{-1} \cdot T_k^{-1}) (T_{s_1} \cdot T_{s_p}) (y_n) =$$
$$(T_{q_1} \cdot T_{q_r}) (T_1^{-1} T_k^{-1}) \cdot (T_{s_1} \cdot T_{s_p}) (y_n) =$$
$$(T_{q_1} T_{q_2} \cdot\cdot T_{q_r} \cdot T_1^{-1} \cdots T_k^{-1} \cdot T_{s_1} \cdot T_{s_p}) (y_n) = S (y_n). \qquad [8\cdot 20]$$

When B_n converges, then $\sum_{p=0}^{p=\infty} \bar\beta_{n,n+p} \leqslant \underset{p \to \infty}{\text{Lt}} B_p = K$ for all n. [8·21]

and since $\sum^{p=\infty} |d_{n,p}| \leqslant K_1$ and $\sum^{p=\infty} |e_{np}| \leqslant K_2$

$\sum^{p=\infty} |\gamma_{n,n+p}| \leqslant k\, k_1\, k_2 = k_4$. Let $\bar{y}_A$ be upper bound of $|y_n|$ for $n \geqslant A$;

then $\sum_{p=A}^{\infty} |\gamma_{n,p} y_p| \leqslant k_4 \bar{y}_A$. Hence (1) of Theorem VII in this case is proved.

Because of (8·21) (2) of Theorem VII in this case is obvious.

THEOREM VIII: Let $\| \bar\beta_{m,n} \| = \| \beta^1_{m,n} \| \cdot \| \beta^2_{m,n} \| \cdots\cdots \| \beta^k_{m,n} \|$ define $(T_1^{-1} T_2^{-1} \cdot T_k^{-1})$.

Let b_ρ be the upper bound of $\bar\beta_{n,n+\rho}$ for all n, and $B_n = \sum_{\rho=0}^{n} b_\rho$ as in Theorem VII.

Let T_{ρ_1}, T_{ρ_2}, T_{ρ_r} be r direct T's and $\| d_{m,n} \| = \| a^{\rho_1}{}_{m,n} \| \cdots\cdot \| a^{\rho_r}{}_{m,n} \|$ define $(T_{\rho_1} \cdot T_{\rho_2} \cdot T_{\rho_r})$.

Let T_0 and T_0^{-1} be defined by $\| a^0{}_{m,n} \|$ and $\| \beta^0{}_{m,n} \|$.

Let $S = (T_{\rho_1} T_{\rho_2} \cdots T_{\rho_r} T_1^{-1} \cdot\cdot T_k^{-1})$ be defined by $\| d_{m,n}, \| \cdot \| \bar\beta_{m,n} \| = \| c_{m,n} \|$ and let $S' = (ST_0^{-1})$ be defined by $\| c_{m,n} \| \cdot \| \beta^0{}_{m,n} \| = \| \gamma_{m,n} \|$.

If $y_n = 0 \left(\frac{1}{B_n}\right)$ {*i.e.*, $(B_n |y_n|)$ is a null sequence}, then

$\{S' [T_0 (y_n)]\} = (S' T_0) (y_n) = S (y_n)$, when the latter exists,

Proof: Let m and p be two any positive integers and $r_{n,n+\mu}$ be $= \sum_{r=\mu+1}^{\infty} | a^0{}_{n,n+r} |$ as in (5·2).

Let $\bar{y}_{n_0}$ be upper bound of $|y_n|$ for all $n \geqslant n_0$.

Let $T_0 (y_n) = z_n$.

Then $z_m = \sum_{r=0}^{\infty} a^0{}_{m,m+r} y_{m+r} = \sum^{r=\rho} + \sum_{\rho+1}^{\infty} = \sum^{r=\rho} + \theta.\ r_{m,m+\rho} \bar{y}_{m+\rho}$

$$k \leqslant \rho, \quad z_{m+k} = \sum^{r=\rho} a^0{}_{m+k,m+r} y_{m+r} + \theta_k r_{m+k,m+\rho} \bar{y}_{m+\rho}.$$
$$z_{m+\rho} = y_{m+\rho} + \theta_\rho r_{n+\rho,n+\rho} \bar{y}_{n+\rho}. \qquad [8\cdot 22]$$

Consider now

$$\sum_{r=0}^{\rho} \gamma_{m,m+r}\, z_{m+r} = \sum^{r=\rho} \gamma_{m,m+r}\cdot \sum^{s=\rho} a^0{}_{m+r,m+s}\, y_{m+s} + \bar{y}_{m+\rho} \sum \theta_r \cdot \gamma_{m,m+r}\, r_{m+r,m+\rho}\cdot$$

$$= \quad A \quad + \quad B$$

$$A = \sum^{s=\rho} \Big(\sum_r \gamma_{m,\,m+r}\, a^0{}_{m+r,\,m+s}\Big)\, y_{m+s} = \sum_{s=0}^{s=\rho} c_{m,\,m+s}\, y_{m+s} \qquad [8\cdot 23]$$

$$|\,B\,| \leqslant \bar{y}_{m+\rho}\cdot \sum_{r=0}^{\rho} |\,\gamma_{m,\,m+r}\,|\cdot r_{m+r,\,m+\rho} = \bar{y}_{m+\rho}\cdot B_2 \qquad [8\cdot 24]$$

Now
$$\gamma_{m,\,m+r} = c_{m,\,m+k}\cdot \beta^0{}_{m+k,\,m+r}$$

$$\therefore\quad B_2 \leqslant \sum_{r=0}^{\rho} \Big(\sum^{k=r} |\,c_{m,\,m+k}\,|\cdot \beta^0{}_{m+k,\,m+r}\Big)\, r_{m+r,\,m+\rho}$$

$$= \sum_{k=0}^{\rho} |\,c_{m,m+k}\,|\cdot \sum_{r=k}^{r=\rho} \beta^0{}_{m+k,m+r}\, r_{m+r,m+\rho}$$

Just as in the course of proof of Theorem IV in (5·4) and (5·5), by (4·6) of Part I

we have $\displaystyle\sum_{r=0}^{\rho} \beta^0{}_{m,\,m+r}\, r_{m+r,\,m+\rho} \leqslant 1$ for all m and all ρ.

$$\therefore\quad B_2 \leqslant \sum_{k=0}^{\rho} |\,c_{m,\,m+k}\,| \text{ and } c_{m,\,m+k} = \sum_{s=0}^{k} d_{m,\,m+s}\cdot \bar{\beta}_{m+s,\,m+k}\cdot$$

Hence
$$\sum^{k=\rho} |\,c_{m,\,m+k}\,| \leqslant \sum^{k=\rho} \cdot \sum^{s=k} |\,d_{m,\,m+s}\,|\cdot \bar{\beta}_{m+s,\,m+k}\cdot$$

Now $\bar{\beta}_{m,\,m+r} \leqslant b_r$. Hence $\displaystyle\sum^{k=\rho} |\,c_{m,\,m+k}\,| \leqslant \sum^{k=\rho} \cdot \sum^{s=k} |\,d_{m,\,m+s}\,|\cdot b_{k-s}\cdot$

$$= \sum^{n=\rho} b_n \cdot \sum_{r=0}^{\rho-n} |\,d_{m,\,m+r}\,|.$$

Now by Lemma of (8·1) we have $\displaystyle\sum_{r=0}^{\infty} |\,d_{m,\,m+r}\,| \leqslant K$, a constant.

Hence $B_2 \leqslant K\cdot B_\rho$. [8·25]

Now $|\,B\,| \leqslant y_{m+\rho}\,|\,B_2\,| \leqslant K\cdot B_\rho \cdot y_{m+\rho} = \dfrac{K\cdot B_\rho\cdot \epsilon_{m+\rho}}{B_{m+\rho}} \leqslant K\cdot \epsilon_{m+\rho}\cdot$

since $y_n = 0\left(\dfrac{1}{B_n}\right)$.

Hence
$$\sum^{r=\rho} \gamma_{m,\,m+r}\, z_{m+r} = \sum^{s=\rho} c_{m,m+s}\, y_{m+s} + \theta K\cdot \epsilon_{m+\rho} \qquad [8\cdot 26]$$

If the second series $\displaystyle\sum^{s=\infty} c_{m,\,m+s}\, y_{m+s}$ converges, then

$$\sum^{r=\infty} \gamma_{m,\,m+r}\, z_{m+r} = \sum^{s=\infty} c_{m,\,m+s}\, y_{m+s} \qquad [8\cdot 27]$$

i.e., $S'\,[T_0\,(y_n)] = (S'\,T_0)\,(y_n) = S\,(y_n)$ when the latter exists [8·28]

THEOREM IX : Let S, S′ and T_0 be as in (8·21) and R be $= (T_{q_1}\ T_{q_2} \cdot T_{q_r})$ defined by

$\| e_{m,n} \|$. Then also (S′) [(T_0 R) (y_n)] = (S″T_0R) (y_n)

= (SR) (y_n) if the latter exists

$y_n = 0\left(\frac{1}{B_n}\right)$.

Proof: Let R $(y_n) = z_n$ and $\bar{y}_{n_0}$ = upper bound of $|y_n|$ for $n \geqslant n_0$

then $|z_n| = \left| \sum_{p=n}^{\infty} c_{np}\, y_p \right| \leqslant \bar{y}_n \cdot \sum_{p=n}^{\infty} |c_{n.p}| \leqslant K\bar{y}_n = 0\left(\frac{1}{B_n}\right)$

Hence $z_n = 0\left(\frac{1}{B_n}\right)$. [8·29]

Now (T_0 R) $(y_n) = T_0\ (z_n)$ and

S′ [T_0 (z_n)] = (S″T_0) (z_n) = S (z_n)

= S [R (y_n)] (if the latter exists). [8·30]

Now the series $\sum_{p=A}^{\infty} |c_{np}|\, y_n| \leqslant y_A \cdot K. = 0\left(\frac{1}{B_A}\right)$

since $\sum^{p=\infty} |c_{np}| \leqslant K$ by 8·1.

Hence by Theorem VI S·R (y_n) = (SR) (y_n) when the latter exists. [8·31]

A particularly interesting case of Theorem IX is when $T_1^{-1} = T_2^{-1} \cdots = T_k^{-1}$ = unit transformation ; then the theorem would be

THEOREM X : If S = (T_{s_1} T_{s_2} T_{s_p}) and y_n a null sequence and R = (T_{q_1} T_{q_2} T_{q_r})

then (ST_0^{-1}) [(T_0R) (y_n)] = (SR) (y_n). [8·32]

§ 9. *On Combinations of Inverse (T^{-1}) Transformations.*

Most of the theorems in this section are either deductions from, or particular cases of theorems of previous section.

THEOREM XI : Let $R_1 = (T_{q_1} T_{q_2} T_{q_r})$ defined by $\|d_{mn}\| = \|a^{q_1}{}_{m,n}\| \cdots \|a^{q_r}{}_{m,n}\|$

$R_2 = (T_{s_1} \cdot T_{s_2} \cdots T_{s_p})$ ..... $\|e_{m,n}\| = \|a^{s_1}{}_{m,n}\| \cdots \|a^{s_p}{}_{m,n}\|$.

$I = (T_1^{-1}\ T_2^{-1}\ T_k^{-1})$ ..... $\|\bar{\beta}_{m,n}\| = \|\beta^1{}_{m,n}\| \cdots \|\beta^k{}_{m,n}\|$

and S = (IR_2) defined by $\|c_{m,n}\| = \|\bar{\beta}_{m,n}\| \cdot \|e_{m,n}\|$.

If $\sum^{n=\infty} c_{m,n}\, y_n$ converges uniformly for all m, then

(R_1)·[S (y_n)] = (R_1S) (y_n). [9·1]

This is a particular case of Theorem V and result follows from (8·7).

THEOREM XII : Let $S = (T_1^{-1} T_k^{-1})$ defined by $\| \bar{\beta}_{m,n} \|$.

If $\sum_{}^{n=\infty} \bar{\beta}_{m,n} y_n$ converges uniformly for all m, then

(1) $(T_l T_{l-1} \cdot T_1)(T_1^{-1} \cdot\cdot T_k^{-1})(y_n) = (T_{l+1}^{-1} \cdots T_k^{-1})(y_n) \quad 0 \leqslant l \leqslant k.$

(2) $(T_k T_{k-1} T_1)(T_1^{-1} \cdot\cdot T_k^{-1})(y_n) = y_n.$

(3) $(T_1^{-1} T_2^{-1} \cdot\cdot T_k^{-1})(y_n) = (T_1^{-1})(T_2^{-1})(T_l^{-1}) \cdot\cdot (T_{l+1}^{-1} \cdots T_k^{-1})(y_n)$
$= (T_1^{-1})(T_2^{-1}) \cdots (T_k^{-1})(y_n).$

Proof :

(1) is a particular case of (8·33) [9·2]

(2) is a particular case of (1) when $l = k$. [9·3]

(3) Let $z_m = \sum_{}^{n=\infty} \bar{\beta}_{mn} y_n$ then by (1) $(T_l T_{l-1} \cdot T_1)(z_m) = (T_{l+1}^{-1} \cdots T_k^{-1})(y_n) = \varpi_m$.

Since $\Sigma \bar{\beta}_{m,n} y_n$ is uniformly convergent by (3) of theorem V it is a null sequence. By repeated application of Theorem IV of Part I as in corollary to it

$$z_m = \sum^{r_1=\infty} \beta^1_{m,m+r_1} \cdot \sum^{r_2=\infty} \beta^2_{m+r,m+r_2} \cdots \sum^{p=\infty} \beta^l_{m+r_{l-1},m+p} \cdot \varpi_p =$$
$$= (T_1)^{-1} \cdots (T_l)^{-1}(\varpi_m) = (T_1)^{-1}(T_2)^{-1} \cdot (T_l)^{-1}(T_{l+1}^{-1} \cdot\cdot T_k^{-1})(y_n) \qquad [9\cdot4]$$

and in particular when $l = k$ $z_m = (T_1)^{-1} \cdot (T_2)^{-1} \cdot\cdot (T_k^{-1})(y_n)$

$$= \sum^{r_1=\infty} \beta^1_{m,m+r_1} \cdot \sum^{r_2=\infty} \beta^2_{m+r_1,m+r_2} \cdots \sum^{p=\infty} \beta^k_{m+r_{k-1},m+p} y_p$$
$$= \sum^{p=\infty} \bar{\beta}_{m,m+p} y_p \qquad [9\cdot5]$$

A particularly interesting case of Theorem VI is the following :

THEOREM XIII : Let $R_1 = (T_{s_1} \cdot T_{s_2} \cdot T_{s_3} \cdot T_{s_p})$, be defined by $\| d_{mn} \|$
and $R_2 = (T_1^{-1} \cdot\cdot T_k^{-1})$ $\| \beta_{m\,n} \|$.

and $S = (T_{p_1}^{-1} \cdots T_{p_l}^{-1} T_{q_1} T_{q_2} \cdots T_{q_r})$ defined by $\| c_{m,n} \|$.

If $\left| \sum_{n=A}^{\infty} c_{m,n} y_n \right| = 0 \left(\frac{1}{B_A} \right)$ for all m

then $(R_1 R_2)[S(y_n)](R_1 R_2 S)(y_n)$ when the latter exists. [9·6]

THEOREM XIV : Let $(T_1^{-1} \; T_2^{-1} \cdot T_k^{-1})$ be defined by $\| \bar{\beta}_{m,n} \|$ then if $\Sigma \bar{\beta}_{mn.} \| y_n \|$ converges for all m, then the terms of the product $(T_1^{-1} \cdot T_k^{-1})$ can be associated in any arbitrary manner, and the resultant of all the associations on y_n is equal to $(T_1^{-1} \; T_2^{-1} \; T_k^{-1})(y_n)$.

We shall take a particular association and prove it, the proof for all other associations being the same.

Let $(T_1^{-1}\cdot T_2^{-1}\cdot T_{l_1}^{-1})$ be defined by $\| \bar\beta'_{m,n} \|$.

$(T_{l_1+1}^{-1} \cdots T_{l_2}^{-1})$ ... $\| \bar\beta''_{m,n} \|$

$(T_{l_2+1}^{-1} \quad T_k^{-1})$ ... $\| \bar\beta'''_{m,n} \|$.

To prove $\sum_{n_1=\infty} \bar\beta'_{m,n_1}\cdot \sum_{n_2=\infty} \bar\beta''_{n_1,n_2}\cdot \sum_{p=\infty} \bar\beta'''_{n_2,p}\, y_p = \sum_{p=\infty} \bar\beta_{m,p}\, y_p$.

Now all the $\beta_{m,n} \geqslant 0$, and $\| \bar\beta'_{m,n} \|\cdot\| \bar\beta''_{m,n} \|\cdot\| \bar\beta'''_{m,n} \| = \| \bar\beta_{m,n} \|$

Since $\Sigma\, \bar\beta_{mp} \mid y_p \mid$ converges

$$\sum^{p=\infty} \bar\beta_{mp}\, y_p = \sum^{p=\infty} (\Sigma\Sigma\, \bar\beta'_{mp_1}\cdot\bar\beta''_{p_1,p_2}\cdot\bar\beta'''_{p_2,p})\, y_p.$$

$$= \sum^{p_1=\infty} \bar\beta'_{m,p_1}\cdot \sum^{p_2=\infty} \bar\beta''_{p_1,p_2}\cdot \sum^{p=\infty} \bar\beta'''_{p_2,p}\, y_p$$

i.e., $(T_1^{-1}\, T_2^{-1}\, T_{l_1}^{-1})\,(T_{l_1+1}^{-1}\cdots T_{l_2}^{-1})\,(T_{l_2+1}^{-1}\cdot T_k^{-1})\,(y_n)$

$= (T_1^{-1} \cdots T_k^{-1})\,(y_n)$. [9·7]

THEOREM XV : $\| \bar\beta_{m,n} \|$ defines $(T_1^{-1}\cdots T_k^{-1})$. If $\Sigma\, \bar\beta_{m,n} \mid y_n \mid$ converges uniformly for all m then (1), (2), (3) of Theorem XII and the conclusion of XIV are all true in this case.

This is obvious. An interesting case of this is when $y_n = 0\left(\dfrac{1}{B_n^{1+\delta}}\right)$ for by Theorem VII $\Sigma\, \bar\beta_{m,n} \mid y_n \mid$ converges uniformly if $y_n = 0\left(\dfrac{1}{B_n^{1+\delta}}\right)$. [9·8]

§ 10. *On the Properties of a Particular Subclass of (T).*

We notice that in the theorems proved till now, in order that a combination of inverse and direct transformations on a sequence $\{y_n\}$ may be valid, there must be some strict restriction on y_n (for *e.g.*, in Theorem IV, $y_n \to 0$ as $n \to \infty$). In general, when a combination involves an inverse, the restriction on y_n is greater than when the combination is purely of direct T's in which latter case the only restriction being y_n be bounded. In § 10 we deal with a subclass of (T) designated by $\bar T_0$. The algebra of combinations involving any number of T's and $(\bar T)^{-1}$ is very simple and exactly like that of direct (T) transformation combinations.

The matrix $\| \bar a_{m,n} \|$ of a $\bar T$ is characterised by the following :

($\bar a$) $\bar a_{nn} = 1$ ($\bar b$) $\bar a_{m,n} = 0$, $n < m$ ($\bar c$) $a_{m,n} \leqslant 0$, $n > m$ and

($\bar d$) $\qquad -\sum_{p=1}^{\infty} \bar a_{m,m+p} \leqslant k < 1$ for all m; where the first three conditions are the same as in (2·1) for a T and ($\bar d$) is a greater restriction than (d) of (2·1) [10·1]

From (4·7) if $\| \bar{\beta}_{mn} \|$ defines a $(\bar{T})^{-1}$ then $\sum_{\rho=0}^{\infty} \bar{\beta}_{m,\, m+\rho} \leqslant \frac{1}{1-k}$. [10·2]

Let us designate the matrix of any T, $\bar{T}$ or $(\bar{T})^{-1}$ by $\| b_{mn} \|$. Then from (10·1), and (10·2) and condition (*d*) of (2·1) we have $\sum_{\rho=0}^{\infty} | b_{m,\, m+\rho} | \leqslant K$ (a constant) for all m. [10·3]

If V stands for any T, $\bar{T}$, or $(\bar{T})^{-1}$ the most general theorem involving T, $\bar{T}$ and $(\bar{T})^{-1}$ will be the following :

THEOREM XVI: If $\{x_n\}$ be $\left\{\frac{\text{bounded}}{\text{null}}\right\}$ sequence so is $(V_1, V_2 \cdot\cdot V_k)(x_n)$, and $(V_1 V_2 V_k)(x_n) = (V_1 V_2 \cdot\cdot V_p) \cdot\cdot (V_{p+1} \cdot\cdot V_l) \cdot\cdot (V_{l+1} \cdot V_k)(x_n)$. Proof is immediate because it means that all $(\bar{T})^{-1}$ behave exactly like the direct transformation T. [10·4]

In particular it is interesting to notice the form which Theorem IV and its complement take in the case of $\bar{T}$ and T^{-1}.

THEOREM XVII : $\{x_n\}$ being a bounded sequence, if $\bar{T}(x_n) = y_n$, then

(1) $x_n = (\bar{T})^{-1}(y_n)$;

(2) Given $\{y_n\}$ the infinite set of equations $T(x_n) = \Sigma\, \bar{a}_{m,\rho}\, x_\rho = y_m$, $m = 0, 1, 2, \cdots n$,—under the restriction x_n be bounded, has utmost one solution.

(3) The necessary and sufficient condition that the infinite set of equations $\Sigma\, \bar{a}_{m,\,\rho}\, x_\rho = y_m$ has one solution $\{x_n\}$ under the restriction that x_n be bounded is that y_n be bounded.

The proofs of (1), (2) and (3) follow very simply from Theorem XI. [10·5]

§ 11. *Two theorems applicable for a special subclass of (T).*

Let T_0 be defined by $\| a^0_{m,\,n} \|$ and T_0^{-1} by $\| \beta^0_{m,\,n} \|$.

Let S be any transformation denoted by $\|s_{m,\,n}\|$ where $s_{m,\,n} = 0$, $n < m$;

THEOREM XVIII : If (1) $\{x_n\}$ be a bounded sequence and (2) $\frac{s_{m,\,m+p}}{B^0_{m,\,m+p}} \to 0$ as $p \to \infty$ for every m and (3) $| s_{m,\,n} | \leqslant K'$ for all m, n, then $S\,[T_0(x_n)] = (S\, T_0)(x_n)$, when the latter exists.

Proof :

Let $|x_n| < K$. Let $\sum_{\rho=p+1}^{\infty} |a^0_{n,\,n+\rho}| = -\sum_{\rho=p+1}^{\infty} a^0_{n,\,n+\rho} = r_{n,\,n+p}$, $p \geqslant 1$ and let $T_0(x_n) = y_n$.

Then $y_n = \sum_{\rho=0}^{\infty} a^0_{n,\,n+\rho}\, x_{n+\rho} = \sum_{\rho=0}^{p} + \sum_{p+1}^{\infty} = \sum^{\rho=p} + \theta \mathrm{K}\cdot r_{n,\,n+p}.$

Similarly $y_{n+k} = \sum^{\rho=p} a^0_{n+k,\,n+\rho}\, x_{n+\rho} + \theta_k \cdot \mathrm{K}\cdot r_{n+k,\,n+p}$ [11·1]

for $k = 0, 1, 2, p.$ and $-1 \leqslant \theta_k \leqslant 1.$

Then $\sum_{\rho=0}^{\rho=p} s_{n,\,n+\rho}\, y_{n+\rho} = \sum^{\rho=p} s_{n,\,n+\rho} \cdot \sum^{\rho'=p} a^0_{n+\rho,\,n+\rho'}\, x_{n+\rho'} +$

$\mathrm{K}\, \Sigma\, \theta_\rho\, s_{n,\,n+\rho} \cdot r_{n+\rho,\,n+p}$

$= \mathrm{A} + \mathrm{B}.$ [11·2]

$$\mathrm{A} = \sum_{\rho'=0}^{\rho'=p} \Big(\sum_{\rho=0}^{\rho=\rho'} s_{n,\,n+\rho} \cdot a^0_{n+\rho,\,n+\rho'} \Big) \cdot x_{n+\rho'}$$

$$|\mathrm{B}| \leqslant \mathrm{K} \cdot \sum_{\rho=0}^{p} |s_{n,\,n+\rho}|\, r_{n+\rho,\,n+p} = \mathrm{K}\,\Big(\sum^{k-1} + \sum_{k}^{p} \Big) = \mathrm{K}\,(\beta_1 + \beta_2)\,;$$

choose k sufficiently large such that for $\rho \geqslant k$, $\dfrac{|s_{n,\,n+\rho}|}{\beta^0_{n,\,n+\rho}} \leqslant \epsilon$

$$\mathrm{B}_2 \leqslant \epsilon \cdot \sum_{\rho=k}^{\rho=p} \beta^0_{n,\,n+\rho} \cdot r_{n+\rho,\,n+p} \leqslant \epsilon \cdot \sum_{\rho=0}^{\rho=p} \beta^0_{n,\,n+\rho} \cdot r_{n+\rho,\,n+p}.$$

Now $r_{n,\,n+p} = \sum_{p+1}^{\infty} - a^0_{n,\,n+\rho} = \sum_{\rho=1}^{\infty} - \sum^{\rho=p} \leqslant \mathrm{R}_{n,\,n+p}$ of (4.6 a)

and by (4·6 a) $\Sigma\, \beta^0_{n,n+\rho}\, \mathrm{R}_{n+\rho,\,n+p} = 1$; Hence $\mathrm{B}_2 \leqslant \epsilon.$ [11·3]

Now $\mathrm{B}_1 \leqslant \mathrm{K}' \sum_{\rho=0}^{\rho=k-1} r_{n+\rho,n+p}$ by hypothesis (3).

Now each of the sequences $r_{n,n+p}, r_{n+1,\,n+p} \cdots r_{n+k-1,\,n+p}$ tends to zero by condition (d) of 2·1; choosing p sufficiently large, $\mathrm{B}_1 \leqslant \epsilon$. Hence for sufficiently large p $|\mathrm{B}| \leqslant \epsilon\,(\rho)$ and we have [11·4]

$$\sum_{\rho=0}^{\rho=p} s_{n,\,n+p} \cdot y_{n+\rho} = \sum_{\rho=0}^{\rho=p} \Big(\sum_{\rho_1} s_{n,n+\rho_1}\, a^0_{n+\rho_1,\,n+\rho}\Big)\, x_{n+\rho} + \theta\epsilon\,(p).$$

$$= \sum_{\rho=0}^{p} d_{n,\,n+\rho}\, x_{n+\rho} + \theta\epsilon\,(p)$$ [11·5]

where $\| d_{m.\,n} \| = \| s_{m,n} \| \cdot \| a^0_{m,\,n} \|$.

If the latter series $\sum^{p=\infty} d_{n\,p}\, x_p$ converges then in the limit making $p \to \infty$

$$\sum^{\rho=\infty} s_{n,\,n+\rho}\, y_{n+\rho} = \sum_{\rho=0}^{\infty} d_{n,n+\rho}\, x_{n+\rho} \quad \text{or}$$

$\mathrm{S}\,[\mathrm{T}_0\,(x_n)] = (\mathrm{S}\,\mathrm{T}_0)\,(x_n)$ when the latter exists. [11·6]

THEOREM XIX. Let T_1 and T_2 and (T_1T_2) be defined by $\| a_{m,n} \|$, $\| b_{m,n} \|$ and $\| c_{mn} \| = \| a_{m,n} \| \cdot \| b_{m,n} \|$.

Let (1) $\sum_{\rho=p+1}^{\infty} |a_{m,m+p}| = r^a_{m,m+p}$

(2) $\sum_{\rho=p+1}^{\infty} |c_{m,m+p}| = r^c_{m,m+p}$

and T_1^{-1} be defined by $\| \beta_{m,n} \|$.

Then if (1) $\dfrac{r^c_{m,m+p}}{r^a_{m,m+p}} \to 0$ as $p \to 0$ uniformly for all m (2) $\beta_{n,n+p} \to 0$ as $p \to \infty$.

and (3) $|x_n| \leqslant K$, then

$$(T_1^{-1}\cdot)\,[(T_1T_2)\,(x_n)] = T_2\,(x_n). \qquad [11\cdot 7]$$

Proof: Let $(T_1T_2)\,(x_n) = y_n$ and p a positive integer

and $y_n = \sum^{\rho=\infty} c_{n,n+\rho}\, x_{n+\rho} = \sum^{\rho=p} + \sum_{p+1}^{\infty} = \sum^{\rho=p} + K.\theta\, r^c_{n,n+p}$

Similarly

$$y_{n+k} = \sum^{\rho=p} c_{n+k,n+\rho}\ x_{n+\rho} + K.\ \theta_k.\ r^c_{n+k,n+p} \qquad [11\cdot 8]$$

for $k = 0, 1, p$ and $-1 \leqslant \theta_k \leqslant 1$.

$$\sum_{\rho=0}^{\rho=p} \beta_{n,n+\rho}\, y_{n+\rho} = \sum^{\rho=p} \beta_{n,n+\rho} \cdot \sum^{s=p} c_{n+\rho,n+s}\, x_{n+s} + K\,\Sigma\,\theta_\rho \cdot \beta_{n,n+\rho} \cdot r^c_{n+\rho,n+p}$$

$$= A + B.$$

$$A = \sum_{s=0}^{p} \cdot \Big(\sum_{\rho=0}^{s} \beta_{n,n+\rho}\, c_{n+\rho,n+s}\Big) \cdot x_{n+s} = \sum^{s=p} b_{n,n+s}\, x_{n+s}$$

$$|B| \leqslant K \cdot \Big(\sum_{\rho=0}^{p-k} \beta_{n,n+\rho}\, r^c_{n+\rho,n+p} + \sum_{p-k+1}^{p}\Big) = K\,(B_1 + B_2) \qquad [11\cdot 9]$$

Choosing k large enough such that $\dfrac{r^c_{n,n+\rho}}{r^a_{n,n+\rho}} \leqslant \epsilon$ for all n and $\rho \geqslant k$

$$B_1 \leqslant \epsilon \cdot \sum_{\rho=0}^{\rho=p-k} \beta_{n,n+\rho}\, r^a_{n+\rho,n+p}$$

$$\leqslant \epsilon \cdot \sum_{\rho=0}^{\rho=p} \beta_{n,n+\rho} \cdot r^a_{n+\rho,n+p} \leqslant \epsilon \quad \text{just as in } (12\cdot 3). \qquad [11\cdot 10]$$

and $B_2 = \sum_{\rho=p-k+1}^{\rho=p} \beta_{n,n+\rho}\, r^c_{n+\rho,n+p} \leqslant K_1 \sum_{\rho=p-k+1}^{\rho=p} \beta_{n,n+\rho}$

since by condition (d) of $(2\cdot 1)$ for T_1 and T_2, $r^c_{n,p} \leqslant K_1$ for all n and p.

Choosing p sufficiently large, by condition (2) of Theorem XIX, we have

$$\sum_{\rho = p - k + 1}^{\rho = p} \beta_{n, n+\rho} \leqslant \epsilon, \text{ i.e., } B_2 \leqslant \epsilon' (p)$$

Hence
$$| B | \leqslant \epsilon'' (\rho) \qquad [11 \cdot 11]$$

and
$$\sum^{\rho = p} \beta_{n, n+\rho}\, y_{n+\rho} = \sum^{\rho = p} b_{n, n+\rho}\, x_{n+\rho} + \theta \epsilon'' (p)$$

and in the limit
$$\sum^{\rho = \infty} \beta_{n, n+\rho}\, y_{n+\rho} = \sum^{\rho = \infty} b_{n, n+\rho}\, x_{n+\rho} \quad \text{or}$$
$$(T_1^{-1}) \left[(T_1 T_2)\, (x_n) \right] = T_2\, (x_n). \qquad [11 \cdot 12]$$

§ *12. On an Integral Equation.*

Corresponding to a T for a sequence, we define the following transformation for the continuous variable.

Let (1) $K(x, t) \geqslant 0$ for all $x \geqslant 0$, $t \geqslant 0$; (2) $K(x, t) = 0$, $t < x$;

(3) $K(x, t)$ being continuous in (x, t), for all $x \geqslant 0$, $t \geqslant 0$.

(4) $\int_0^\infty K(x, t)\, dt \leqslant 1$ for all x. [12·1]

Let $u(t)$ be a given continuous function defined for $t \geqslant 0$, then the transformation T will be $T(u) = u(t) - \int_0^\infty K(t, r)\, u(r)\, dr = v(t)$. [12·2]

We discuss here only the existence of the inverse of this transform and show that under certain conditions $u(t) = T^{-1} \{v(t)\}$. The main point of interest in the solution of the integral equation in u of (12·2) is that the range is infinite. We shall show that the solution can be given in terms of Volterra's reciprocal of $K(x, t)$ just as in the ordinary case of finite range when $v(t)$ satisfies a certain condition.

Theorem XX: (1) If $u(x) \to 0$ as $x \to \infty$ and

$$T(u) = u(x) - \int_0^\infty K(x, t)\, u(t)\, dt = v(x),$$

then
$$u = T^{-1} [(v(x)] = v(x) + \int_0^\infty \overline{K}(x, t) .\, v(t)\, dt.$$

where
$$[\overline{K}(x, t)] = K(x, t) + K_1(x, t) + \cdots K_n(x, t) + \cdots$$

where
$$[\overline{K}_n(x, t)] = \int_0^t K_{n-1}(x, u)\, K(u, t)\, dt.$$

(2) Given $v(x)$, the equation $T(u) = v$ will have utmost one solutio under the restriction $u(x) \to 0$ as $x \to \infty$.

(3) Given $v(x)$ the necessary and sufficient condition that the equatior $T(u) = v$ has one solution under the restriction $u(x) \to 0$ as $x \to \infty$ is that

$\int_0^\infty \overline{K}(x, t)\, v(t)\, dt$ converges uniformly for all $x \geqslant 0$.

Lemmas :

(1) $K_n(x, t) = 0 \quad t < x$ (proof by mathematical induction and (2) o (12·1) [12·3

(2) $K_n(x, t) \geqslant 0$ for x and t, obvious from definition. [12·4

(3) By definition $K_n(x, t) = \int_0^t K_{n-1}(x, u).\ K(u, t)\, du$. To prove

$$K_n(x, t) = \int_0^t K_{n-1}(u, t).\ K(x, u)\, du, \qquad [12\cdot 5$$

suppose this is true for 1, 2, $\cdots$ $n - 1$, then we will prove it is true for n and hence for all $n \geqslant 1$ since it is true for $n \geqslant 1$.

By definition $K_n(x, t) = \int_0^t K(u, t) \cdot K_{n-1}(x, u)\, du$

and by hypothesis $K_{n-1}(xu) = \int_0^t K_{n-2}(u, t) \cdot K(xu)\, du$.

$$\text{Hence } K_n(x, t) = \int_0^t K(u, t) \cdot du \cdot \int_0^u K(xv) \cdot K_{n-2}(v, u)\, dv$$

$$= \int_0^t K(x, v)\, dv \int_v^t K(u, t)\, K_{n-2}(vu)\, du$$

$$= \int_0^t K(x, v)\, dv \int_0^t K(u, t)\, K_{n-2}(v, u)\, du$$

since $K_n(v, u) = 0 \ u < v$

$$= \int_0^t K(x, v)\, K_{n-1}(vt)\, dt \qquad [12\cdot 5$$

for $n = 1$. (12·5) is obvious since $K_n(x, t) \equiv K(x, t)$. Hence (12·5) is tru for $n \geqslant 1$.

(4) $\sum_0^\infty K_n(n, t)$ converges. This follows from continuity of $K(x, t)$ i: $x \geqslant 0, t \geqslant 0$ [12·6

(5) $\overline{K}(x, t)$ is continuous ; this follows from uniform convergence of Σ K and continuity of K_n. [12·7

(6) (1) $\overline{K}(x, t) \geqslant 0$ for all x, t; (2) $\overline{K}(x, t) = 0 \quad t < x$. (1) is obvious, (2) follows from (12·3).

(7) The reciprocal function $\overline{K}(x, t)$ satisfies two integral equations.

$$\overline{K}(x, t) - K(x, t) = \int_0^t \overline{K}(x, u)\, K(u, t)\, dt = \int_0^t \overline{K}(u, t) \cdot K(x, u)\, du.$$

This follows immediately from definition of K_n and (12·5), and term by term integration of $\Sigma\, K_n(x)$. [12·8]

(8) Let $R(x, A) = 1 - \int_0^A K(x,t)\, dt$. To prove

$$\int_0^A \overline{K}(x, u) \cdot R(u, A)\, du = 1 - R(x, A). \qquad [12\cdot 9]$$

Proof: $\int_0^A \overline{K}(x, u) \left\{1 - \int_0^A K(u, t)\, dt\right\} du = \int_0^A \overline{K}(x, u)\, du$

$$- \int_0^A \overline{K}(x, u) \left\{\cdot \int_0^A K(u, b)\, dt\right\} du = \alpha - \beta.$$

$$\beta = \int_0^A \overline{K}(x, u)\, du \int_u^A K(u, t)\, dt, \text{ since } K(u, t) = 0 \quad t < u$$

$$= \int_0^A dt \cdot \int_0^t \overline{K}(x, u)\, K(u, t)\, du = \int_0^A \{\overline{K}(x, t) - K(x, t)\}\, dt \text{ by } (12\cdot 8)$$

Hence $\alpha - \beta = \int_0^A \overline{K}(x, t)\, dt = 1 - R(x, A)$

$$= \int_0^A \overline{K}(x, u)\, R(u, A) \qquad [12\cdot 9]$$

(12·9) corresponds to formula (4·7) in Part I.

Proof (1) of Theorem XX :—

$$\int_0^A \overline{K}(xt)\, v(t)\, dt = \int_0^A \overline{K}(x, t)\, u(t)\, dt - \int_0^A \overline{K}(x, t)\, dt \int_0^\infty \overline{K}(t, z)\, u(z)\, dz \text{ by } (12\cdot 2)$$

$$= B_1 - B_2. \qquad [12\cdot 10]$$

Let $\bar{u}(z_0)$ be the upper bound of $|u(z)|$ for $z \geqslant z_0$

then $\int_0^\infty \overline{K}(tz)\, u(z)\, dz = \int_0^A + \int_A^\infty = \int_0^A + \theta \cdot r(t, A) \cdot \bar{u}(A_0)$, [12·11]

where $r(t,A) = \int_A^\infty K(tz)\, dz$ and $-1 \leqslant \theta \leqslant 1$.

Hence $B_2 = \int_0^A \bar{K}(x, t)\, dt \int_0^A K(tz)\, dz + \bar{u} \cdot (A) \cdot \int_0^A \theta \cdot \bar{K}(x, t) \cdot r(t, A)\, dt$

$$= C_1 + C_2. \qquad [12 \cdot 12)$$

Now $r(t, A) = \int_0^\infty K(t, z)\, dz - \int_0^A K(tz)\, dz \leqslant 1 - \int_0^A = R(t, A)$.

Hence $| c_2 | \leqslant \bar{u}(A) \cdot \int_0^A \bar{K}(x, t) \cdot R(t, A)\, dt = \bar{u}(A) \{1 - R(x, A)\}$ by (12·9)

and $c_1 = \int_0^A \bar{K}(x, t)\, dt \int_t^Z K(tz) \cdot u(z)\, dz$ since $K(tz) = 0, \quad z < t$

$$= \int_0^A u(z)\, dz \cdot \int_0^Z \bar{K}(x, t)\, K(t, z)\, dt$$

$$= \int_0^A (\bar{K}(x, z) - K(x, z) \cdot u(z)\, dz. \qquad [12 \cdot 13]$$

Hence $B_1 - B_2 = \int_0^A K(x, z)\, u(z) + \theta' u(A) \{1 - R(x, A)\}$ [12·14]

$$= \int_0^\infty K(x, z)\, \bar{u}(z) + \theta'' R(x, A)\, u(A) + \theta' \cdot \{1 - R(x, A)\}\, \bar{u}(A)$$

$$= \int_0^\infty \bar{K}(x, z)\, u(z)\, dz + \theta'''\, \bar{u}(A) \qquad [12 \cdot 15]$$

Hence $\int_0^A \bar{K}(xt)\, v(t)\, dt = \int_0^\infty K(x,z)\, u(z)\, dz + \theta'''\, \bar{u}(A)$

$$= u(x) - v(x) + \theta'''\, \bar{u}(A). \qquad [12 \cdot 16]$$

Since $\bar{u}(A) \to 0, \quad A \to \infty$ by hypothesis, we have

$$u(x) = v(x) + \int_0^\infty K(x, t)\, v(t)\, dt = T^{-1} \{v(x)\}. \qquad [12 \cdot 17]$$

Proofs of (2) and (3) of Theorem XX follow almost exactly by the same way as the corresponding portion of the Theorem IV of Part I and complement of Theorem IV of Part I.