ON LINEAR TRANSFORMATIONS OF BOUNDED SEQUENCES—III.

By K. S. K. Iyengar.

(*From the Department of Mathematics, University of Mysore, Bangalore.*)

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Part III.

§ *13. General Remarks.*

This part deals with a subclass of T [T as defined in $2 \cdot 1$ of Part I of this paper]. We designate the direct and inverse transformations of this class by U and U^{-1}, and prove that these transformations, and others defined by their products are commutative. We further shew that transformations corresponding to differences of any real order form a subclass of the group defined by U, U^{-1}, and their products. In (16) we shew that some important theorems of Anderson (A. 1)* are, either deducible from, or particular cases of, theorems of Parts I and II of this paper. In (17) we discuss the generalization of Knopp's results on "Mehrfach monotone folgen" $(K \cdot 2)$.†

§ *14. A Class of Commutative Transformations.*

Let $\| a_{m,n} \|$ define a U. Then besides the four conditions of $(2 \cdot 1)$ of Part I, condition (*e*), namely, $a_{n,n+p} = a_p$ for all n, *i.e.*, $a_{0p} = a_{1,p+1} = = a_{n,n+p} = \cdots$, characterizes a_{mn}; so that $a_{m,n}$ for U is characterized as follows:

$(a')\ a_{n,n} = 1 \quad (b')\ a_{mn} = 0, n < m \quad (c')\ a_p \leqslant 0$ for all $p \geqslant 1$ $\qquad [14 \cdot 1]$

$(d')\ -\sum_{p=1}^{\infty} a_p \leqslant 1$. We see that a U is defined completely by a sequence $\{a_p\}$ satisfying (c') and (d').

$(A \cdot 1)$*. A. F. Anderson *Studier over Cesaro's summabilitets methode* (Danish). See the second chapter entitled "Om differencer".

$(K \cdot 2)$†. K. Knopp, "Mehrfach Monotone folgen," *Mathematisсhe Zeitschrift*, 1925, 22, 75-85.

In section 2 of Part I we established the existence of a unique reciprocal matrix $\| \beta_{m,n} \|$ such that $\| \beta_{m,n} \| \cdot \| a_{m,n} \| = \| \delta_{m,n} \|$ (unit matrix). Since any U is a T it follows that in this case also $\| \beta_{m,n} \|$ the reciprocal matrix exists.

Further from section 2 of Part I, we obtain

$$\beta_{n,n+\rho} = - \sum_{k=1}^{\rho} a_{n,n+k} \cdot \beta_{n+k,n+\rho} \quad (2 \cdot 6)$$

We can at once deduce $\beta_{n,n+\rho} = b_\rho$ for all n; and

$$b_\rho = - \sum_{k=1}^{\rho} a_k b_{\rho-k}. \qquad [14 \cdot 2]$$

We obtain the following results also easily. $b_0 = 1$, $b_n > 0$, and from $(14 \cdot 2)$ it follows that b_n is given by the equation

$$\left(\sum_0^\infty b_n x^n\right) \cdot \left(1 + \sum_{n=1}^\infty a_n x^n\right) = 1. \qquad [14 \cdot 3]$$

If U_1 and U_2 are defined by $\{a_n^1\}$ and $\{a_n^2\}$, it is easy to prove (1) $U_1 U_2 = U_2 U_1$; (2) if $\| c_{m,n} \|$ defines $U_1 U_2$ then $c_{n,n+p} = a_p^3$ for all n, (3) a_p^3 is given by the equation

$$1 + \sum_{\rho=1}^\infty a_\rho^3 x^\rho = \left(1 + \sum_1^\infty a_\rho^1 x^\rho\right)\left(1 + \sum_1^\infty a_\rho^2 x^p\right). \qquad [14 \cdot 4]$$

The matrix of any product of U's and U^{-1}'s is always characterized by condition (*e*) of $14 \cdot 1$. If $\{\bar{a}_\rho\}$ defines the product $\bar{a}_\rho$ can be calculated in all cases from an equation of the type of $(14 \cdot 4)$. It is quite easy to shew that the commutative property is true for any product of U's and U^{-1}'s.

§ 15. *Differences of any Real Order.*

THEOREM: Transformations defined by differences of any real order form a subclass of the class formed by U's, U^{-1}'s, and their products.

Lemma 1. If $0 \leqslant \gamma \leqslant 1$ then we shall prove that $\Delta^\gamma \equiv U(\gamma)$.

Formally the difference

$$\begin{aligned}\Delta^\gamma v_n &= v_n - \gamma v_{n+1} + \frac{\gamma(\gamma-1)}{\underline{|2}} v_{n+2} - \frac{\gamma(\gamma-1)(\gamma-2)}{\underline{|3}} v_{n+3} + \cdots\cdot \\ &= v_n - \gamma v_{n+1} - \frac{\gamma(1-\gamma)}{\underline{|2}} v_{n+2} - - \cdot \\ &\quad - \frac{\gamma(1-\gamma)\cdots(p-1-\gamma)}{\underline{|p}} v_{n+p} - \cdot\end{aligned}$$

Consider a transformation $U(\gamma)$ defined by $\{a_n\}$ as follows:

$$a_1 = -\gamma \quad a_2 = -\frac{\gamma(1-\gamma)}{\underline{|2}} \cdots a_p = -\frac{\gamma(1-\gamma)(p-1-\gamma)}{\underline{|p}}$$

then, $a_p < 0$ for $p \geqslant 1$ and $-\sum_1^\infty a_p = 1$.

Hence conditions (c') and (d') of $(14 \cdot 1)$ are fulfilled and we have

$$\Delta^{\gamma} \equiv \mathrm{U}(\gamma). \qquad [15 \cdot 1]$$

Lemma 2. If $0 \leqslant \gamma \leqslant 1$ $\quad \Delta^{-\gamma} = \{\mathrm{U}(\gamma)^{-1}\}$ $\qquad [15 \cdot 2]$

By $(14 \cdot 3)$ this is obvious.

Proof of Theorem : Let $\mathrm{U}(1) \equiv \Delta^{1}$ as in $(15 \cdot 1)$

Let $[\mathrm{U}(1)]^{-1} \equiv \Delta^{-1}$ as in $(15 \cdot 2)$

Then if Δ^{p} be a difference of any positive order, consider the transformation

$$\mathrm{S} = [\mathrm{U}(1)]^{m} \cdot \mathrm{U}(\gamma)$$

where $\quad m = [p]$ and $\gamma = (p)$

and $\quad \mathrm{U}(\gamma) \equiv \Delta^{\gamma}.$

If $\{a_{\rho}\}$ defines S, it is given as in $(14 \cdot 4)$ by

$$1 + \Sigma\, a_{\rho}\, x^{p} = (1-x)^{m} \cdot \left(1 - \gamma x - \gamma \frac{(1-\gamma)}{\underline{|2}} x^{2} \cdots \right)$$

$$= (1-x)^{m+\gamma} = (1-x)^{p}$$

so that

$$a_{n} = (-1)^{n} \frac{p\,(p-1)\,(p-n-1)}{\underline{|n}}$$

i.e.,

$$\mathrm{S} \equiv \Delta^{p} \equiv [\mathrm{U}(1)]^{m}\; \mathrm{U}(\gamma). \qquad [15 \cdot 3]$$

If p is negative we prove in exactly the same way as above

$$\Delta^{p} \equiv \{[\mathrm{U}(1)]^{-1})\}^{m} \cdot \{\mathrm{U}(\gamma)\}^{-1}$$

where $\quad m = (-p)$ and $\gamma = (-p). \qquad [15 \cdot 4]$

Hence the theorem.

§ 16. *Deductions of Some Theorems of Anderson.*

We propose in this section to derive Soetning 3, 4, and 5 of Anderson on differences from theorems of Part I and II of this paper.

Soetning III (page 20 of Anderson's Book A $\cdot$ 1).

(*a*) If $x_n = \mathrm{O}(1)$ $\quad r > 0 \quad s > -1$ and $r + s > 0$

then $\quad \Delta^{s}\{\Delta^{r}(x_n)\} = \Delta^{r+s}(x_n)$

(*b*) If $x_n = 0\,(1)$ $\quad r > 0 \quad s \geqslant -1$ and $r + s \geqslant 0$

then $\quad \Delta^{s}\{\Delta^{r}(x_n)\} = \Delta^{r+s}(x_n).$

Proof of (*a*) : Leaving aside the trivial case of $s > 0$ we shall shew that (*a*) is a particular case of Theorem XVIII of section 11§, Part II of this paper.

Let $s < 0$ and $s = -q$ $\quad q < 1$ choose q_1 such that $q < q_1 < 1$ and $q_1 < r$ so that $r = q_1 + t,\; t > 0$.

Then by Theorem of 15§ $\quad \Delta^{t} \equiv \mathrm{U}_{\rho_1} \cdot \mathrm{U}_{\rho_2} \cdots \mathrm{U}_{\rho_k}$

and $\quad \Delta^{r} = \mathrm{U}(q_1) \cdot \mathrm{U}_{\rho_1} \cdots \mathrm{U}_{\rho_k}$

where $$U(q_1) = \Delta^{q_1} \quad \text{as in } (15\cdot 1)$$
and $$\Delta^s = [U(q)]^{-1}.$$
Since $\{x_n\}$ is bounded by Theorem II of Part I so is $(U_{\rho_1} U_{\rho_2} U_{\rho_k})(x_n) = y_n$.
We shall now shew that $[U(q)]^{-1}\cdot[U(q_1)(y_n)] = \{[U(q)]^{-1}\cdot U(q_1)\}(y_n)$.
Let $\| \beta_{mn} \|$ define $[U(q_1)]^{-1}$ and $\| s_{mn} \|$ define $[U(q)]^{-1}$

then $$s_{n,n+p} = s_p = \frac{q(q+1)(q+p-1)}{\underline{|p}} = O(p^{q-1})$$

and $$\beta_{n,n+p} = b_p = \frac{q_1(q_1+1)(q_1+p-1)}{\underline{|p}} = 0(p^{q_1-1})$$

therefore $$\frac{s_{n,n+p}}{\beta_{n,n+p}} = 0(p^{q-q_1}) \quad \therefore \quad \frac{s_{n,n+p}}{\beta_{n,n+p}} \to 0 \quad \text{as } p \to \infty$$

Obviously s_p is bounded and y_n bounded. Hence the three conditions of Theorem XVIII are fulfilled and we have

$$[U(q)]^{-1}\cdot[U(q_1)(y_n)] = \{[U(q)]^{-1}\cdot U(q_1)\}(y_n) = \Delta^{q_1-q} y_n$$
$$= U(q_1 - q)(y_n).$$

But $y_n = U_{\rho_1} \cdots U_{\rho_k}(x_n)$
and by Theorem II of Part I

$$[U(q_1-q)]\,[U_{\rho_1} U_{\rho_2} U_{\rho_k}(x_n)] = [U(q_1-q)\cdot U_{\rho_1} \cdots U_{\rho_k}](x_n)$$
$$= \Delta^{r+s}(x_n).$$

Proof of (*b*) : Leaving aside the trivial case of $s > 0$, this is a particular case of Theorem X of section 8§, Part II (refer 8·32).

Let $$s = -q \quad q > 0 \quad r = q + t \quad t \geqslant 0$$
then $$\Delta^s = [U(q)]^{-1} \quad \text{and} \quad \Delta^r = \Delta^q\cdot\Delta^t$$
$$= U(q)\cdot U_{\rho_1} U_{\rho_2} \cdots U_{\rho_k}.$$

Since x_n is a null sequence by 8·32 of Part II

$$[U(q)]^{-1}[U(q)\,U_{\rho_1} U_{\rho_2} \cdots U_{\rho_k}(x_n)] = U_{\rho_1} U_{\rho_2} \cdots U_{\rho_k}(x_n)$$
or
$$\Delta^s[\Delta^r(x_n)] = \Delta^{r+s}(x_n).$$

Statement Soetning IV and V of Anderson :—

Soetning IV (*a*)—If, $x_n = O\left(\frac{1}{n^a}\right)$, $a > 0, r > -a,\ s > -1-a, r+s > -a$

then $$\Delta^s[\Delta^r(x_n)] = \Delta^{r+s}(x_n).$$

Soetning IV (*b*)—If $x_n = 0\left(\frac{1}{n^a}\right)$, $a > 0\ r > -a, s \geqslant -1-a, r+s > -a$

then $$\Delta^s[\Delta^r(x_n) = \Delta^{r+s}(x_n).$$

Soetning V (*c*)—If $x_n = 0\left(\frac{1}{n^a}\right)$, $a > 0, r > -a, s \geqslant -1-a, r+s \geqslant -a$

$$\Delta^s[\Delta^r(x_n)] = \Delta^{r+s}(x_n) \quad \text{when the latter exists.}$$

We shall shew that these are particular cases of Theorems VI, VII, VIII and IX of Part II of this paper. There is a considerable amount of overlapping of the various cases occurring in (*a*), (*b*), (*c*). In any particular case of (*a*) we shall have

(a') $s \geqslant -1-\alpha_2$, $r \geqslant -\alpha_2$, and $r+s \geqslant -\alpha_2$, where $0 < \alpha_3 < \alpha$.

The cases of (*b*) which do not occur under (*a*) are

(b') $s = -1-\alpha$, $r > -\alpha$, $r+s > -\alpha$.

The cases of (*c*) which do not occur under (*a*) and (*b*) are

(c') $r+s = -\alpha$, $r > -\alpha$, $s \geqslant -1-\alpha$.

We propose to further divide (a'), (b') and (c') as follows :

(a') :— (a_1') when $s \geqslant -\alpha_2$; (a_2') $s = -\alpha_2 - q$ $0 \leqslant q \leqslant 1$

(b') :— only one case (b'_1)

(c') :— (c_1') when r and s are negative

i.e., $r = -\alpha_3$ $s = -\alpha_4$ $\alpha_3 > 0$ $\alpha_4 > 0$ and $\alpha_3 + \alpha_4 = \alpha$

(c_2') when $0 \leqslant r \leqslant 1$ and $s = -\alpha - r$.

We shall shew that (a_1') is a particular case of Theorem VII of Part II

(a'_2), (b_1') and (c_2') are particular cases of Theroem IX of Part II

and (c_1') is a particular case of Theorem VI of Part II.

Proof : (a_1') $r \geqslant -\alpha_2$ $s \geqslant -\alpha_2$ and $r+s \geqslant -\alpha_2$.

By 15·4 $\Delta^{-\alpha_2} = U_1^{-1} \cdot U_2^{-1} \cdots U_k^{-1}$ and the most general way of taking Δ^r and Δ^s would be $\Delta^r = (U_1^{-1} \cdot U_2^{-1} \cdot U_l^{-1} U_{\rho_1} \cdot U_{\rho_2} \cdots U_{\rho_r})$ and $\Delta^s = (U_{q_1} U_{q_2} \cdots U_{q_s} \cdot U_{l+1}^{-1} \cdot U_{l+2}^{-1} \cdots U_k^{-1})$.

If $\| \beta_{n,n+p} \|$ defines $(U_1^{-1} \cdots U_k^{-1})$ then $\beta_{n,n+p} = b_p =$

$$\frac{\alpha_2(\alpha_2+1)\cdots(\alpha_2+p-1)}{\underline{|p}} = O(p^{\alpha_2 - 1})$$

and
$$\sum_{p=0}^{n} b_p = B_n = O(n^{\alpha_2})$$

Let $\alpha_2(1+\delta) = \alpha$; since $\alpha_2 < \alpha$ $\delta > 0$;

By hypothesis
$$x_n = O\left(\frac{1}{n^\alpha}\right) = O\left(\frac{1}{B_n^{1+\delta}}\right).$$

Hence by (8·20) of Theorem VII of Part II

$$\Delta^s[\Delta^r(x_n)] \equiv (U_{q_1} U_{q_2} \cdot U_{q_s} \cdot U_{l+1}^{-1} \cdots U_k^{-1})[(U_1^{-1} \cdots U_l^{-1} U_{\rho_1} \cdots U_{\rho_r}(x_n)]$$
$$= (U_{q_1} \cdots U_{q_r} \cdot U_{l+1}^{-1} \cdots U_k^{-1} \cdots U_1^{-1} \cdot U_l^{-1} U_{\rho_1} \cdots U_{\rho_r})(x_n) = \Delta^{r+s}(x_n).$$

Proof of (a_2') : $s = -\alpha_2 - q$ $0 \leqslant q \leqslant 1$ $r = q + t$ $t \geqslant 0$

since $x_n = O\left(\frac{1}{n^\alpha}\right)$ as above $x_n = O\left(\frac{1}{B_n^{1+\delta}}\right) = 0\left(\frac{1}{B_n}\right)$.

Let $\Delta^{-a_2} = U_1^{-1} \cdots U_k^{-1}.$

$\Delta^q = U(q)$

then $\Delta^s = (U_1^{-1} \cdots U_k^{-1})\,[U(q)]^{-1}$

$\Delta^r = U(q)\, U_{\rho_1} \cdot U_{\rho_2} \cdots U_{\rho_r}.$

If $\| \bar{\beta}_{n,n+p} \|$ defines $(U_1^{-1} \cdots U_k^{-1})$ and $b_\rho = \beta_{n,n+p}$ and $B_n = \sum_{\rho>0}^{n} b_\rho$

since by hypothesis $x_n = 0\left(\frac{1}{B_n}\right)$ we have by (8·30) and (8·31) of Theorem IX of Part II

$$\begin{aligned}\Delta^s[\Delta^r(x_n)] &\equiv [U_1^{-1} \cdots U_k^{-1}\{U(q)\}^{-1}]\,(U(q)\,U_{\rho_1}\,U_{\rho_2} \cdots U_{\rho_r})\,(x_n)\\ &= (U_1^{-1} \cdots U_k^{-1} \cdot U_{\rho_1}\,U_{\rho_r})\,(x_n)\\ &= \Delta^{r+s}(x_n) \text{ when the latter exists.}\end{aligned}$$

But the latter exists by Theorem VII since $x_n = 0\left(\frac{1}{B_n^{1+\delta}}\right)$, $\delta > 0$

Hence (a_2') is established.

Proof of (b'): $s = -1-a \quad r = 1+t \quad t > 0 \quad x_n = 0\left(\frac{1}{n^a}\right)$

Let $\Delta^1 = U(1)$, $\Delta^{-a} = (U_1^{-1} \cdots U_k^{-1})$

then if $\| \bar{\beta}_{mn} \|$ defines $(U_1^{-1} \cdots U_k^{-1})$ $\bar{\beta}_{n,n+p} = b_p$

$$= \frac{a(a+1)(a+p-1)}{\lfloor p} = 0\,(p^{a-1})$$

$$\text{and } B_n = \sum_{\rho=0}^{n} b_\rho = 0\,(n^a).$$

We therefore have $\Delta^r = U(1) \cdot U_{\rho_1}\,U_{\rho_2} \cdots U_{\rho_r}.$

$\Delta^s = [U(1)]^{-1}\,U_1^{-1}\,U_k^{-1}$

and since $x_n = 0\left(\frac{1}{B_n}\right)$

$$\Delta^s[\Delta^r(x_n)] = [\{U(1)\}^{-1} \cdot U_1^{-1} \cdots U_k^{-1}]\,[U(1) \cdot U_{\rho_1} \cdots U_{\rho_r}]$$

$= (U_1^{-1} \cdots U_k^{-1} \cdot U_{\rho_1} \cdots U_{\rho_r})\,(x_n) = \Delta^{r+s}(x_n)$ if the latter exists by (8·30) and (8·31) of Theorem IX of Part II.

But the latter exists since $x_n = 0\left(\frac{1}{n^a}\right)$ and $r+s > -a$ by Theorem VII.

Proof of (c_2'): In this case argument is identical with that of (b') except in the last step where it must be noted that the equality will be valid if $\Delta^{-a} x_n$ exists.

Proof of (c'_1): $r = -a_3 \quad s = -a_4 \quad a_3 + a_4 = a \quad a_3 > 0 \quad a_4 > 0 \quad x_n = 0\left(\frac{1}{n^a}\right)$

Let $\| \beta_{n,\rho} \|$ define $\Delta^{-\alpha_3}$. We shall prove that

$$\left| \sum_{\rho = A}^{\infty} \beta_{n,\rho} \mid x_\rho \mid \right| = 0\left(\frac{1}{A^{\alpha_4}}\right)$$

Proof: $\beta_{n,n+\rho} = b_\rho = 0\,(p^{\alpha_3 - 1})$ and $\sum^{n} b_\rho = B_n = O\,(n^{\alpha_3})$

and $x_n = \dfrac{\epsilon\,(n)}{n^\alpha} = \dfrac{\epsilon\,(n)}{n^{\alpha_3(1+\delta)}} = \dfrac{\epsilon\,(n)}{B_n^{1+\delta}}$ where $\epsilon\,(n) \to 0$ as $n \to \infty$.

Hence by result (1) of Theorem VII.

$$\left| \sum_{\rho = A}^{\infty} \beta_{n,\rho} \,| x_\rho | \right| \leqslant \frac{\epsilon^1\,(A)}{B_A^{\delta}} = \frac{\epsilon'\,(A)}{A^{\alpha_4}} = 0\left(\frac{1}{A^{\alpha_4}}\right).$$

If $\| \beta'_{n,\rho} \|$ defines $\Delta^{-\alpha_4}$ then $\beta'_{n,n+p} = b_\rho{}' = O\,(p^{\alpha_4 - 1})$

and $\sum^{n} b_\rho{}' = B_n{}' = O\,(n^{\alpha_4})$.

Hence $\left| \sum_{A}^{\infty} \beta_{n,\rho} \,| x_\rho | \right| = 0\left(\frac{1}{B_A{}'}\right)$ uniformly for all n.

Hence the conditions of Theorem VI are fulfilled and we have by the same theorem

$$\Delta^{-\alpha_4}\,[\Delta^{-\alpha_3}\,(x_n)] = \Delta^{-\alpha}\,(x_n) \text{ when the latter exists.}$$

§ 17. *Generalization of some Theorems of Knopp.*

His results in the paper (K. 1) are as follows:

Given $x_n > 0$ and $x_n = 0\,(1)$ then

I. If $\Delta^\alpha\,(x_n) \geqslant 0$ for all n, then $\Delta^\beta\,(x_n) \geqslant 0$ for $0 \leqslant \beta \leqslant \alpha$. (Satze 6 of his paper).

II. If $\alpha \geqslant 1$ and $0 \leqslant \beta \leqslant \alpha - 1$ and $x_n > 0$ $x_n = 0$ (1) and $\Delta^\alpha\,(x_n) > 0$ for all n

then $\Delta^\beta\,(x_n) = 0\left(\frac{1}{n^\beta}\right)$ [Satze 9 of his paper]

and two particular cases of II are also given as Satze 7 and Satze 8 of his paper.

It will be shewn that,

if $\alpha > 0$ $(x_n) = y_n$ $y_n > 0$

then for all $0 \leqslant \beta \leqslant \alpha$ $\Delta^\beta(x_n) = \Delta^{-(\alpha-\beta)}\,(y_n)$ [17·1]

and in particular $x_n = \Delta^{-\alpha}\,y_n$, is an immediate consequence of Theorem IV of Part I.

In particular $x_0 = \Delta^{-\alpha} y_0 = \Sigma\, A_n^{\alpha-1}\,\Delta^\alpha\,(x_n) = \Sigma\, A_n^{\alpha-1}\,|\,\Delta^\alpha\,(x_n)\,|$. Hence the conditions of Hjaelpsoetning III. B of Anderson on page 34 of his book (A·1) are satisfied and result II of Knopp follows at once as a particular case of the theorem of Anderson. It is rather remarkable that Knopp has not noticed this. We will here give generalizations of results I and II applicable to U's.

THEOREM I : Let $S = U_1 U_2 \cdot U_3 \cdot U_k$ and $S(x_n) = y_n$

and $S' = U_{r+1} \cdot U_{r+2} \cdots U_k$ and $S'(x_n) = z_n$

If $x_n = 0\ (1)$ and $y_n > 0$ for all n,

then $S'(x_n) = (U_1^{-1} U_2^{-1} \cdots U_r^{-1})(y_n)$

and $S'(x_n) > 0$ for all n [17·2]

Proof:
$$y_n = S(x_n) = (U_1 U_2 U_r)(U_{r+1} \cdots U_k)(x_n)$$
$$= (U_1 U_2 U_r)\{S'(x_n)\} \text{ by Theorem II of Part I}$$
$$= (U_1 U_2 U_r)(z_n).$$

Also since $x_n = 0\ (1)$ so is $z_n = 0\ (1)$ by Theorem II of Part I.

Hence by repeated application of Theorem IV of Part I just as in the corollary to it.

$$z_n = (U_r)^{-1} \cdot (U_{r-1})^{-1} \cdots (U_1)^{-1}(y_n)$$
$$= \Sigma\, \beta^r_{n,\, n+p_1} \cdot \Sigma\, \beta^{r-1}_{n+p_1,\, n+p_2} \cdots \Sigma\, \beta'_{n+p_{r-1},\, n+p}\, y_{n+p} \qquad [17 \cdot 3]$$

where $\| \beta^r_{m,n} \|$ defines $(U_r)^{-1}$. The multiple series on the right can be summed up in any manner since $\beta_{m,n} \geqslant 0$ and $y_n \geqslant 0$.

$$\therefore \quad z_n = (\Sigma\, \beta^r_{n,n+p_1} \cdot \beta^{r-1}_{n+p_1 \cdot n+p_2} \cdots \beta'_{n+p_{r-1},\, n+p}) \cdot y_{n+p}$$
$$= \sum_{p=0}^{\infty} \bar{\beta}_{n+p}\, y_{n+p}$$

where $\| \bar{\beta}_{m,n} \| = \| \beta^r_{m,n} \| \cdot \| \beta^{r-1}_{m,n} \| \cdots \| \beta'_{m,n} \|$.

Hence $z_n = (U_1^{-1} \cdot U_2^{-1} \cdots U_r^{-1})(y_n) = (U_{r+1} \cdots U_k)(x_n)$

since $y_n > 0$ and all $\bar{\beta}_{mn} \geqslant 0$ we have $z_n > 0$

and in particular we have $x_n^r = (U_1^{-1} \cdots U_k^{-1})(y_n)$, when $r = k$. [17·4]

When the U's are Δ's we have as a deduction from above the result of (17·1) namely :—If $\Delta^\alpha(x_n) = y_n > 0$ for all n $\alpha > 0$

then $\Delta^\beta (x_n) = \Delta^{-(\alpha-\beta)}\, y_n$

and in particular $x_n = \Delta^{-\alpha}\, y_n$.

THEOREM II : Let $U(1) \equiv \Delta^1$ and $S = [U(1) \cdot U_1 U_2 U_3 U_k]$

$S' = U_1 U_2 \cdots U_k$.

Let $\| \bar{\beta}_{m,n} \|$ define $U_1^{-1} U_2^{-1} \cdots U_k^{-1}$

Now $\bar{\beta}_{n,n+p} = b_p$. Let $\sum_0^n b_\rho = B_n$

then if

$x_n = 0\ (1)$ and $S(x_n) = y_n$ be > 0 for all n, then $S'(x_n) = 0\left(\frac{1}{B_n}\right)$. [17·5]

Proof: By 17·4 $x_n = [\{U(1)\}^{-1} \cdot U_1^{-1} \cdot U_2^{-1} \cdots U_k^{-1}](y_n)$

$$= \{U(1)\}^{-1} \cdot (U_1^{-1} \cdots U_k^{-1})(y_n)$$

$$= U(1)^{-1} \cdot (\Sigma\, \beta_{n,\, n+p}\, y_{n+p})$$

$$= U(1)^{-1} \left(\sum_0^\infty b_p \cdot y_{n+p}\right)$$

$$= \sum_0^\infty B_p\, y_{n+p}$$

and
$$x_0 = \sum_0^\infty B_p\, y_p$$

and
$$S'(x_n) = [U(1)]^{-1}(y_n) = \sum_n^\infty y_p.$$

Now
$$\sum_n^\infty y_\rho = \sum_n^\infty \frac{B_\rho\, y_\rho}{B_\rho} \leqslant \frac{1}{B_\rho} \cdot \Sigma\, B_\rho \cdot y_\rho = 0\left(\frac{1}{B_n}\right)$$

since $\Sigma\, B_\rho\, y_\rho$ converges and $B_0 \leqslant B_1 \leqslant \cdots \leqslant B_n \leqslant \cdots$

Hence
$$S'(x_n) = 0\left(\frac{1}{B_n}\right).$$
Thus proving (17·5).

Result II of Knopp follows immediately from this

for let $U(1) = \Delta^1 \quad U_1\, U_2\, U_k = \Delta^\alpha \quad \alpha > 0$

If $\Delta^{1+\alpha}(x_n) > 0$ for all n

then
$$\Delta^\alpha x_n = 0\left(\frac{1}{n^\alpha}\right)$$

for B_n in this case $= O(n^\alpha)$.

Knopp's Satze 7 is an immediate consequence of this. His Satze 8 takes the following interesting form in terms of U's.

If $U(x_n) = y_n$ be > 0 for all n, $x_n > 0$ and $x_n = 0(1)$ and $\sum_0^\infty x_n$ convergent, then,

$B_n\, x_n \rightarrow 0$ as $n \rightarrow \infty$ where $\{b_n\}$ defines U^{-1} as in (14·3) and $B_n = \sum_{\rho=0}^{n} b_\rho$. Putting $r_n = x_n + x_{n+1} + \cdots$

we have $\Delta\, r_n = x_n = U(1)(r_n)$.

Hence $[U(1) \cdot U](r_n) = y_n > 0$ for all n.

Hence
$$U(r_n) = 0\left(\frac{1}{B_n}\right) \text{ by } (17\cdot5).$$

But $U(r_n) = \sum_n^\infty x_\rho + a_1 \sum_{n+1}^\infty x_\rho + a_2 \sum_{n+2}^\infty x_\rho + \cdots$, where $\{a_n\}$ defines U

The right-hand side is an absolutely convergent double series since

$$\sum_{1}^{\infty} | a_n | = - \sum_{1}^{\infty} a_n \leqslant 1 \text{ and } \Sigma\, x_n \text{ is convergent.}$$

Hence $\quad U(r_n) = x_n + x_{n+1}(1 + a_1) + x_{n+2}(1 + a_1 + a_2) + \cdots$

since $1 + a_1 + a_2 + a_n \geqslant 0$ by condition (d') of 14·1

we have $x_n < U(r_n)$

Hence
$$x_n = 0\left(\frac{1}{B_n}\right). \qquad [17 \cdot 6]$$

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