ON A PROBLEM RELATED TO THE CAUCHY-MACLAURIN INTEGRAL TEST

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Received February 8, 1939

1. Given: $f(x) > 0$ and $f(x) \to 0$ as $x \to \infty$; $f'(x) \leqslant 0$, $f''(x) \geqslant 0$; let $r(x)$ be defined by $r(x) = \sum_{n=0}^{\infty} \left\{ \int_{x+n}^{x+n+1} f(t)\,dt - f(x+n+1) \right\}$. It is the purpose of this note to discuss the behaviour of $\frac{r(x)}{f(x)}$ and its relation to the value of $\frac{f'(x)}{f(x)}$ for large x.

2. Theorem I: $\frac{1}{2} f(x+1) \leqslant r(x) \leqslant \frac{1}{2} f(x)$.
Since $f''(x) \geqslant 0$, for $x \leqslant t \leqslant x+1$ we shall have

$$\frac{f(x+1) - f(x)}{1} \leqslant \frac{f(t) - f(x+1)}{t - (x+1)} \leqslant \frac{f(x+2) - f(x+1)}{+1}$$

$$\therefore \quad \int_x^{x+1} \{f(t) - f(x+1)\}\,dt \leqslant [f(x) - f(x+1)] \int_x^{x+1} (x+1-t)\,dt$$
$$= \tfrac{1}{2} \{f(x) - f(x+1)\},$$

and similarly $\quad \int_x^{x+1} \{f(t) - f(x+1)\}\,dt \geqslant \frac{1}{2} \{f(x+1) - f(x+2)\}$.

Therefore, $$\tfrac{1}{2} f(x) \geqslant r(x) \geqslant \tfrac{1}{2} f(x+1) \tag{A}$$

Corollary: $\sum^{\infty} r(x+n)$ and $\sum^{\infty} f(x+n)$ converge or diverge together.

3. Theorem II: Besides the assumptions on $f(x)$ given above, let $\frac{f'}{f} \to -\theta \quad (\theta > 0)$.

Then, $$\frac{r(x)}{f(x)} \to \frac{1}{\theta} - \frac{1}{e^{\theta} - 1} = \lambda, \text{ say, as } x \to \infty$$

Proof: Since $\frac{f'}{f} \to -\theta$ as $x \to \infty$, $\dfrac{\int_x^{x+1} f(t)\,dt}{-\int_x^{x+1} f'(t)\,dt} = \dfrac{\int_x^{x+1} f(t)\,dt}{f(x) - f(x+1)}$

$$= \frac{1}{\theta + \epsilon}$$

and

$$\frac{f(x+1)}{f(x)} = e^{-(\theta + \epsilon')}, \text{ so that } \frac{f(x+1)}{f(x) - f(x+1)} = \frac{1}{e^{\theta + \epsilon'} - 1}.$$

Hence

$$\frac{\int_x^{x+1} f(t)\,dt - f(x+1)}{f(x) - f(x+1)} = \frac{1}{\theta+\epsilon} - \frac{1}{e^{\theta+\epsilon'}-1} = \frac{1}{\theta} - \frac{1}{e^{\theta}-1} + \epsilon_x'' \qquad \text{(B)}$$

$$\therefore \quad r(x) = \sum_{n=0}^{\infty} \left\{ \int_{x+n}^{x+n+1} f(t)\,dt - f(x+n+1) \right\}$$

$$= \sum_{n=0}^{\infty} (\lambda + \epsilon''_{x+n}) \{ f(x+n) - f(x+n+1) \}.$$

$$= (\lambda + \epsilon_x''') f(x)$$

whence $$\frac{r(x)}{f(x)} \to \lambda \text{ as } x \to \infty.$$

If $\theta = 0$, $\quad \dfrac{f(x+1)}{f(x)} \to 1$ as $x \to \infty$,

Hence from (A), $\quad \dfrac{r(x)}{f(x)} \to \frac{1}{2}$ as $x \to \infty$.

If $\theta = \infty$, it is obvious from (B) that $\dfrac{r(x)}{f(x)} \to 0$ as $x \to \infty$.

4. *The Converse of Theorem II ; an inequality.*

Suppose it is given that $\dfrac{r(x)}{f(x)} \to \lambda$ as $x \to \infty$, where $0 \leqslant \lambda \leqslant \frac{1}{2}$, the assumptions on f, f' and f'' being the same as in § (1),

then $$r(x) - r(x+1) = \int_x^{x+1} f(t)\,dt - f(x+1) = \lambda \{ f(x) - f(x+1) \}$$
$$+ \epsilon_x f(x) - \epsilon_{x+1} f(x+1).$$

Now, since $f''(x) \geqslant 0$,

$$f(x+\tfrac{1}{2}) \leqslant \int_x^{x+1} f(t)\,dt = \lambda f(x) + \mu f(x+1) + \epsilon_x f(x) - \epsilon_{x+1} f(x+1)$$

$$\therefore \quad f(x+\tfrac{1}{2}) - f(x+1) \leqslant \lambda \{ f(x) - f(x+1) \} + \epsilon_x f(x) - \epsilon_{x+1} f(x+1)$$

$$f(x+1) - f(x+\tfrac{3}{2}) \leqslant \lambda \{ f(x+\tfrac{1}{2}) - f(x+\tfrac{3}{2}) \} + \epsilon_{x+\frac{1}{2}} f(x+\tfrac{1}{2})$$
$$- \epsilon_{x+} \ f(x+\tfrac{3}{2})$$

$$\dots\dots\dots\dots\dots, \text{etc.}$$

Hence $$f(x+\tfrac{1}{2}) \leqslant \lambda \{ f(x) + f(x+\tfrac{1}{2}) \} + \epsilon_x f(x) + \epsilon_{x+\frac{1}{2}} f(x+\tfrac{1}{2})$$

and $$f(x+\tfrac{1}{2}) \leqslant \left(\frac{\lambda}{\mu} + \epsilon_x' \right) f(x). \qquad \text{(C)}$$

Assume $\lambda \neq 0$. Then, since $f''(x) \geqslant 0$, if A is $\{x, f(x)\}$, P is $\{x+\lambda, f(x+\lambda)\}$ and B is $\{x+1, f(x+1)\}$, the curve composed of the chords AP, PB lies above the curve $y = f(x)$.

Hence $\lambda\cdot\dfrac{f(x)+f(x+\lambda)}{2}+\mu\dfrac{f(x+\lambda)+f(x+1)}{2}\geqslant\displaystyle\int_x^{x+1}f(t)\,dt$

$$=\lambda f(x)+\mu f(x+1)+\epsilon_x f(x)-\epsilon_{x+1}f(x+1)$$

$$\therefore\quad f(x+\lambda)\geqslant\lambda f(x)+\mu f(x+1)+2\,\epsilon_x f(x)-2\,\epsilon_{x+1}f(x+1)\qquad\text{(D)}$$

Also $\qquad f(x+\lambda)\geqslant(\lambda-\epsilon)\,f(x)$

and $\qquad \dfrac{f(x+1)}{f(x)}\geqslant(\lambda-\epsilon)^{\left[\frac{1}{\lambda}\right]+1}=\lambda^{\left[\frac{1}{\lambda}\right]+1}-\epsilon'=\mathrm{K}_\lambda-\epsilon$

Hence, when $\lambda\neq 0$,

$$\left(\frac{\lambda}{\mu}\right)^2+\epsilon'\geqslant\frac{f(x+1)}{f(x)}\geqslant\mathrm{K}_\lambda-\epsilon\qquad\text{(E)}$$

From this it is easy to deduce that $-\dfrac{f'}{f}$ oscillates, if at all, finitely, between two positive values for large x. There are two interesting cases when $\lambda=0$ and $\lambda=\frac{1}{2}$, where we can prove that $\dfrac{f'}{f}$ definitely converges as $x\to\infty$.

THEOREM III : *If, as* $x\to\infty$, $\dfrac{r(x)}{f(x)}\to 0$, *then* $\dfrac{-f'}{f}\to\infty$, *and if* $\dfrac{r(x)}{f(x)}\to\frac{1}{2}$ *then* $\dfrac{-f'}{f}\to 0$ *(under the same assumptions about f, f' and f'' as before).*

Proof: (i) $\lambda=0$.

From (C) or (A), $\qquad \dfrac{f(x+1)}{f(x)}=\epsilon_x$

$$\therefore\quad r(x)=\int_x^\infty f(t)\,dt-\sum_{n=1}^{\infty}f(x+n)=\int_x^\infty f(t)\,dt-\epsilon_x' f(x)$$

therefore $\qquad \displaystyle\int_x^\infty f(t)\,dt=\epsilon_x'' f(x).$

Now the area between the X-axis, the lines $\mathrm{X}=x$ and the tangent $\mathrm{Y}-f(x)=f'(x)(\mathrm{X}-x)$

is $\qquad -\dfrac{f^2(x)}{2f'(x)}<\displaystyle\int_x^\infty f(t)\,dt=\epsilon_x''\,f(x).$

Hence $\qquad \dfrac{f}{f'}\to 0$, as $x\to\infty$.

(ii) $\lambda=\frac{1}{2}$.

From (D) we have

$$f(x+\tfrac{1}{2})\geqslant\tfrac{1}{2}\{f(x)+f(x+1)\}+2\,\epsilon_x f(x)-2\,\epsilon_{x+1}f(x+1)$$

$$f(x+1)\geqslant\tfrac{1}{2}\{f(x+\tfrac{1}{2})+f(x+\tfrac{3}{2})\}+2\,\epsilon_{x+\frac{1}{2}}f(x+\tfrac{1}{2})-2\,\epsilon_{x+\frac{3}{2}}f(x+\tfrac{3}{2})$$

$$\cdots\cdots\cdots\cdots,\ \text{etc.}$$

Writing these in the form

$$f(x+\tfrac{1}{2})-\tfrac{1}{2}\{f(x)+f(x+1)\}\geqslant 2\,\epsilon_x f(x)-2\,\epsilon_{x+1}f(x+1), \text{ etc.}$$

and adding, we get $-\{f(x)-f(x+\frac{1}{2})\}\geqslant 4\{\epsilon_x f(x)+\epsilon_{x+\frac{1}{2}}f(x+\frac{1}{2})\}$

[It is to be noticed that, from (A), all the ϵ's are negative.]

Hence $$\epsilon_x' \geqslant \frac{f(x)-f(x+\frac{1}{2})}{f(x)}\geqslant 0$$

i.e., $$\frac{f(x+\frac{1}{2})}{f(x)}\to 1 \quad \text{as } x\to\infty.$$

Also $$2\epsilon_x' \geqslant \left\{\frac{f_0(x)-f(x+\frac{1}{2})}{\frac{1}{2}f(x)}\right\}=\frac{-f'\left(x+\frac{\theta}{2}\right)}{f(x)}\geqslant\frac{-f'(x+\frac{1}{2})}{f(x)}$$
$$=\frac{-f'(x+\frac{1}{2})}{f(x+\frac{1}{2})}(1+\epsilon_x).$$

Hence $$\frac{-f'}{f}\to 0 \text{ as } x\to\infty.$$

5. The problem of the asymptotic behaviour of $\frac{f'}{f}$ when $\frac{r(x)}{f(x)}\to\lambda$, as $x\to\infty$, has been partially solved in § 4. It has been there shown that, in two particular cases, *viz.*, $\lambda=0$ and $\lambda=\frac{1}{2}$, the behaviour of $\frac{f'}{f}$ is definite. This raises the interesting problem : *Under the restrictions on $f(x)$ given above, let $\frac{r(x)}{f(x)}\to\lambda$, as $x\to\infty$, where $0<\lambda<\frac{1}{2}$. Then, does $\frac{f'}{f}$ necessarily tend to a definite limit as $x\to\infty$?*