A NEW PROOF OF THE FORMULA FOR THE GENERATING FUNCTION OF LAGUERRE POLYNOMIALS AND OTHER RELATED FORMULÆ

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1. Let $\phi_n(x) = \dfrac{e^x D^n (e^{-x} \cdot x^n)}{|n}$. The main object of this note is to give a new proof of the formula

$$\sum_{n=0}^{\infty} t^n \phi_n(x) . \phi_n(y) = \frac{e^{\frac{-(x+y)t}{1-t}}}{1-t} I_0\left(\frac{2\sqrt{xyt}}{1-t}\right) \quad \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

Professor Watson, in his note* on Laguerre functions in (*J.L.M.S.*, Vol. 8, 1933), after pointing out that (1) Hille's (general case) and Wigert's (my case) demonstration of the formula involved thI use of Infinite integrals containing Bessel functions and that (2) Hardy's proof of the general case was by use of Mellin's inversion formula, gives his own proof which involves Saalschutz's formula in the theory of generalized hypergeometric functions. My proof of (A) is elementary as it involves no knowledge of special functions or transformations. It involves only (i) a very elementary knowledge of functions of a complex variable, (ii) recurrence formula for ϕ_n and ϕ_n', and (iii) elementary Analysis for justifying term by term differentiation. Besides the proof of (A), some other interesting formulæ that arise from parts of the proof of (A) are also given.

2. Let ϕ_n be as in 1. Then, I prove the following:

(A) Formula (A) of 1.

(B) $\displaystyle \operatorname*{Lim}_{n \to \infty} \frac{\sum_{1}^{n} \phi_r^2}{\sqrt{n}} = \frac{e^x}{\pi \sqrt{x}}$.

(C) $\displaystyle \int_0^x \phi_n^2 \, dx = O\left(\frac{\sqrt{x} \cdot e^x}{\sqrt{n} + 1}\right)$.

* [Notes on Generating Functions of Polynomials, "Laguerre Polynomials," *Jour. Lond. Math. Soc.*, 1933, 8, Part 3.]

3. We shall here prove the particular case of (A) when $y = x$.

It is known that $\dfrac{e^{\frac{-xt}{1-t}}}{1-t} = \Sigma\, t^n \phi_n(x)$ for $|t| < 1$.

By integrating on $|t| = r < 1$, we obtain

$$\Sigma\, \phi_n^2\, r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{e^{\frac{-xt}{1-t}}}{1-t}\right|^2 d\theta = \frac{e^x}{\pi}\int_0^{\pi} \frac{e^{\frac{-x(1-r^2)}{1+r^2-2r\cos\theta}}}{1+r^2-2r\cos\theta}\, d\theta = \mathrm{I}\cdot\frac{e^x}{\pi} \cdot (3\cdot1)$$

Putting $\dfrac{1-r^2}{1+r^2-2r\cos\theta} = \dfrac{1+r^2}{1-r^2} + \dfrac{2r\sin\phi}{1-r^2}$ or $\cos\theta = \dfrac{(1+r^2)\sin\phi+2r}{1+r^2-2r\sin\phi}$,

$$\text{I becomes } \frac{e^{\frac{-x(1+r^2)}{1-r^2}}}{1-r^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\frac{-2rx}{1-r^2}\sin\phi}\, d\phi = \frac{\pi}{1-r^2}\, e^{\frac{-x(1+r^2)}{(1-r^2)}}\, \mathrm{I}_0\left(\frac{2rx}{1-r^2}\right) \quad (3\cdot2)$$

where $\mathrm{I}_0(z) = \mathrm{J}_0(iz)$

$$\text{Hence } \Sigma\, \phi_n^2\, r^{2n} = \frac{e^{\frac{-2r^2}{1-r^2}x}}{1-r^2}\, \mathrm{I}_0\left(\frac{2rx}{1-r^2}\right) \quad (3\cdot4)$$

Putting $r^2 = t$ we obtain the particular case of (A) when $x = y$.

4. We shall here prove two lemmas.

Lemma 1. If $\beta_n(x, y) = x\,\phi_n'(x).\,\phi_n(y) - y\,\phi_n'(y)\cdot\phi_n(x)$, then

$$\beta_n - \beta_{n-1} + (x-y)\,\phi_{n-1}(x).\,\phi_{n-1}(y) = 0 \quad (4\cdot1)$$

Since

$$x\,\phi_n'(x) = n\,(\phi_n - \phi_{n-1})$$

$$\beta_n(x, y) = n\,\{\phi_n(x)\,\phi_{n-1}(y) - \phi_n(y)\,\phi_{n-1}(x)\}$$

and

$$\beta_n - \beta_{n-1} = \phi_{n-1}(y)\,\{n\cdot\phi_n(x) + (n-1)\,\phi_{n-2}(x)\} - \phi_{n-1}(x)\,(n\cdot\phi_n(y) + (n-1)\,\phi_{n-2}(y)\} \quad (4\cdot2)$$

Since $n\phi_n + (n-2)\,\phi_{n-2} = (2n-1-x)\,\phi_{n-1}$

$\beta_n - \beta_{n-1} = -(x-y)\cdot\phi_{n-1}(x)\cdot\phi_{n-1}(y)$, thus proving 4·1 (4·3)

Lemma 2. If $\mathrm{F}(x, y) = e^{\frac{r^2}{1-r^2}(x+y)}\, \Sigma\, \phi_n(x)\cdot\phi_n(y)\cdot r^{2n}$, then

F (x, y) is a function of (xy) (4·4)

$$\textit{Proof}: \quad \frac{\partial \mathrm{F}}{\partial x} = e^{\frac{r^2}{1-r^2}(x+y)}\, \Sigma\, \phi_n'(x)\cdot\phi_n(y)\, r^{2n} + \frac{r^2}{1-r^2}\,\mathrm{F}(xy) \quad (4\cdot5)$$

and a similar formula for $\dfrac{\partial \mathrm{F}}{\partial y}$.

And $$x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} = e^{\frac{r^2}{1-r^2}(x+y)} \Sigma\, \beta_n(x, y) r^{2n} + \frac{(x-y)}{1-r^2} \cdot r^2 F(x, y) \qquad (4\cdot6)$$

$$= \frac{e^{\frac{r^2}{1-r^2}(x+y)}}{1-r^2}\{\Sigma\, [\beta_n - \beta_{n-1} + (x-y)\,\phi_{n-1}(x)\,\phi_{n-1}(y)]\, r^{2n}\}$$

$$= 0 \text{ by Lemma 1.} \qquad (4\cdot7)$$

Hence $F(x, y)$ is a function of xy (4·8)

[*Note.*—Term by term differentiation is justified since the differentiated series is uniformly convergent in any finite range of x or y for $r < 1$ since $|\phi_n| \leqslant e^{x/2}$ and $\phi'_n(x) \leqslant n\, e^{\frac{x}{2}}$.]

5. We shall now prove (A).

By Lemma 2 $$F(x, y) = F(xy) \qquad (5\cdot1)$$

By (3·4) $$F(x^2) = \frac{I_0\left(\frac{2rx}{1-r^2}\right)}{1-r^2} \qquad (5\cdot2)$$

Hence $$F(x, y) = \frac{I_0\left(\frac{2r}{1-r^2}\sqrt{xy}\right)}{1-r^2} \qquad (5\cdot3)$$

Hence $$\Sigma\, \phi_n(x)\cdot\phi_n(y) r^{2n} = e^{\frac{-r^2(x+y)}{1-r^2}}\, I_0\left(\frac{2r}{1-r^2}\sqrt{xy}\right) \qquad (5\cdot4)$$

and putting $r^2 = t$ we obtain (A).

6. We shall prove (B).

Putting $t = \dfrac{1-r^2}{1+r^2-2r\cos\theta}$ the integral I, of (3·1), namely,

$$\int_0^{\pi} \frac{e^{\frac{-x(1-r^2)}{1+r^2-2r\cos\theta}}}{1+r^2-2r\cos\theta}\, d\theta, \text{ it becomes } \frac{1}{1-r^2} I_2, \text{ where}$$

$$I_2 = \int_{\frac{1}{a}}^{a} \frac{e^{-xt}}{\sqrt{\left(t-\frac{1}{a}\right)(a-t)}}\, dt, \; a \text{ being } = \frac{1+r}{1-r} \qquad (6\cdot1)$$

Putting $t = \dfrac{1}{a} + \dfrac{v^2}{x}$

$$I_2 = 2\, e^{\frac{-x}{a}} \int^{\sqrt{bx}} \frac{e^{-v^2}}{\sqrt{bx - v^2}}\, dv = 2e^{\frac{-x}{a}} I_3, \text{ where } b = \frac{4r}{1-r^2} \qquad (6\cdot2)$$

and $I_3 = \int_0^{(bx)^{\frac{1}{4}}} + \int_{(bx)^{\frac{1}{4}}}^{(bx)^{\frac{1}{2}}} = I_4 + I_5$ (say).

$$I_4 = \frac{1}{\sqrt{(bx) - \theta^2 (bx)^{\frac{1}{2}}}} \int_0^{(bx)^{\frac{1}{4}}} e^{-v^2}\, dv = \frac{\left(\frac{\sqrt{\pi}}{2} + \epsilon\right)}{\sqrt{bx}} \text{ for large } bx. \tag{6·3}$$

$$I_5 < e^{-(bx)^{\frac{1}{2}}} \int_0^{\sqrt{bx}} \frac{dv}{\sqrt{bx - v^2}} = \frac{\pi}{2}\; e^{-(bx)^{\frac{1}{2}}} \tag{6·4}$$

$$\text{Hence } I_2 = \frac{(\sqrt{\pi} + \epsilon)}{\sqrt{\dfrac{4x}{1 - r^2}}} \quad \text{where } (1 - r) \text{ is small and } x \text{ is fixed} \tag{6·5}$$

$$\text{and} \quad I = \frac{\sqrt{\pi}}{2\sqrt{x}\,(1 - r^2)} \quad (1 + \epsilon) \tag{6·6}$$

$$\text{and from (3·1)} \quad \Sigma\, \phi_n^2\, r^{2n} \sim \frac{e^x}{2\sqrt{\pi x}\,(1 - r^2)} \qquad r \to 1 - 0. \tag{6·7}$$

Hence by the well-known Tauberian theorem of Littlewood we obtain

$$\lim_{n \to \infty} \frac{\sum^{n} \phi_n^2}{\sqrt{n}} = \frac{e^x}{2\sqrt{\pi}\; x\; \big|\frac{3}{2}} = \frac{e^x}{\pi\sqrt{x}} \tag{6·8}$$

thus proving (B).

7. *Proof of (C)*:

Since $\frac{x}{n} \cdot \frac{d}{dx} \phi_n = \phi_n - \phi_{n-1}$,

$$\frac{x}{n} \cdot \frac{d}{dx} \phi_n^2 = 2\,\phi_n (\phi_n - \phi_{n-1}) = \phi_n^2 - \phi_{n-1}^2 + (\phi_n - \phi_{n-1})^2. \tag{7·1}$$

Integrating in (o, x) we obtain,

$$\frac{x}{n} \cdot \phi_n^2 = \left(1 + \frac{1}{n}\right) \int_0^x \phi_n^2\, dx - \int_0^x \phi_n^2\, dx + \int_0^x (\phi_n - \phi_{n-1})^2\, dx \tag{7·2}$$

Let $I_n = \int_0^x \phi_n^2\; dx$. Then,

$$(n + 1)\, I_n - n\, I_{n-1} < x\, \phi_n^2 \tag{7·3}$$

Hence

$$(n+1)\,\mathrm{I}_n < \mathrm{I}_0 + x \sum_1^n \phi_n{}^2 < x + \frac{e^x}{\pi} \sqrt{x}\,(1+\epsilon_n)\,\sqrt{n} \tag{7·4}$$

Hence

$$\mathrm{I}_n = 0\left(\frac{\sqrt{x}\cdot e^x}{\sqrt{n}+1}\right) + 0\left(\frac{x}{n+1}\right) = \mathrm{O}\left(\frac{\sqrt{x}\,e^x}{\sqrt{n}+1}\right) \tag{7·5}$$

$$\text{as } \frac{x}{n+1} < \frac{\sqrt{x}\cdot e^x}{\sqrt{n}+1}.$$

thus proving (C).