A NEW PROOF OF MEHLER'S FORMULA AND OTHER THEOREMS ON HERMITIAN POLYNOMIALS

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1. Many proofs of Mehler's formula, namely,

$$\sum_0^\infty t^n \psi_n(x)\cdot\psi_n(y) = \frac{1}{\sqrt{\pi}\,(1-t^2)}\, e^{\frac{2xyt}{1-t^2} - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1+t^2}{1-t^2}\right)},$$

$$\text{where } \psi_n(x) = \frac{(-1)^n\cdot e^{\frac{x^2}{2}}\, D^n\,(e^{-x^2})}{\{2^n\cdot \lfloor n \cdot \sqrt{\pi}\}^{\frac{1}{2}}},$$

have been given, which for the most part have been indirect and elaborate as suggested by Prof. Watson in his Notes* in *J.L.M.S.* (1933). Of the three proofs given by Prof. Watson in his Notes, the first one implies a knowledge of certain results in the theory of Laguerre and Bessel functions, the second a knowledge of the formula of Saalschutz for generalised hypergeometric functions and the third, something of the theory of Fourier Transforms and of absolutely convergent infinite integrals. I have here given a proof that is different from all the known proofs of Mehler's formula which involves only a knowledge of (i) recurrence formula for $H_n(x)$, (ii) elementary analysis of differentiating a series term by term. Hence I believe that the proof given herein besides being new is the most elementary one.

Besides this, some interesting equalities (given in 2) giving the order of $\psi_n(x)$ for all n and x are derived by elementary methods.

2. Let $\psi_n(x)$ be as above and $a_n = \frac{1}{\sqrt{\pi}}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$, $n \geqslant 1$

$$a_0 = \frac{1}{\sqrt{\pi}}.$$

We shall prove the following results :

$$\text{(A)}\quad a_{\left[\frac{n}{2}\right]} - 4\int_0^x \psi_n^2\, e^{-x^2}\cdot x\cdot dx = e^{-x^2}\,(\psi_n^2 + \psi_{n-1}^2) = a_{\left[\frac{n}{2}\right]}\, e^{-2x^2} + 4e^{-2x^2}\cdot\int_0^x \psi_{n-1}^2\, e^{x^2}\cdot x dx$$

* [Notes on Generating Functions of Polynomials, "Hermite Polynomials," *Jour. Math. Soc., London*, July 1933, Part 3.]

(A′) (A) obviously implies $\psi_n^2 < \frac{k}{\sqrt{n+1}} e^{x^2}$ for all n and x

(B) $\sum_{r=1}^{n} \psi_r^2 = \frac{\sqrt{2n}}{\pi} + 0\,(1) + 0\left(\frac{e^{x^2}}{\sqrt{n}}\right)$

(C) From (A) is deduced the known Equation

$$\sum_0^\infty t^n \psi_n^2 = \frac{e^{-x^2\frac{(1-t)}{1+t}}}{\sqrt{\pi(1-t^2)}}$$

(D) If $F(x, y, t) = e^{\frac{-2xyt}{1-t^2}} \Sigma\, t^n \psi_n(x) \cdot \psi_n(y)$, it is proved that F is a function of $x^2 + y^2$ and Mehler's formula (as stated in the Introduction) is then deduced from (C).

3. We shall first establish :

$$D\{(\psi_n^2 + \psi_{n-1}^2)\, e^{-x^2}\} = -4xe^{-x^2}\psi_n^2 \tag{3·1}$$

$$D\{(\psi_n^2 + \psi_{n-1}^2)\, e^{x^2}\} = 4xe^{x^2}\psi_{n-1}^2 \tag{3·2}$$

Proof of (3·1) : Let $H_n = (-1)^n e^{x^2} D^n (e^{-x^2})$ and $K_n^2 = \frac{1}{2^n \cdot \underline{|n} \cdot \sqrt{\pi}}$;

then $D\{\psi_n^2 e^{-x^2}\} = D\{K_n^2 H_n^2 e^{-2x^2}\} = 2K_n^2 e^{-2x^2} H_n \{D H_n - 2x H_n\}$

$$= -2K_n^2 e^{-2x^2} H_n \cdot H_{n+1} \text{ (since } D H_n - 2x H_n = -H_{n+1}) \tag{3·3}$$

Similarly $D\{\psi_{n-1}^2 e^{-x^2}\} = -2 K_{n-1}^2 \cdot e^{-2x^2} \cdot H_{n-1} H_n$

$$= -4n K_n^2 e^{-2x^2} H_{n-1} \cdot H_n \tag{3·4}$$

From (3·3) and (3·4) we obtain,

$D\{(\psi_n^2 + \psi_{n-1}^2)\, e^{-x^2}\} = -2K_n^2 e^{-2x^2} H_n \{H_{n+1} + 2n H_{n-1}\}$

$$= -4 K_n^2 e^{-2x^2} H_n^2 \cdot x \text{ (since } H_{n+1} + 2n H_{n-1} = 2x H_n) \tag{3·5}$$

thus establishing (3·1).

Writing (3·1) as $D\{(\psi_n^2 + \psi_{n-1}^2)\, e^{-x^2}\} + 4x\,(\psi_n^2 + \psi_{n-5}^2)\, e^{-x^2}$

$= 4xe^{-x^2}\, \psi_{n-1}^2$

and multiplying the Equation by e^{2x^2} we obtain after rearrangement,

$$D\{(\psi_n^2 + \psi_{n-1}^2)\, e^{x^2}\} = 4x \cdot e^{x^2}\, \psi_{n-1}^2 \tag{3·6}$$

thus proving 3·2.

4. *Proof of* (A) : Integrating (3·1) in (0, x) we obtain,

$$\psi_n^2(0) + \psi_{n-1}^2(0) - e^{-x^2}(\psi_n^2 + \psi_{n-1}^2) = 4\int_0^x \psi_n^2 e^{-x^2} \cdot x \cdot dx.$$

Since $\psi_{2n}(0) = \frac{1}{\sqrt{\pi}} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} = a_n$ and $\psi_{2n+1}(0) = 0$

Hence

$$a_{[\frac{n}{2}]} - e^{-x^2}(\psi_n^2 + \psi_{n-1}^2) = 4\int_0^x \psi_n^3 \cdot e^{-x^2} \cdot x \cdot dx \tag{4·1}$$

Similarly by integrating (3·2) in $(0, x)$ we obtain,

$$e^{x^2}(\psi_n^2 + \psi_{n-1}^2) - a_{\left[\frac{n}{2}\right]} = 4\int_0^x \psi_{n-1}^2 e^{x^2} \cdot x \cdot dx \tag{4·2}$$

Combining (4·1) and (4·2) we obtain (A).

5. *Proof of* (B) : Let $S_n = \sum_{r=1}^{n} \psi_n^2$.

From (3·1) we obtain,

$$\sum_{r=1}^{n} D\{(\psi_r^2 + \psi_{r-1}^2)\, e^{-x^2}\} = D\{e^{-x^2}(2S_n + \psi_0^2 - \psi_n^2)\}$$

$$= -4xe^{-x^2} S_n \tag{5·1}$$

i.e., $D(e^{-x^2} S_n) + 2x \cdot e^{-x^2} S_n = \frac{1}{2} D\{e^{-x^2}(\psi_n^2 - \psi_9^2)\}$.

The L.H.S. $= e^{-x^2} D\, S_n$. Hence we have

$$D\, S_n = e^{x^2} \cdot \tfrac{1}{2} \cdot D\{e^{-x^2}(\psi_n^2 - \psi_9^2)\} = \frac{D\psi_n^2}{2} - x\,\psi_n^2 + \frac{2x}{\sqrt{\pi}} e^{-x^2} \tag{5·2}$$

Since $\psi_0^2 = \dfrac{e^{-x^2}}{\sqrt{\pi}}$

Integrating (5·2) in $(0, x)$ we obtain,

$$S_n(x) = S_n(0) + \tfrac{1}{2}\{\psi_n^2 - \psi_n^2(0)\} - \int_0^x \psi_n^2 \cdot x \cdot dx + \frac{1}{\sqrt{\pi}}(1 - e^{-x^2}) \tag{5·3}$$

$$= S_n(0) + \alpha + \beta + \gamma$$

By (A′) α and β are of order $0\left(\dfrac{e^{x^2}}{\sqrt{n}}\right)$ (5·4)

Let $n = 2p$ or $2p + 1$, then

$$S_n(0) = \sum_1^n \psi_r^2(0) = \sum_1^p a_n = \frac{1}{\sqrt{\pi}}\left\{1 + \tfrac{1}{2} + \cdots \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots 2p} - 1\right\}$$

$$= \frac{1}{\sqrt{\pi}}\left\{\frac{3\cdot 5\cdots(2p+1)}{2\cdot 4\cdots 2p} - 1\right\}$$

$$= \frac{2\sqrt{p}}{\pi} + 0\left(\frac{1}{\sqrt{p}}\right) - \frac{1}{\sqrt{\pi}}$$

$$= \frac{\sqrt{2n}}{\pi} + 0\,(1)$$

Hence (5·3) becomes

$$S_n(x) = \frac{\sqrt{2n}}{\pi} + 0\,(1) + 0\left(\frac{e^{x^2}}{\sqrt{n}}\right) \tag{5·5}$$

thus proving (B).

6. *Proof of* (C) : Let $F(x, t) = e^{-x^2} \cdot \sum_0^{\infty} t^n \psi_n^2$. Since $\psi_n^2 = 0\left(\dfrac{e^{x_2}}{\sqrt{n+1}}\right)$

the series for $F(x, t)$ converges for $|t| < 1$ and all x.

Then

$$\frac{d}{dx}\{(1+t)\,F(x,t)\} = \frac{d}{dx}\{\Sigma\,[(\psi_n^2+\psi_{n+1}^2)\,e^{-x^2}t^n]\}$$

$$= \sum_0^\infty t^n \cdot \frac{d}{dx}\{(\psi_n^2+\psi_{n-1}^2)\,e^{-x^2}\}$$

$$= -\Sigma\,(t^n \cdot 4x \cdot e^{-x^2}\psi_n^2) = -4x\,F(x,t) \qquad (6\cdot 1);$$

term by term differentiation being justified since $\Sigma\,(4x\,\psi_n^2\,e^{-x^2}\cdot t^n)$ converges uniformly in any finite range for x, for $|t| < 1$.

Integrating $(6\cdot 1)$ in $(0, x)$, we obtain:

$$F(x,t) = F(0,t)\,e^{\frac{-2x^2}{1+t}}$$

But $$F(0,t) = \frac{1}{\sqrt{\pi}}\left\{1+\sum_1^\infty \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot t^{2n}\right\} = \frac{1}{\sqrt{\pi}(1-t^2)}$$

Hence $$F(x,t) = \frac{e^{\frac{-2x^2}{1+t}}}{\sqrt{\pi}\,(1-t^2)} \qquad (6\cdot 2)$$

and $$\Sigma\,t^n\,\psi_n^2(x) = \frac{e^{\frac{-x^2(1-t)}{1+t}}}{\sqrt{\pi}\,(1-t^2)} \qquad (6\cdot 3)$$

thus proving (C).

7. Before proving (D) we shall prove the following two Lemmas:

LEMMA I. Let $H_n(x)$ and K_n be as defined in 3; and let

$$\beta_n(x,y) = K^2_{n-1}\{y\,H_n(y)\cdot H_{n-1}(x) - x\,H_n(x)\cdot H_{n-1}(y)\}$$

Then we shall prove the identity:

$$\beta_n(x,y) - \beta_{n-2}(x,y) = 2(y^2-x^2)\,K^2_{n-1}\,H_{n-1}(x)\,H_{n-1}(y) \qquad (7\cdot 1)$$

Proof of $(7\cdot 1)$: Now $H_n(x) = 2x\cdot H_{n-1}(x) - 2(n-1)\,H_{n-2}(x) \qquad (7\cdot 2)$

Hence $$y\,H_n(y)\cdot H_{n-1}(x) = 2y^2\cdot H_{n-1}(y)\,H_{n-1}(x) - 2(n-1)\,y\,H_{n-2}(y)\,H_{n-1}(x)$$

$$x\,H_n(x)\cdot H_{n-1}(y) = 2x^2\cdot H_{n-1}(x)\cdot H_{n-1}(y) - 2(n-1)\,x\cdot H_{n-2}(x)\cdot H_{n-1}(y) \qquad (7\cdot 3)$$

Hence $$\beta_n(x,y) = 2(y^2-x^2)\,K^2_{n-1}\,H_{n-1}(x)\,H_{n-1}(y) + K^2_{n-1}\{x\,H_{n-2}(x)\cdot H_{n-1}(y) - y\,H_{n-2}(y)\,H_{n-1}(x)\} \qquad (7\cdot 4)$$

Making another application of $(7\cdot 2)$ to the second term on the right side of $(7\cdot 4)$, we have:

$$K^2_{n-1}\{x\,H_{n-2}(x)\cdot H_{n-1}(y) - y\,H_{n-2}(y)\,H_{n-1}(x)\}$$

$$= K^2_{n-1}\{y\,H_{n-2}(y)\,H_{n-3}(x) - x\,H_{n-2}(x)\,H_{n-3}(y)\} = \beta_{n-2}(x,y) \qquad (7\cdot 5)$$

Hence,

$$\beta_n(x,y) - \beta_{n-2}(x,y) = 2(y^2-x^2)\,K^2_{n-1}\,H_{n-1}(x)\cdot H_{n-1}(y),$$

thus establishing (7·1)

LEMMA II. Let $F_1(x, y, t) = e^{\frac{-2xyt}{1-t^2}} \sum_0^\infty K_n^2 H_n(x) \cdot H_n(y) \cdot t^n$

we shall prove here $\quad y \dfrac{\partial F_1}{\partial x} - x \dfrac{\partial F_1}{\partial y} = 0 \qquad (7\cdot6)$

Proof of (7·6) :

$$\frac{\partial F_1}{\partial x} = e^{\frac{-2xyt}{1-t^2}} \sum_0^\infty K_{n-1}^2 H_{n-1}(x) H_n(y) t^n - \frac{2yt}{1-t^2} F_1(x,y,t) \qquad (7\cdot7)$$

since $\dfrac{d H_n(x)}{dx} = 2n H_{n-1}$ and $K_n^2 \cdot 2n = K_{n-1}^2$.

Similarly,

$$\frac{\partial F_1}{\partial y} = e^{\frac{-2xyt}{1-t^2}} \sum_0^\infty K_{n-1}^2 H_n(x) \cdot H_{n-1}(y) t^n - \frac{2xt}{1-t^2} F_1(x, y, t) \qquad (7\cdot8)$$

Hence,

$$y \frac{\partial F_1}{\partial x} - x \frac{\partial F_1}{\partial y}$$

$$= e^{\frac{-2xyt}{1-t^2}} \Sigma\, \beta_n(x, y)\, t^n - \frac{2(y^2 - x^2)\cdot t}{1-t^2} F_1(x, y, t)$$

$$= \frac{e^{\frac{-2xyt}{1-t^2}}}{(1-t^2)} \{ (1-t^2)\, \Sigma\, \beta_n t^n - 2(y^2 - x^2)\, \Sigma\, K_n^2 H_n(x) \cdot H_n(y)\, t^{n+1} \}$$

$$= \frac{e^{\frac{-2xyt}{1-t^2}}}{1-t^2} \{ \Sigma\, t^n [\beta_n - \beta_{n-2} - 2 K_{n-1}^2 \overline{x^2 - y^2} H_{n-1}(x) \cdot H_{n-1}(y)] \}$$

$= 0$ by Lemma I [or (7·1)] $\qquad (7\cdot9)$

term by term differentiation being permissible since the differentiated series is uniformly convergent in any finite range of (x, y) for some t such that $|t| < 1$, by property A′

8. *Proof of Mehler's formula* (D) :

Since by Lemma II $\; y \dfrac{\partial F_1}{\partial x} - x \dfrac{\partial F_1}{\partial y} = 0,$

$$F_1 = \phi_t(x^2 + y^2) \qquad (8\cdot1)$$

and if $F(x, y, t)$ be defined as in 2 (D), then

$$F(x, y, t) = F_1 \cdot e^{\frac{-x^2+y^2}{2}} = \Phi_t(x^2 + y^2) \qquad (8\cdot2)$$

i.e., $\quad \Sigma\, t^n \psi_n(x) \cdot \psi_n(y) = e^{\frac{2txy}{1-t^2}} \Phi_t(x^2 + y^2).$

Putting $x = y$ and applying (C) we obtain

$$\Phi_t\,(2\,x^2)\; e^{\frac{t}{1-t^2}\cdot 2x^2} = \frac{e^{\frac{-x^2(1-t)}{1+t}}}{\sqrt{\pi\,(1-t^2)}}$$

i.e.,

$$\Phi_t\,(2\,x^2) = \frac{1}{\sqrt{\pi}\,(1-t^2)}\; e^{\frac{-x^2(1+t^2)}{(1-t^2)}}$$

Hence

$$\Sigma\, t^n\,\psi_n\,(x)\cdot\psi_n\,(y) = \frac{1}{\sqrt{\pi}\,(1-t^2)}\cdot e^{\frac{2\,txy}{1-t^2} - \frac{(x^2+y^2)}{2}\,\frac{(1+t^2)}{(1-t^2)}}$$

thus proving (D).