A NEW PROOF OF MEHLER'S FORMULA AND OTHER THEOREMS ON HERMITIAN POLYNOMIALS

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1. Many proofs of Mehler's formula, namely,

\[ \sum_{\nu=0}^{\infty} \nu^{n} \psi_{\nu}(x) \cdot \psi_{\nu}(y) = \frac{1}{\sqrt{\pi} (1 - \nu^2)} e^{\frac{x^2 + y^2}{2} \left( \frac{1 + \nu^2}{1 - \nu^2} \right)}, \]

where \( \psi_{\nu}(x) = \frac{(-1)^{\nu} \cdot e^{x^2} D^{\nu} (e^{-x^2})}{\nu \cdot \sqrt{\pi} \nu^{\nu}}, \)

have been given, which for the most part have been indirect and elaborate as suggested by Prof. Watson in his Notes* in J.L.M.S. (1933). Of the three proofs given by Prof. Watson in his Notes, the first one implies a knowledge of certain results in the theory of Laguerre and Bessel functions, the second a knowledge of the formula of Saalschutz for generalised hypergeometric functions and the third, something of the theory of Fourier Transforms and of absolutely convergent infinite integrals. I have here given a proof that is different from all the known proofs of Mehler's formula which involves only a knowledge of (i) recurrence formula for \( H_{\nu}(x), \) (ii) elementary analysis of differentiating a series term by term. Hence I believe that the proof given herein besides being new is the most elementary one.

Besides this, some interesting equalities (given in 2) giving the order of \( \psi_{\nu}(x) \) for all \( n \) and \( x \) are derived by elementary methods.

2. Let \( \psi_{\nu}(x) \) be as above and \( a_{\nu} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{\sqrt{\pi} \cdot 2 \cdot 4 \cdot 6 \cdots 2n} \), \( n \geq 1 \)

\[ a_{0} = \frac{1}{\sqrt{\pi}}. \]

We shall prove the following results:

\[ (A) \ a_{\left[ \frac{n}{2} \right]} = 4 \int_{0}^{x} \psi_{\nu}^{2} e^{-x^2} \cdot x \cdot dx = e^{-x^2} (\psi_{\nu}^{2} + \psi_{\nu-1}^{2}) = a_{\left[ \frac{n}{2} \right]} e^{-2x^2} + 4e^{-2x^2} \int_{0}^{x} \psi_{\nu-1}^{2} e^{x^2} \cdot x \cdot dx \]

(A') (A) obviously implies \( \psi_n^2 < \frac{k}{\sqrt{n} + 1} e^{x^2} \) for all \( n \) and \( x \)

(B) \( \sum_{r=1}^{n} \psi_r^2 = \frac{\sqrt{2} n}{\pi} + 0 \cdot 1 + 0 \left( \frac{e^{x^2}}{\sqrt{n}} \right) \)

(C) From (A) is deduced the known Equation

\[
\sum_{0}^{\infty} t^n \psi_n^2 = \frac{e^{-x^2(1-t) \over 1+t}}{\sqrt{\pi}(1-t^2)}
\]

(D) If \( F(x, y, t) = e^{2xyt} \sum t^n \psi_n(x) \cdot \psi_n(y) \), it is proved that \( F \) is a function of \( x^2 + y^2 \) and Mehler's formula (as stated in the Introduction) is then deduced from (C).

3. We shall first establish:

\[
D \{ (\psi_n^2 + \psi_{n-1}^2) e^{-x^2} \} = -4xe^{-x^2} \psi_n^2 \quad (3.1)
\]

\[
D \{ (\psi_n^2 + \psi_{n-1}^2) e^x \} = 4xe^{-x^2} \psi_n^2 \quad (3.2)
\]

**Proof of (3.1):** Let \( H_n = (-1)^n e^{x^2} D^n (e^{-x^2}) \) and \( K_n^2 = {1 \over 2^n \cdot 1 ! \cdot \sqrt{\pi}} \); then

\[
D \{ \psi_n^2 e^{-x^2} \} = D \{ K_n^2 H_n^2 e^{-2x^2} \} = 2K_n^2 e^{-2x^2} H_n \{ D H_n - 2x H_n \}
\]

\[
= -2K_n^2 e^{-2x^2} H_n \cdot H_n + 1 \quad (\text{since } D H_n - 2x H_n = -H_n + 1)
\]

Similarly

\[
D \{ \psi_{n-1}^2 e^{-x^2} \} = -2K_{n-1}^2 e^{-2x^2} H_{n-1} H_n
\]

\[
= -4n K_n^2 e^{-2x^2} H_{n-1} \cdot H_n
\]

From (3.3) and (3.4) we obtain,

\[
D \{ (\psi_n^2 + \psi_{n-1}^2) e^{-x^2} \} = -2K_n^2 e^{-2x^2} H_n \{ H_n + 1 + 2n H_{n-1} \}
\]

\[
= -4K_n^2 e^{-2x^2} H_n \cdot x \quad (\text{since } H_n + 1 + 2n H_{n-1} = 2x H_n)
\]

thus establishing (3.1).

Writing (3.1) as

\[
D \{ (\psi_n^2 + \psi_{n-1}^2) e^{-x^2} \} = -4xe^{-x^2} \psi_n^2 \psi_{n-1}
\]

and multiplying the Equation by \( e^{2x^2} \) we obtain after rearrangement,

\[
D \{ (\psi_n^2 + \psi_{n-1}^2) e^{x^2} \} = 4xe^{-x^2} \psi_n^2 \psi_{n-1}
\]

thus proving 3.2.

4. **Proof of (A):** Integrating (3.1) in \( (0, x) \) we obtain,

\[
\psi_n^2 (0) + \psi_{n-1}^2 (0) - e^{-x^2} (\psi_n^2 + \psi_{n-1}^2) = 4 \int_0^x \psi_n^2 e^{-x^2} \cdot x \cdot dx.
\]

Since

\[
\psi_{2n} (0) = \frac{1}{\sqrt{n}} \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} = a_n \quad \text{and} \quad \psi_{2n+1} (0) = 0
\]

Hence

\[
\alpha_n - e^{-x^2} (\psi_n^2 + \psi_{n-1}^2) = 4 \int_0^x \psi_n^2 \cdot e^{-x^2} \cdot x \cdot dx \quad (4.1)
\]
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Similarly by integrating (3.2) in (0, x) we obtain,

\[ e^{x^2} \left( \psi_n^2 + \psi_{n-1}^2 \right) - a_{[n]} = 4 \int_0^x \psi_{n-1}^2 e^{x^2} \cdot x \cdot dx \quad (4.2) \]

Combining (4.1) and (4.2) we obtain (A).

5. Proof of (B): Let \( S_n = \sum_{r=1}^n \psi_r^2 \).

From (3.1) we obtain,

\[ \sum_{r=1}^n D \{ (\psi_r^2 + \psi_{r-1}^2) e^{-x^2} \} = D \{ e^{-x^2} (2S_n + \psi_0^2 - \psi_n^2) \} = -4xe^{-x^2} \ S_n \quad (5.1) \]

i.e., \( D \{ e^{-x^2} S_n \} + 2x \cdot e^{-x^2} S_n = \frac{1}{2} D \{ e^{-x^2} (\psi_n^2 - \psi_0^2) \} \).

The L.H.S. = \( e^{-x^2} D \ S_n \). Hence we have

\[ D \ S_n = e^{x^2} \cdot \frac{1}{2} \cdot D \{ e^{-x^2} (\psi_n^2 - \psi_0^2) \} = \frac{D\psi_n^2}{2} - x \psi_n^2 + \frac{2x}{\sqrt{\pi}} e^{-x^2} \quad (5.2) \]

Since \( \psi_0^2 = \frac{e^{-x^2}}{\sqrt{\pi}} \)

Integrating (5.2) in (0, x) we obtain,

\[ S_n (x) = S_n (0) + \frac{1}{2} \{ \psi_n^2 - \psi_0^2 (0) \} - \int_0^x \psi_n^2 \cdot x \cdot dx + \frac{1}{\sqrt{\pi}} (1 - e^{-x^2}) \quad (5.3) \]

By (A) \( \alpha \) and \( \beta \) are of order \( \left( \frac{e^{x^2}}{\sqrt{n}} \right) \quad (5.4) \)

Let \( n = 2p \) or \( 2p + 1 \), then

\[ S_n (0) = \sum_{r=1}^n \psi_r^2 (0) = \sum_{r=1}^p a_{2r} = \frac{1}{\sqrt{\pi}} \left\{ \frac{1 + \frac{1}{2} + \cdots \frac{1}{2} \cdot \cdots \cdot \frac{2p}{2} - 1}{2 \cdot 4 \cdots 2p} - 1 \right\} \]

\[ = \frac{1}{\sqrt{\pi}} \left\{ \frac{3 \cdot 5 \cdots (2p + 1)}{2 \cdot 4 \cdots 2p} - 1 \right\} \]

\[ = \frac{2 \sqrt{p}}{\pi} + O \left( \frac{1}{\sqrt{p}} \right) - \frac{1}{\sqrt{\pi}} \]

\[ = \frac{\sqrt{2n}}{\pi} + O (1) \]

Hence (5.3) becomes

\[ S_n (x) = \frac{\sqrt{2n}}{\pi} + O (1) + O \left( \frac{e^{x^2}}{\sqrt{n}} \right) \quad (5.5) \]

thus proving (B).

6. Proof of (C): Let \( F (x, t) = e^{-x^2} \sum_0^\infty t^n \psi_n^2 \). Since \( \psi_n^2 = 0 \left( \frac{e^{x^2}}{\sqrt{n} + 1} \right) \)

the series for \( F (x, t) \) converges for \( | t | < 1 \) and all \( x \).
Then
\[
\frac{d}{dx} \left\{ (1 + t) F(x, t) \right\} = \frac{d}{dx} \left\{ \sum (\psi_n^2 + \psi_{n+1}^2) e^{-x^2 \cdot t^n} \right\}
\]
\[
= \sum_0^\infty t^n \cdot \frac{d}{dx} \left\{ (\psi_n^2 + \psi_{n+1}^2) e^{-x^2} \right\}
\]
\[
= - \sum (t^n \cdot 4x \cdot e^{-x^2} \psi_n^2) = -4x \int (x, t) \quad (6.1),
\]
term by term differentiation being justified since \( \sum (4x \psi_n^2 e^{-x^2 \cdot t^n}) \) converges uniformly in any finite range for \( x \), for \( |t| < 1 \). Integrating (6.1) in \((0, x)\), we obtain:

\[
F(x, t) = F(0, t) e^{-\frac{2x^2}{1+t}}
\]

But \( F(0, t) = \frac{1}{\sqrt{\pi}} \left\{ 1 + \sum_1^\infty \frac{1 \cdot 3 \cdot (2n - 1)}{2 \cdot 4 \cdot 2n} \cdot t^{2n} \right\} = \frac{1}{\sqrt{\pi}(1 - t^2)} \)

Hence \( F(x, t) = \frac{e^{-\frac{2x^2}{1+t}}}{\sqrt{\pi}(1 - t^2)} \) (6.2)

and \( \sum t^n \psi_n^2(x) = \frac{e^{-\frac{x^2(1-t)}{1+t}}}{\sqrt{\pi}(1 - t^2)} \) (6.3)

thus proving (C).

7. Before proving (D) we shall prove the following two Lemmas:

**Lemma I.** Let \( \mathcal{H}_n(x) \) and \( \mathcal{K}_n \) be as defined in 3; and let

\[
\beta_n(x, y) = \mathcal{K}_{n-1} \{ y \mathcal{H}_n(y) \cdot \mathcal{H}_{n-1}(x) - x \mathcal{H}_n(x) \cdot \mathcal{H}_{n-1}(y) \}
\]

Then we shall prove the identity:

\[
\beta_n(x, y) = \beta_{n-2}(x, y) = 2(y^2 - x^2) \mathcal{K}_n \mathcal{H}_{n-1}(x) \mathcal{H}_{n-1}(y) \quad (7.1)
\]

**Proof of (7.1):** Now \( \mathcal{H}_n(x) = 2x \cdot \mathcal{H}_{n-1}(x) - 2(n - 1) \mathcal{H}_{n-2}(x) \) (7.2)

Hence \( y \mathcal{H}_n(y) \cdot \mathcal{H}_{n-1}(x) = 2y^2 \cdot \mathcal{H}_{n-1}(y) \mathcal{H}_{n-1}(x) - 
\]
\[
\quad 2(n - 1)y \mathcal{H}_{n-2}(y) \mathcal{H}_{n-1}(x)
\]

\[
x \mathcal{H}_n(x) \cdot \mathcal{H}_{n-1}(y) = 2x^2 \cdot \mathcal{H}_{n-1}(x) \cdot \mathcal{H}_{n-1}(y) - 
\]
\[
\quad 2(n - 1)x \cdot \mathcal{H}_{n-2}(x) \cdot \mathcal{H}_{n-1}(y)
\]

Hence \( \beta_n(x, y) = 2(y^2 - x^2) \mathcal{K}_n \mathcal{H}_{n-1}(x) \mathcal{H}_{n-1}(y) + 
\]
\[
\mathcal{K}_{n-1} \{ x \mathcal{H}_{n-2}(x) \cdot \mathcal{H}_{n-1}(y) - y \mathcal{H}_{n-2}(y) \mathcal{H}_{n-1}(x) \}
\]

Making another application of (7.2) to the second term on the right side of (7.4), we have:

\[
\mathcal{K}_{n-1} \{ x \mathcal{H}_{n-2}(x) \cdot \mathcal{H}_{n-1}(y) - y \mathcal{H}_{n-2}(y) \mathcal{H}_{n-1}(x) \}
\]

\[
= \mathcal{K}_{n-1} \{ y \mathcal{H}_{n-2}(y) \mathcal{H}_{n-3}(x) - x \mathcal{H}_{n-2}(x) \mathcal{H}_{n-3}(y) \} = \beta_{n-2}(x, y) \quad (7.5)
\]

Hence,

\[
\beta_n(x, y) = \beta_{n-2}(x, y) = 2(y^2 - x^2) \mathcal{K}_{n-1} \mathcal{H}_{n-1}(x) \cdot \mathcal{H}_{n-1}(y),
\]
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thus establishing (7·1)

Lemma II. Let $F_1(x, y, t) = e^{\frac{-2xyt}{1-t^2}} \sum_0^\infty K_n^2 H_n(x) \cdot H_n(y) \cdot t^n$

we shall prove here $y \frac{\partial F_1}{\partial x} - x \frac{\partial F_1}{\partial y} = 0 \quad (7·6)$

Proof of (7·6):

$$\frac{\partial F_1}{\partial x} = e^{\frac{-2xyt}{1-t^2}} \sum_0^\infty K_n^2 H_{n-1}(x) H_n(y) t^n - \frac{2yt}{1-t^2} F_1(x, y, t) \quad (7·7)$$

since $d \frac{H_n(x)}{dx} = 2n H_{n-1}$ and $K_n^2 \cdot 2n = K_{n-1}^2$.

Similarly,

$$\frac{\partial F_1}{\partial y} = e^{\frac{-2xyt}{1-t^2}} \sum_0^\infty K_n^2 H_n(x) \cdot H_{n-1}(y) t^n - \frac{2xt}{1-t^2} F_1(x, y, t) \quad (7·8)$$

Hence,

$$y \frac{\partial F_1}{\partial x} - x \frac{\partial F_1}{\partial y} = e^{\frac{-2xyt}{1-t^2}} \sum \beta_n(x, y) t^n - \frac{2(y^2 - x^2)}{1-t^2} F_1(x, y, t)$$

$$= e^{\frac{-2xyt}{1-t^2}} \left\{ \frac{(1-t^2)}{1-t^2} \sum \beta_n t^n - 2(y^2 - x^2) \sum K_n^2 H_n(x) \cdot H_n(y) t^n+1 \right\}$$

$$= e^{\frac{-2xyt}{1-t^2}} \left\{ \frac{(1-t^2)}{1-t^2} \sum t^n \left[ \beta_n - \beta_n - 2 - 2 K_{n-1}^2 x^2 - y^2 H_{n-1}(x) \cdot H_{n-1}(y) \right] \right\}$$

$$= 0 \text{ by Lemma I [or (7·1)]} \quad (7·9)$$

term by term differentiation being permissible since the differentiated series is uniformly convergent in any finite range of $(x, y)$ for some $t$ such that $|t| < 1$, by property $A'$

8. Proof of Mehler's formula (D):

Since by Lemma II $y \frac{\partial F_1}{\partial x} - x \frac{\partial F_1}{\partial y} = 0$,

$$F_1 = \phi_t(x^2 + y^2) \quad (8·1)$$

and if $F(x, y, t)$ be defined as in 2 (D), then

$$F(x, y, t) = F_1 \cdot e^{\frac{-x^2 + y^2}{2}} = \phi_t(x^2 + y^2) \quad (8·2)$$

i.e.,

$$\sum t^n \psi_n(x) \cdot \psi_n(y) = e^{\frac{-x^2 + y^2}{2}} \phi_t(x^2 + y^2).$$
Putting $x = y$ and applying (C) we obtain

$$
\Phi_t \left( 2 \ x^2 \right) \ e^{\frac{t}{1 - t^2} \cdot 2x^2} = e^{\frac{-x^2(1 - t)}{1 + t}} \frac{1}{\sqrt{\pi} \ (1 - t^2)}
$$

i.e.,

$$
\Phi_t \left( 2 \ x^2 \right) = \frac{1}{\sqrt{\pi} \ (1 - t^2)} \ e^{\frac{-x^2(1 + t^2)}{(1 - t^2)}
$$

Hence

$$
\Sigma \ t^n \ \psi_n \ (x) \cdot \psi_n \ (y) = \frac{1}{\sqrt{\pi} \ (1 - t^2)} \cdot e^{\frac{2 \ txy}{1 - t^2} - \frac{(x^2 + y^2)}{2} \ (1 + t^2)}
$$

thus proving (D).