A TAUBERIAN THEOREM AND ITS APPLICATION TO CONVERGENCE OF FOURIER SERIES

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1. Let $\left(1+\frac{x}{2}+\frac{x^2}{3}+\cdots\frac{x^n}{n+1}+\text{ to }\infty\right)^k=\sum_0^\infty b_{k,n}\, x^n$, k being a positive integer and let $\sum_{r=0}^n b_{k,r}=\mathrm{B}_{k,n}$. Whenever $\sum_0^n b_{k,n-r}\cdot \mathrm{S}_r/\mathrm{B}_{k,n}$ converges to a limit say S as $n\to\infty$, I shall indicate that fact by the symbol $\mathrm{S}_n\to\mathrm{S}$ (N, k) as $n\to\infty$. (1·1)

The main object of this note is the proof of two theorems (I and II) the first of these being a Tauberian theorem on mean convergence or summability (N, k), the second being the summability (N, 1) of the Fourier series of a summable (L) function $f(x)$ at a point x_0 at which, $\{(f(x_0+h)-f(x_0))\cdot\log|h|)\}\to 0$ as $h\to 0$. From these two theorems, which are new, it is noteworthy that the truth of the convergence criterion, of Hardy and Littlewood (1932) of the Fourier series of a summable function $f(x)$ at a point x_0, namely (1) $\{(f(x_0+h)-f(x_0))\log|h|\}\to 0$ as $h\to 0$ (2) the Fourier coefficients of $f(x)$ or order $n\leqslant \mathrm{A}\cdot n^{-\delta}$, $(\mathrm{A}>0,\ \delta>0)$, (1·2) follows at once. Before proceeding further it should be noted that summability (N, k) is more stringent than Cesaro means of any positive order, for it can be established that (N, k) implies (C, r) when $k>0$ and $r>0$, but not conversely.

2. *Statement of Theorems*

Theorem I. If $\mathrm{S}_n\to\mathrm{S}$ (N, k) k being a positive integer and $\mathrm{S}_n-\mathrm{S}_{n-1}=\delta\mathrm{S}_n\leqslant \mathrm{A}n^{-\mu}$ $(\mathrm{A}>0,\ 1>\mu>0)$, then S_n converges ordinarily to S. (2·1)

Note 1.—If $\mu=1$, in view of the remark at the end of 1, the theorem reduces itself to Cesaro-Tauber theorem.

Note 2.—As we should expect from Hardy-Littlewood's paper,[1] we should be able to prove that whenever $\mathrm{S}_n\to\mathrm{S}$ (N, k), k being a positive integer then $\mathrm{S}_n\to\mathrm{S}$ also by Valeron means (V, k_1) of order k_1 where

$$1\leqslant k_1<2,\ i.e.,\ \frac{1}{\sqrt{\pi\cdot\mathrm{M}}}\sum_{p=0}^{\infty}\mathrm{S}_p\, e^{\frac{-(p-n)^2}{\mathrm{M}}}\to\mathrm{S}\text{ as }n\to\infty,\ \mathrm{M}=2\cdot n^{2-k_1};\quad (2\cdot 2)$$

then the Tauberian Theorem I follows. But the direct proof of Theorem I, which is given here, apart from its intrinsic interest is much simpler than the proof of the fact that (N, k) implies (V, k_1) for k a positive integer and $1 \leqslant k_1 < 2$. However for completeness the fact of (N, k) implying (V, k_1) is stated as Theorem III, and the proof of this theorem under the title "On the relation between summability by Norlund means of a certain type and summability by Valeron Means," will appear shortly in the *Journal of the Mysore University*.

THEOREM II. If $f(x)$ be a periodic (2π) L integeble function and $\{(f(x_0+h)-f(x_0))\log|h|\}\to 0$ as $h\to 0$, then the Fourier series of $f(x)$ at x_0 is summable (N, 1) to $f(x_0)$. (2·3)

THEOREM III. If $S_n \to S$, (N, k) k being a positive integer then

$$S_n \to S,\ (V, k_1) \qquad 1 \leqslant k_1 < 2 \text{ (for all such } k_1). \tag{2·4}$$

3. *Proof of Preliminary Lemmas*

Lemma I. Let $b_{k,n}$ and $B_{k,n}$ be as in (1·1), and $a_{k,n}$ be defined by

$$\left(1-\sum_{r=1}^{\infty} a_{k,r}\, x^r\right)\left(\sum_{0}^{\infty} b_{k,r}\, x^r\right)=1 \tag{3·1}$$

then

$$\left.\begin{aligned} &(a)\ b_{k,n}=\sum_{r=1}^{n} a_{k,r}\cdot b_{k,n-r}, \qquad (b)\ B_{k,n}=1+\sum_{r=1}^{n} a_{k,r}\, B_{k,n-r} \\ &(c)\ b_{k,n}=O\left(\frac{(\log n)^{k-1}}{n}\right) \qquad (d)\ a_{k,n}=O\left(\frac{1}{n\cdot(\log n)^{1+k}}\right). \end{aligned}\right\} \tag{3·2}$$

(a) and (b) follow from the definition in (3·1) and (c) can be deduced by induction from

$$b_{k,n}\, x^n=\left(1+\sum_{1}^{\infty}\frac{x^r}{r+1}\right)^k. \tag{3·3}$$

Proof of (d) *of* (3·2)

Let $V_n(a)=\dfrac{(1-a)(2-a)\cdots(n-1-a)}{\lfloor n}$; we shall here establish by induction that

$$a_{k,n}=\frac{1}{\lfloor k-1}\int_0^1 V_{n-k}\cdot a^k\cdot da + O\left(\frac{1}{n^2\cdot(\log n)^2}\right). \tag{3·4}$$

$$[k \geqslant 1]$$

Let $I=\displaystyle\int_0^1 (1-x)^a\cdot\frac{(1-a)^{k-1}}{\lfloor k-1}\, da$; integrating directly and expanding

$(1-x)^\alpha$ and integrating term by term, we obtain

$$\mathrm{I}=\frac{(1-x)-\sum_{r=0}^{k-1}\frac{(\log 1-x)^r}{\underline{|r}}}{(\log 1-x)^k}$$

$$=\frac{1}{\underline{|k}}-\sum_{n=1}^{\infty}\left[\int_0^1 \mathrm{V}_n(\alpha)\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\,d\alpha\right\}\cdot x^n \tag{3·5}$$

Multiplying by $(-x)^{k-1}$, we obtain from 3·5

$$\left(\frac{-x}{\log 1-x}\right)^k=\sum_{r=1}^{k-1}\frac{(-x)^{k-1-r}}{\underline{|k-r}}\cdot\left(\frac{-x}{\log 1-x}\right)^r$$

$$+(-1)^{k-1}\left\{\frac{x^{k-1}}{\underline{|k}}-\sum_{n=1}^{\infty}\left\{\mathrm{V}_n(\alpha)\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k}-1}\cdot d\alpha\right\}x^{n+k-1}\right\} \tag{3·6}$$

From 3·1 and 1·1 definitions we have

$$\left(\frac{-x}{\log 1-x}\right)^k=1-\sum_{n=1}^{\infty}a_{k,n}\,x^n \tag{3·7}$$

so that for $n>k$ we obtain from 3·6

$$a_{k,n}=a_{k-1,n}-\frac{a_{k-2,n-1}}{\underline{|2}}+\cdots+\cdots$$

$$+(-1)^{k-2}\frac{a_{1,n-k+2}}{\underline{|k-1}}+(-1)^{k-1}\cdot\int_0^1 \mathrm{V}^{(\alpha)}_{n-k+1}\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\,d\alpha. \tag{3·8}$$

It is easy to see that (3·4) is true for $k=1$; assume it true for $k=1,2,\cdots k-1$, then from 3·8

$$a_{k,n}=\int_0^1 \mathrm{V}_{n-k+1}\left(\frac{\alpha^{k-1}}{\underline{|k-2}}-\frac{\alpha^{k-2}}{\underline{|2}\,\underline{|k-3}}+\cdots\frac{(-1)^{k-2}}{\underline{|k-1}}\cdot\alpha\right.$$

$$\left.+(-1)^{k-1}\cdot\alpha\cdot\frac{(1-\alpha)^{k-1}}{\underline{|k-1}}\cdot\right)d\alpha+\mathrm{O}\left(\frac{1}{n^2\,(\log n)^2}\right) \tag{3·9}$$

$$=\int_0^1 \mathrm{V}^{(\alpha)}_{n-k+1}\cdot\alpha^k\,d\alpha+\mathrm{O}\left(\frac{1}{n^2\cdot(\log n)^2}\right). \tag{3·10}$$

Now $\qquad \mathrm{V}_n\leqslant\frac{\mathrm{K}}{n^{1+\alpha}}$ and $\mathrm{V}_{n-k+1}=\mathrm{V}_{n-k}\left\{1+\mathrm{O}\left(\frac{1}{n}\right)\right\}$.

Hence $\qquad \int_0^1 \mathrm{V}_{n-k+1}\,\alpha^k\,d\alpha=\int_0^1 \mathrm{V}_{n-k}\cdot\alpha^k\,d\alpha+\mathrm{O}\left\{\frac{1}{n}\cdot\int_0^1 \mathrm{V}_n\cdot\alpha^k\,d\alpha.\right\}$.

$$=\int_0^1 \mathrm{V}_{n-k}\cdot\alpha^k\cdot d\alpha+\mathrm{O}\left(\frac{1}{n^2\cdot(\log n)^{1+k}}\right) \tag{3·11}$$

Hence $$a_{k,n}=\int_0^1 \mathrm{V}_{n-k}\, a^k\cdot da+\mathrm{O}\left(\frac{1}{n^2\cdot(\log n)^2}\right) \text{ for } k\geqslant 1 \tag{3·12}$$

thus proving (3·4)

From (3·12)

$$a_{k,n}=\mathrm{O}\left(\frac{1}{n\cdot(\log n)^{1+k}}\right)+\mathrm{O}\left(\frac{1}{n^2\cdot(\log n)^2}\right)$$

$$=\mathrm{O}\left(\frac{1}{n\cdot(\log n)^{1+k}}\right) \text{ thus proving } (d) \text{ of } (3\cdot2) \tag{3·13}$$

4. *Lemma II*

If $\mathrm{S}_n\to 0$ (N, k) where k is a positive integer and $\mathrm{N}=\mathrm{N}_n=[\mathrm{K}\cdot n^\delta]$ where $0>\delta>1$, and K is a positive fixed number, then

$$\left[\sum_{r=0}^{n} b_{k,r+\mathrm{N}}\cdot \mathrm{S}_{n-r}\Big/\sum_0^n b_{k,r+\mathrm{N}}\right]\to 0 \text{ as } n\to\infty \tag{4·1}$$

[Since in what follows k is a fixed integer, a_n, b_n, and B_n, shall stand for $a_{k,n}$, $b_{k,n}$ and $\mathrm{B}_{k,n}$ respectively and whenever x_n and y_n are of the same order we shall indicate by $x_n=\mathrm{O}(\mathrm{S}_n)$ (big O) and whenever $\frac{x_n}{y_n}\to 0$, we indicate by $x_n=0\,(y_n)$ (little 0).] (3·2)

From 3·2 and the condition about N it is easy to deduce that

$$\sum_{r=0}^{n} b_{k,r+\mathrm{N}}=\sum_0^n b_{r+\mathrm{N}}\,(\text{by, } 4\cdot 2)=\mathrm{O}\,(\mathrm{B}_{k,n})=\mathrm{O}\,[(\log n)^k] \tag{4·3}$$

and we have therefore to prove $\sum\limits_{r=0}^{n} b_{r+\mathrm{N}}\ \mathrm{S}_{n-r}=0\,(\mathrm{B}_n)$. (4·4)

Let $\sum\limits_0^n b_r\ \mathrm{S}_{n-r}/\mathrm{B}_n=y_n$; then from relation (3·1) we obtain

$$\mathrm{S}_n=\mathrm{B}_n\, y_n-\sum_{r=1}^{n} a_r\cdot \mathrm{B}_{n-r}\, y_{n-r}. \tag{4·5}$$

Substituting (4·5) in the l.h.s. of 4·4 we obtain

$$\sum_{r=0}^{n} b_{r+\mathrm{N}}\,\mathrm{S}_{n-r}=\sum_{r=0}^{n} \mathrm{A}_{n,r}\cdot \mathrm{B}_r\cdot y_r \tag{4·6}$$

where $$\mathrm{A}_{n,r}=b_{n-r+\mathrm{N}}-a_1\, b_{n-r-1+\mathrm{N}}-a_{n-r}\, b_{\mathrm{N}}. \tag{4·7}$$

We shall now consider r.h.s of 4·6 in three parts.

$$\sum_{r=0}^{n}\mathrm{A}_{n,r}\cdot\mathrm{B}_r\cdot y_r=\sum_{r=0}^{r=n_1}+\sum_{n_1+1}^{[n-\mathrm{N}^{1\cdot}\epsilon]}+\sum_{[n-\mathrm{N}^{1\cdot}\epsilon]+1}^{n}=\mathrm{A}+\mathrm{B}+\mathrm{C}. \tag{4·8}$$

where n_1 is a conveniently chosen fixed integer.

Since by (3·2) $\Sigma\,|\alpha_r|$ converges, we have from (4·7) and (3·2)

$$|\,A_{n,\,r}\,| \leqslant O\left(\frac{(\log N)^{k-1}}{N}\right). \tag{4·9}$$

Hence $$|\,C\,| \leqslant K\cdot \sum_{0\leqslant n-r\leqslant N^{1-\epsilon}} A_{n,\,r}\, B_r \cdot \leqslant K'\cdot N^{1-\epsilon}\cdot\frac{(\log N)^{k-1}}{N}\cdot B_n = 0\,(B_n). \tag{4·10}$$

Consider now B. From 4·7 and 3·2 (a) we have

$$A_{n,\,r} = a_{n-r+1}\, b_{N-1} + \cdots + a_{n-r+N} \tag{4·11}$$

Since in B

$$n-r \geqslant N^{1-\epsilon}-1 \text{ and } a_n = O\left(\frac{1}{n\cdot(\log n)^{1+k}}\right) \text{ and } b_n = O\left(\frac{(\log n)^{k-1}}{n}\right)$$

and $N = [K\cdot n^{\delta}]$ from, (3·2)
$0 < \delta < 1$

we obtain from (4·11) $|\,A_{n,\,r}| \leqslant K_1 \displaystyle\sum_{p=1}^{p=N} \frac{(\log N-p+2)^{k-1}}{(N-p+2)\,(q+p)\,[\log\,(q+p)]^{1+k}}$

where $q = n-r$.

$$\leqslant \frac{K_2}{(\log n)^2}\cdot\sum_{p=1}^{p=N}\frac{1}{(q+p)\,(N-p+2)} \tag{4·12}$$

$$\leqslant \frac{K_3}{\log n\cdot(N+q+2)} \tag{4·13}$$

so that $$|\,B\,| \leqslant \underset{(n_1+1\leqslant r\leqslant n-N^{1-\epsilon})}{\text{Max of } y_r}\cdot\frac{K_3\cdot B_n}{\log n}\cdot\sum^{q=n}\frac{1}{N+2+q}$$

$$\leqslant \underset{n_1+1\leqslant r}{\text{Max of } y_r}\cdot K_4\cdot B_n. \tag{4·14}$$

Consider now A. From (4·13) for fixed r as $n\to\infty$ we have

$$|\,A_{n,\,r}\,| \leqslant \frac{K_4}{n\cdot\log n} \tag{4·15}$$

so that for fixed r $$A_{n,\,r}\cdot B_r = 0\,(B_n) \tag{4·16}$$

so that from (4·10), (4·14) and (4·16) and hypothesis that $y_n\to 0$ as $n\to\infty$ (*i.e.*, 4·1) we obtain $A+B+C = 0\,(B_n)$, thus proving 4·4 and establishing Lemma II. (4·17)

5. *Proof of Theorem I*

Without loss of generality we assume that

$$\sum_0^n b_{n-r}\, S_r/B_n \to 0, \text{ as } n\to\infty. \tag{5·1}$$

[As in 4, a_n, b_n, B_n stand for $a_{k,\,n}$, $b_{k,\,n}$ and $B_{k,\,n}$].

Suppose $\liminf S_n = -b$, $b > 0$ (5·2)

Then there exists a sequence $n_0 < n_1 < \cdots < n_r \to \infty$ such that $S_{n_r} \leqslant -b_1$
$0 < b_1 < b$. (5·3)

Since $\delta S_n \leqslant A \cdot n^{-\mu}$ $(0 < \mu < 1)$ it can be easily shown that

$$S_n \leqslant -b_1/2 \text{ when } n_r \leqslant n \leqslant n_r + N_{n_r} \tag{5·4}$$

$$\text{where } N_n = N = \left[\frac{b_1}{2} \cdot \frac{n^{\mu}}{A}\right]$$

Now $$\sum_{r=0}^{n+N} b_{n+N-r} S_r = \sum_{r=0}^{r=n} + \sum_{r=n+1}^{n+N} = \Sigma_1 + \Sigma_2 \tag{5·5}$$

From Lemma II $$\Sigma_1 = 0\,(B_{n+N}). \tag{5·6}$$

By taking $n = n_r$, in view of 5·4

we have $$\Sigma_2 \leqslant -\frac{b_1}{2} \cdot \sum_{r=n+1}^{r=N+n} b_{n+N-r} = -\frac{b_1}{2} \cdot \sum_{p=0}^{N-1} b_p = -\frac{b_1}{2} \cdot B_{N-1}. \tag{5·7}$$

In view of condition about N, it can be shewn that $B_{N-1} \geqslant K_1 \cdot B_{n+N}$ (5·8)
where K_1 is a positive constant.

Hence $$\Sigma_2 / B_{n_r + N_{n_r}} \leqslant -\frac{b_1}{2} \cdot K_1 \cdot \text{so that}$$

$$\lim\inf \frac{\sum_0^n b_{n-r} S_r}{B_n} \leqslant -\frac{b_1}{2} K_1 \text{ contradicting } 5·1. \tag{5·9}$$

Hence *lim inf* $S_n = 0$, the case of lim inf $S_n = -\infty$ being easily disposed of. (5·10)
similarly lim sup $S_n = 0$, thus establishing Theorem I. (5·11)

6. *Proof of Theorem II*

Let $S_n = \sum_0^n (a_r \cos rx + b_r \sin rx)$, a_r, b_r, being the Fourier coefficients of $f(x)$, and $$\psi(t) = \frac{f(x_0 + t) + f(x_0 - t) - 2f(x_0).}{2} \tag{6·1}$$

Then $$S_n(x_0) - f(x_0) = \frac{1}{\pi} \int_0^{\pi} \psi(t) \cdot \frac{\sin(n + \frac{1}{2})t}{\sin t/2} \cdot dt = \frac{1}{\pi} \int_0^{\delta} + \epsilon(n, \delta)$$

where δ is so chosen that $|\psi(t) \log t| \leqslant \epsilon$ $0 \leqslant t \leqslant \delta$; $\epsilon(n, \delta) \to 0$ as $n \to \infty$ (6·2)

Now $$\sum_{r=0}^{n} \frac{S_{n-r}(x_0)}{r+1} \Big/ \sum_{r=0}^{n} \frac{1}{r+1} - f(x_0) = \frac{1}{\pi L_n} \int_0^{\delta} \psi(t) \cdot \frac{K_n(t)}{\sin t/2}\, dt + \epsilon'(n, \delta) \tag{6·3}$$

where $$K_n(t) = \sum_{r=0}^{n} \frac{\sin(r + \frac{1}{2})t}{n + 1 - r}, \text{ and } L_n = \sum_{r=0}^{n} \frac{1}{1+r} \tag{6·4}$$

We shall first prove that

$$\left.\begin{aligned} &\left|\frac{K_n(t)}{\sin t/2}\right| \leqslant L_n \cdot 2n+1 \text{ in } 0 \leqslant t \leqslant \frac{\pi}{n} \\ \text{and} \quad &|K_n(t)| \leqslant K\,|\log t| \text{ in } \frac{\pi}{n} \leqslant t \leqslant \delta < 1 \end{aligned}\right\} \tag{6·5}$$

The first part follows from the fact that $\left|\frac{\sin pt}{\sin t}\right| \leqslant p$ when p is an integer.

To prove the second part we note that

$$K_n(t) = \text{Imaginary part of} \left\{ - e^{-i(n+3/2)t} \cdot \sum_{p=1}^{n+1} \frac{e^{ipt}}{p} \right\} =$$

$$= \text{Imaginary part of} \left\{ + e^{-i(n+3/2)t} \cdot \left(\log(1-e^{it}) + \sum_{p=n+2}^{\infty} \frac{e^{ipt}}{p} \right) \right\}$$

$$= \text{Imaginary part of} \left\{ e^{-i(n+3/2)t} (P_1 + P_2) \right\} \tag{6·6}$$

Now $|P_1| = O\{|\log t|\}$ when $0 \leqslant t \leqslant \delta < 1$

and in $\frac{\pi}{n} \leqslant t \leqslant \delta < 1$ $|P_2| \leqslant \frac{K}{n \cdot t} = O(1)$, so that second part of 6·5 is established. (6·7)

Now $$\int_0^{\delta} \psi(t) \frac{K_n(t)}{\sin t/2}\, dt = \int_0^{\pi/n} + \int_{\pi/n}^{\delta}$$

$$\left.\begin{aligned} &\text{(6·2) and (6·5) imply that } \left|\int_0^{\pi/n}\right| \leqslant K \cdot \epsilon \cdot L_n \text{ and} \\ &\left|\int_{\pi/n}^{\delta}\right| \leqslant K_1 \int_{\pi/n}^{\delta} \frac{|\psi(t)\log t|}{t}\, dt \leqslant K_2 \cdot \epsilon \cdot \log n \end{aligned}\right\} \tag{6·8}$$

(6·3) and (6·8) implies that $S_n(x_0) \to f(x_0)$, (N, 1) thus establishing Theorem II.

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