LIE RINGS IN PATH SPACE

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Communicated by Oswald Veblen, May 8, 1949

1. In studying the tensor analysis of the system of differential equations

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0; \quad i = 1, 2, \ldots, n; \quad \dot{x}^i = dx^i/dt, \text{ etc.,} \tag{1.1}$$

their equations of variation

$$\theta u^i \equiv \ddot{u}^i + \alpha^i{}_{;r}\dot{u}^r + \alpha^i{}_{,r}u^r = 0 \tag{1.2}$$

have been found to be a prime tool of exploration. The notation used is, for any function $\varphi(x, \dot{x}, t)$,

$$\varphi_{,r} = \frac{\partial \varphi}{\partial x^r}; \quad \varphi_{;k} = \frac{\partial \varphi}{\partial \dot{x}^k}; \quad \dot{\varphi} = \frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \varphi_{,r}\dot{x}^r - \varphi_{;r}\alpha^r \tag{1.3}$$

and the tensor summation convention is followed for indices repeated in subscript and superscript.

These equations of variation (1.2) to be regarded as partial differential equations in u^i are obtained from (1.1) by the "infinitesimal change," $\bar{x}^i = x^i + u^i\delta\tau$, where $\bar{x}$ and x are supposed to be coördinates of points on "nearby" paths. Thus, $\delta x^i = u^i\delta\tau$ and the convention usually adopted is $\delta\dot{x}^i = \dot{u}^i\delta\tau$, which is derived from the assumption $d\delta - \delta d = 0$; this appears in all classical texts without further clarification. The α^i are assumed to be arbitrarily differentiable, which allows an expansion in which second and higher powers of the parameter $\delta\tau$ are to be neglected.

2. The only meaning (unless we fix the basic path in 1.2) that can be attached to $\delta x^i = u^i\delta\tau$ is of an infinitesimal transformation of a one-parameter Lie group. But the solutions of $\theta u^i = 0$ are generally functions of $x, \dot{x}, t$, whence the most general such one-parameter group must be taken as operating in the V_{2n+1} of $t, x, \dot{x}$, where $\dot{x}^i$ is to be regarded as a fiber-bundle attached to a generic point of the base-space defined by coördinates x^i. The only operator which can be used here must necessarily be of "the first extension," i.e.,

$$X \equiv u^r \frac{\partial}{\partial x^r} + \dot{u}^r \frac{\partial}{\partial \dot{x}^r} \tag{2.1}$$

and the usual infinite series expansion

$$\bar{\psi} = \psi + \tau X\psi + \frac{\tau^2}{2} X^2\psi + \dots \tag{2.2}$$

can then be obtained, at least symbolically, in the case where the α^i are analytic, to which we restrict our entire discussion. Given several distinct solutions $u^i, v^i, w^i, \dots$ of the equations of variation with the associated (extended) operators $X, Y, Z, \dots$ the question then naturally arises as to the existence of a Lie group, with more than one parameter being formed in some way out of these. For this, the alternant $\{X, Y\} = XY - YX$ must itself be an operator of the first extension with respect to the paths. That is, if

$$XY - YX = \mu^r \frac{\partial}{\partial x^r} + \lambda^r \frac{\partial}{\partial \dot{x}^r}, \text{ then } \dot{\mu}^r - \lambda^r = 0. \tag{2.3}$$

Furthermore, the Jacobi condition must also be satisfied:

$$\{X, \{Y, Z\}\} + \{Y, \{Z, X\}\} + \{Z, \{X, Y\}\} = 0 \quad \text{for all } X, Y, Z. \tag{2.4}$$

After this, we can see whether the solutions of the equations of variation form a Lie algebra, and over what fields.

Direct calculation gives us

THEOREM 1. *Two (extended) operators X, Y associated with vectors u^i, v^i alternate to give one of the same type if the vectors concerned are (each) solutions* *of the equations of variation* $\theta u^i = 0$ *or of* $u^i_{;j} = 0$. *Similarly for the Jacobi condition on three operators.*

The proofs can be shortened considerably by noting another result

THEOREM 2. *The solutions of* $\theta u^i = 0$ *are just those whose associated operators permute with the linear operator* d/dt *of 1.3.*

Proof: We have

$$Xd/dt - dX/dt \equiv \theta u^r \frac{\partial}{\partial \dot{x}^r}, \tag{2.5}$$

which suffices for the second theorem. To use this result, we may put $Z = XY - YX$, and note that

$$dZ/dt - Zd/dt = dXY/dt - dYX/dt - XYd/dt + YXd/dt \equiv 0. \tag{2.6}$$

The vanishing identically follows from the permutability of d/dt with both X and Y, and then proves that $XY - YX$ is also permutable with d/dt, whence the associated operator for Z must be formed from the solutions of the equations of variation, provided of course it was of the first extension. For solutions of $u^i_{;j} = 0$, the proof is trivial. Similarly for the Jacobi identity.

THEOREM 3. *The solutions of* $\theta u^i = 0$ *form a vector space which gives a Lie ring over the set of functions* φ *with* $\dot{\varphi} = 0$ *(i.e., constant along any path) and a Lie algebra (and therefore defines a Lie group in the analytic case under discussion) over the field of all real constants.*

The latter statement follows from well-known results in Lie groups. For the former case, we have merely to note that φu^i is a solution of the equations of variation with u^i provided $\dot{\varphi} = 0$. However, we only have here $\{\varphi X, Y\} = \varphi\{X, Y\} - (Y\varphi)X$, whence we get a *ring* over the set $\dot{\varphi} = 0$ defined by the vector space of solutions of $\theta u^i = 0$. The basis of the ring is clearly of dimension $2n + 1$. For the Lie group, however, if we take the most general case, the dimension cannot be finite, and the question remains open whether the infinite Lie algebra and group thus obtained are equivalent to E. Cartan's infinite Lie groups.

3. We now consider the subgroups (over the field of all real constants) leaving the base space of the x as well as the paths invariant; the generators must now satisfy $\partial u^i/\partial t = 0$; $u^i_{;j} = 0$; $\theta u^i = 0$. This leads to

THEOREM 4. *The Lie group leaving the base space as well as the paths invariant is of order* $\leq n(n + 1)$.

It suffices to show that the total number of arbitrary parameters in the solutions is finite, $\leq n(n + 1)$ for then these can be specialized to give that number of independent basic solutions and the linearity of the equations allows a general solution to be formed out of linear combinations of these basic solutions. That is, the number of essential parameters being determined, they can be taken to occur in the linear combinations alone.

In this case, we have $\dot{u}^i = u^i_{,r}\dot{x}^r$ and $\ddot{u}^i = u^i_{,j,k}\dot{x}^j\dot{x}^k u^i_{,r}\alpha^r$. Now the equations $\theta u^i = 0$ may be differentiated successively, because of $\partial u^i/\partial t = 0$ to give a succession of homogeneous linear conditions on $u^i_{,j}$, u^k. Differentiating the equations of variation with respect to $\dot{x}^s$ successively gives on the *second* differentiation an explicit equation for $u^i_{,j,k}$, and thereafter linear homogeneous restrictions in u^i, $u^i_{,r}$. From the original equations $\theta u^i = 0$, and from the first $\dot{x}$ derivative thereof, we may therefore eliminate $u^i_{,j,k}$, and obtain two more linear homogeneous restrictions on $u^i_{,r}, u_j$. The problem therefore is reduced to solving a system of first order partial differential equations for the variables u^i and $v^i_j = u^i_{,j}$ (adjoining this last differential equation to the system), along with linear homogeneous restrictions upon the variables v^i_j, u^k, to which may be added others derived from the compatibility conditions. In any case, the solution $u^i = v^j{}_k = 0$ always exists, corresponding to the identity as the sole Lie group for the path-space. But it is well known (cf., E. Goursat, chapter 1 of "Leçons sur . . . équations aux dérivées partielles") that the total number of arbitrary parameters in the general solution is equal to the number of variables which cannot be eliminated from the conditions of compatibility and the linear restrictions, which in no case can exceed the total original number of the variables. Here, that number is n for the u^i plus n^2 for the v^i_j, proving our theorem.

It is easy to see that *the maximum number of parameters for the group may actually be attained.* The simplest example is of $\ddot{x}^i = 0$, the paths being straight lines. The group is then that of the translations with n parameters, plus the linear transformations leaving the origin invariant, of order n^2. In the Riemannian case, for example, as with the equations of Killing, something more is demanded, namely the invariance of a quadratic form as well, whence the maximum order is half the above. For the path-space of straight lines, if we impose, say, a Euclidean metric, the group is then translations plus rotations, order $n + n(n - 1)/2 = n(n + 1)/2$.

4. Further extensions of the previous results are possible in several directions, e.g.:

The group whose generating vectors satisfy only $u^i_{;j} = 0$, $\theta u^i = 0$, *thus transforming into itself the space* (x, t), *while leaving the absolute parameter* t *unchanged, has a number of parameters* $\leq n(n + 2)$.

The point transformations corresponding to this are Cartan's group B, and their tensor invariants can be found now in an obvious way. The proof parallels the above step by step.

These processes can be carried out also for systems of ordinary differential equations of higher order, as well for partial differential equations, the sole condition being that they be explicitly soluble for the derivatives of highest total order. The d/dt operator for ordinary differential equations has to be again defined as total differentiation along the paths, whereas for partial differentiation we have as many such operators as there are independent variables. We can then prove as before:

THEOREM 5: *The results of theorems 1–3 are valid also for the systems*

$$\frac{d^{\sigma+1}x^i}{dt^{\sigma+1}} + \alpha^i\left(t, x, \frac{dx^i}{dt}, \ldots, \frac{d^\sigma x^i}{dt^\sigma}\right) = 0, \qquad (4.1)$$

the equations of variation being defined by all those (extended) operators which permute with d/dt, itself defined as:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{dx^r}{dt}\frac{\partial}{\partial x^r} + \ldots + \frac{d^\sigma x^r}{dt^\sigma}\frac{\partial}{\partial x^{(\sigma-1)r}} - \alpha^r\frac{\partial}{\partial x^{(\sigma)r}}. \qquad (4.2)$$

The maximum number of parameters in the group leaving base-space as well as paths invariant cannot then exceed the sum of the first $(\sigma + 1)$ terms of the series

$$n\left\{1 + n + \frac{n(n+1)}{2} + \frac{n(n+1)(n+2)}{3} + \ldots\right\}. \qquad (4.3)$$

For partial differential equations, we consider only the second order system:

$$\frac{\partial^2 x^i}{\partial t^2 \partial t^\beta} + H^i_{\alpha\beta}(t, x, p^j_\gamma) = 0; \qquad \begin{aligned} \alpha, \beta &= 1, \ldots m, \\ i, j, \ldots &= 1, \ldots n, \\ p^i_\alpha &= \partial x^i/\partial t^\alpha. \end{aligned}$$

Here the operators corresponding to d/dt are the set ∂_α defined by

$$\partial_\alpha \equiv \partial/\partial t^\alpha + p^r_\alpha \partial/\partial_x r \div H^r_{\alpha\beta}\partial/\partial p^r_\beta. \qquad (4.5)$$

THEOREM 6. *The equations of variation have as solutions those vectors and only those whose associated (extended) operators form the ring that permutes with all operators ∂_α. The maximum number of parameters for the group leaving base-space and paths invariant cannot exceed $n(n + 1)$, as before.*

The proof is by following the case of ordinary differential equations step by step, and the condition for composition of two operators as well as the Jacobi condition may be derived by direct calculation. The number of parameters is again from considerations of the variation vector u^i being a function of the same number of variables x, as is seen directly from the structure of the equations of variation. However, in this case, there are also equations of variation for the independent variables t^ρ, and it should be made clear that the base-space is of the variables x^i.

For equations (1.1) which are deducible from a metric, i.e., the extremals of a regular problem of the calculus of variations, we have the following formula. If the (inverse) Eulerian equations be abbreviated by $\delta_i f = 0$, then the result of any operation X of the base-space group is:

$$\delta_i(Xf) = X(\delta_i f) + u^r_{,i}\delta_r f + f_{;i;r}\theta u^r. \tag{4.6}$$

This shows what would otherwise have been expected:

THEOREM 7: *If a metric exists for the paths, it is carried into metric by any transformation of the group preserving base space and paths.*

For the simple case $\ddot{x}^i = 0$, a general metric is any arbitrary function $f(x)$ with the determinant $|f_{;i;j}| \neq 0$, and not containing the x^i at all. This is carried into another of the type by any linear homogeneous transformation, and into itself by any translation. The only possible additive terms are necessarily of type dh/dt, where $h(x, x)$ is homogeneous of degree zero in $\dot{x}$, as can easily be shown.

It is to be noted, in conclusion, that the conditions of theorem 1 are not necessary. For the alternant of two extended operators associated with vectors u, v to give an extended operator the precise condition is

$$v^i_{;r}\theta u^r - u^i_{;r}\theta v^r = 0. \tag{4.7}$$

Similarly, the Jacobi condition for three such operators is satisfied if and only if

$$u^i_{;r}\{v^r_{;k}\theta w^k - w^r_{;k}\theta v^k\} + \text{(two more terms by cyclic rotation)} = 0. \tag{4.8}$$

This shows, in particular, that it even suffices to have one of the two vectors a solution of both $\theta u^i = 0$ and $u^i_{;j} = 0$; in the Jacobi condition, it is again sufficient for one of the three to satisfy both these equations. Thus, the infinitesimal transformations not containing $\dot{x}^i$, and in particular the subgroup leaving both base-space and paths invariant, have a special position.

The second remark is about the possibility of defining a Lie differential operator that carries tensors into others of the same type, but of defining it in a manner that can be carried over to more general classes of transformations, such as those over the entire path-space. To this end, we may note that any such transformation may be regarded either as a change of variables, or a charge of coördinates. *The (infinitesimal) Lie operation gives the difference of the (infinitesimal) changes in any geometric object due first to regarding the transformation as a change of variables, and then as a change of coördinates.* Thus, for the tensor T^i_j of weight p under transformations preserving the base-space, we have, when $u^i_{;j} = 0$,

$$LT^i_j \equiv T^i_{j,r}u^r + T^i_{j;r}u^r - T^r_j\dot{u}^i_{,r} + T^i_r u^r_{,j} + pT^i_j u^r_{,r}. \tag{4.9}$$

Moreover, there is the infinite series expansion as in (2.2), and we have the tensor carried over into another of the same sort, independently of any connection that may be assigned to the x-space. The main definition is obviously extensible for more general transformations.