HOMOGENEOUS METRICS.

By D. D. Kosambi,
Fergusson College, Poona.

Received May 10, 1935.

Using the notation of my former papers on the same subject,[1] I discuss the necessary and sufficient conditions for the existence of a metric homogeneous in the direction components $\dot{x}^i$. The general result may be restated as follows: *for trajectories described without an external force* ($\epsilon^i = 0$), *the velocity is homogeneous in the velocity components* $\dot{x}^i$ *of degree one or zero.* This is a loose dynamical equivalent of the discussion which follows in geometrical terminology. For the presupposed results and unproved identities, the paper referred to will be useful.[2]

The Euler equations associated with $\delta \int f(x, \dot{x})\, dt = 0$ may be written in the tensor-invariant form

[1] $$\delta_i f \equiv \mathcal{D} f_{;i} - f_{|i} \equiv (\mathcal{D} f)_{;i} - 2 f_{|i} \equiv - \alpha^r f_{;r;i} + \dot{x}^r f_{,r;i} - f_{,i} = 0$$

Suppose the paths of the space to be given by

[2] $$\ddot{x}^i + \alpha^i (x, \dot{x}) = 0 \qquad i = 1 \cdots n$$

Then [1] may be regarded as a system of partial differential equations for the unknown metric f. This system of the second order breaks up into $n+1$ equations of the first order if and only if $\mathcal{D} f = 0$ *i.e.* $f =$ const. along the paths. The resulting equations

[3] $$\mathcal{D} f \equiv - \alpha^r f_{;r} + \dot{x}^r f_{,r} = 0 \qquad f_{|i} \equiv f_{,i} - \tfrac{1}{2} \alpha^r{}_{;i} f_{;r} = 0$$

may be discussed by the standard methods for linear partial differential equations of the first order, and the existence of f is a matter of pure algebra, the conditions involving merely the coefficient of successive derived equations:

[4] $$\begin{aligned} \epsilon^r \ f_{;r} &= 0 \\ \mathrm{P}^r{}_i \ f_{;r} &= 0 \\ \mathrm{R}^r{}_{ij} \ f_{;r} &= 0 \end{aligned}$$

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1 *The Quarterly Journal of Mathematics* [Oxford Series], 1935, **6**, 1-12.

2 *Journal Ind. Math. Society, Jubilee Volume*, 185-188, will also be of use in this connection. It should be kept in mind that the parameter t is taken as nowhere explicitly present in this note. The identities as used derive from those in the two papers cited, often for $\epsilon^i = 0$.

The matrix of all these coefficients must be of rank less than n for a solution to exist.

These special metrics have the fundamental property that any function of any given set of possible metrics which satisfy [3] is also a metric of the same type. We call these metrics "invariant metrics".

All metrics, whether invariant or not, must satisfy the energy integral

[5] $$\dot{x}^r f_{;r} - f = \text{const. along the paths.}$$

Now if this and the integral $f =$ const. (which an invariant metric allows) be not independent, there must subsist a relation of the type

[6] $$\dot{x}^r f_{;r} = \phi(f)$$

In this case, we apply the following lemma:

LEMMA: *A necessary and sufficient condition that a function $f(y_1, y_2, \cdots y_n)$ be of the type $f(\lambda y_i) = \psi(\lambda, f)$ is that $y_i \, \partial f / \partial y_i = \phi(f)$. In this case, either f is homogeneous of degree zero in y ($\phi = 0$), or there exist functions of f homogeneous of any given degree, one of the first degree being $H(f)$ where $H(z) = exp. \int dz/\phi(z)$.*

This shows that if the two first integrals available when an invariant metric exists be not independent, there is a homogeneous metric for the space.

The foregoing results are necessary preliminaries to the main deductions of this note. It is clear that if there is a homogeneous metric of any degree but the first, then $\epsilon^i = 0$. This can be shown directly, as such a metric is automatically an invariant metric with related integrals as above. From [6], we obtain

[7] $$\dot{x}^r f_{;r;i} = (\phi' - 1) f_{;r}$$

Applying the operator $\mathcal{D}$, and keeping in mind the results $\mathcal{D} f_{;i;j} = 0$ (for any metric) and $\mathcal{D} f_{;i} = 0$ (for invariant metrics) we obtain

[8] $$\mathcal{D} \{\dot{x}^r f_{;r;i}\} = -\epsilon^r f_{;r;i} = 0$$

This gives $\epsilon^i = 0$, provided the metric is non-degenerate *i.e.* $|f_{;i;j}| \neq 0$ a necessary assumption. The Finslėr metric, the most important of all, is left out of this deduction. But if we recall the fact that for a Finsler space the corresponding problem of the calculus of variations is not regular, and that it is usually regularized by the assumption that $f =$ const. along the extremals, we see that the Finsler metric in use is also an invariant metric, to which the discussion then applies. Therefore, $\epsilon^i = 0$ is a necessary condition for the homogeneity of the metric.

There is nothing new in this result, except the approach. It is the converse which is of interest and importance. Suppose $\epsilon^i = 0$. Then for any tensor,

$$\mathcal{D}\, T_{ij} = T_{ij|r}\, \dot{x}^r + \partial T_{ij}/\partial t \tag{9}$$

This gives, for the fundamental tensor, $f_{;i;j|r}\, \dot{x}^r = 0$. For any vector, we have $u_{i;j|k} - u_{i|k;j} = \frac{1}{2}\, u_r\, \alpha^r{}_{;i;j;k}$. For any metric, $f_{;i|j} - f_{;j|i} = 0$. These lead to $f_{|i;j;r}\, \dot{x}^r = 0$, *i.e.* either $f_{|r} = 0$, or f is the sum of a function homogeneous of degree one in $\dot{x}$, and a function independent of $\dot{x}$. It can be seen directly from [1] that when $\epsilon^i = 0$, the α^i being homogeneous of degree two, the second part of f must be identically a constant. If there exist a metric, then, either it is an invariant metric, or a Finsler metric.

If the metric for $\epsilon^i = 0$ is an invariant one, we can show at once that it must admit the integral $\dot{x}^r f_{;r} = \phi\,(f)$. In fact, regard this as a further equation for f, and add it to the system [3]. In the system [3], the first equation is a consequence of the remaining n when $\epsilon^i = 0$. The compatibility conditions for $f_{|i} = 0$ and $\dot{x}^r f_{;r} = \phi$ are identically satisfied. The condition $\dot{x}^r f_{;r} = \phi$ and the equations [4] are seen to be compatible in virtue of the identities $P^i{}_r\, \dot{x}^r = 0$, $R^i{}_{jk;r}\, \dot{x}^r = R^i{}_{jk}$, etc. which can be proved from the fundamental identities that hold between the differential invariants of the space, keeping in mind that here, $\epsilon^i = 0$, $\partial\alpha^i/\partial t = 0$. Thus, if [3] has a solution, it has one with $\dot{x}^r f_{;r} = \phi$ added on.

Therefore, even if the metric is an invariant metric, it must be homogeneous. All the results may be summed up as follows:

THEOREM: *If there exist a metric for a given path space where* $\dfrac{\partial \alpha^i}{\partial t} = 0$, *a necessary and sufficient condition for the existence of a homogeneous metric giving the same paths as geodesics is that* $\alpha^i - \frac{1}{2}\, \dot{x}^r\, \alpha^i{}_{;r} = 0$. *In this case, either a Finsler metric is available for the space, or the only possible metric is homogeneous of degree zero in* $\dot{x}$.