AN AFFINE CALCULUS OF VARIATIONS.

By D. D. Kosambi,
Fergusson College, Poona.

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I use the word *affine* here in the sense of *non-metric*, and not with reference to the transformation group of a parameter. More specifically, we shall deal with the differential geometry of a system of n equations of order $\sigma+1$:

(I) $x^{(\sigma+1)} + \alpha^i(x, x^{(1)}, x^{(2)}, \cdots, x^{\sigma}, t) = 0 \quad (i = 1, 2, \cdots, n.)$

Besides the tensor summation convention, we shall use the notation:

$$u^{(0)i} = u^i \qquad u^{\nu i} = u^{(\nu)i} = \frac{du^{(\nu-1)i}}{dt} \qquad f_{,i} = f_{(0)i} = \frac{\partial f}{\partial x^i} \qquad f_{\nu i} = f_{(\nu)i} = \frac{\partial f}{\partial x^{\nu i}}$$

Greek indices, when summed, will range over the values $0, 1, 2, \cdots, \sigma$.

We may assume various kinds of parallelisms or differential operators for this geometry. But it will be seen[1] that the simplest differential operator which carries a tensor into another of the same rank and is distributive in addition, is necessarily of the type

$$(1) \qquad \mathcal{D}u^i = \frac{du^i}{dt} + \gamma^i_{\ r}u^r \qquad \frac{d}{dt} = -\alpha^r \frac{\partial}{\partial x^{\sigma r}} + \sum_{\nu=0}^{\sigma-1} x^{(\nu+1)r} \frac{\partial}{\partial x^{\nu r}},$$

where the coefficients $\gamma^i_{\ j}$ have the proper law of transformation when coordinates undergo a non-singular change independent of the absolute parameter t. The difference of any two sets of coefficients $\gamma^i_{\ j} - \gamma'^i_{\ j}$ would be a tensor of the rank indicated as an invariant of the connection. But to determine an *intrinsic* connection and the set of differential in variants arising therefrom, we shall have to refer to the equations of variation of (I), *i.e.*,

(II) $u^{(\sigma+1)i} + u^{\nu r}\alpha^i_{\ \nu r} = 0$

(I) and (II) are assumed, of course, to be tensor invariant in form, with u^i a vector variation which carries *paths* [solutions of (I)] into other paths, when multiplied by an infinitesimal factor as usual. To bring out the tensor invariance of (II), they may be written in the form

$$(2) \qquad \mathcal{D}^{(\sigma+1)}u^i + \mathcal{D}^{\nu}u^r \underset{(\nu)}{\mathrm{P}^i_{\ r}} = 0 \quad \mathcal{D}^{\nu} = \mathcal{D}(\mathcal{D}^{\nu-1}), \text{ etc.}$$

The coefficient of $\mathcal{D}^{\sigma}u^r$ is $\underset{(\sigma)}{\mathrm{P}^i_{\ r}} = \alpha^i_{\ \sigma r} - (\sigma+1)\gamma^i_{\ r}$. If this is taken to be zero, the intrinsic connection is at once determined as $\gamma^i_{\ j} = \frac{1}{\sigma+1}\alpha^i_{\ j}$. The

[1] D. D. Kosambi, "Parallelism and Path-Spaces," *Math. Zeit.*, 1933, pp. 608-618.

remaining coefficients $\underset{(0)}{\mathrm{P}}{}^i_j$, $\underset{(1)}{\mathrm{P}}{}^i_j$, $\cdots$, $\underset{(\sigma-1)}{\mathrm{P}}{}^i_j$ calculated with this determination of the connection, are intrinsic differential invariants of the space, occupying the position of primary curvature tensors. The full set of differential invariants may be obtained by alternating the fundamental differential operations

$$\frac{\partial}{\partial x^\sigma} \qquad \mathfrak{D} \qquad \frac{\partial}{\partial t}$$

It is the first two that are most productive; we obtain a secondary set of differential operators corresponding to the ordinary covariant derivative:

$$(3) \qquad \begin{aligned} \underset{(1)j}{\nabla} u^k &= (\mathfrak{D}u^k)_{\sigma j} - \mathfrak{D}u^k{}_{\sigma j} \\ \underset{(2)j}{\nabla} u^k &= \underset{(1)j}{\nabla}\mathfrak{D}u^k - \mathfrak{D}\underset{(1)j}{\nabla} u^k \\ &\cdots\cdots\cdots \\ \underset{(\sigma)j}{\nabla} u^k &= \underset{(\sigma-1)j}{\nabla}\mathfrak{D}u^k - \mathfrak{D}\underset{(\sigma-1)}{\nabla} u^k \end{aligned}$$

In the actual computation of each of these, it must be noted that certain tensorial terms may occur, due to the fact that $a^i{}_{\sigma j \sigma k}$ is a tensor of the space when the order $\sigma+1$ is greater than two. Such terms are left out altogether without affecting the tensorial nature of the operation. The operators $\underset{(\nu)j}{\nabla}$ as well as $\mathfrak{D}$ are extensible to tensors of any rank, as in the usual covariant differentiation. It should be noted that whereas only $\partial/\partial x^{(\sigma)}$ is tensorial, any $\partial/\partial x^\nu$ will give a tensor with an additional covariant index when applied to a tensor containing no higher derivative of x^i than $x^{\nu i}$. But the general processes are as given above.

Further alternations of the primary and secondary differential operators will give relations involving these operators with coefficients which give the differential invariants of the space, of which a full set may thus be obtained. The $\underset{(\nu)}{\mathrm{P}}{}^i_j$ and $a^i{}_{\sigma j \sigma k}$ with $x^{(1)i}$ itself constitute the primary set, when $\sigma > 1$.

The calculus of variations has always been a guide in the construction of such generalized differential geometries. It corresponds to the metric case. A necessary condition for the existence of a metric is that the equations (II) be self-adjoint. It is, presumably, a sufficient condition also, though the general theorem is not easy to prove. For $\sigma = 1$, it has been demonstrated, and, for that matter, the complete differential geometry of the space has been worked out on the lines sketched above.[2] I treat here

[2] D. D. Kosambi, "Systems of Differential Equations of the Second Order," *Quarterly Journal of Math.*, Oxford Series, 1935, pp. 1-12.

the case $\sigma = 0$, as it is so simple as to give only a few of the features of a complete differential geometry, and for that reason, will not be considered elsewhere.

The paths being given by

$$\frac{dx^i}{dt} + a^i(x, t) = 0 \qquad i = 1, 2, \cdots, n$$

the vector a^i itself is the essential differential invariant of the space. If now a tensor f_{ij} be sought which associates covariant and contravariant indices according to the usual law, the equations of variation of

$$f_{ir}\{\dot{x}^r + a^r\} = 0$$

become self-adjoint if and only if

$$(4) \quad f_{ij} + f_{ji} = 0$$

$$(5) \quad f_{ij,k} + f_{jk,i} + f_{ki,j} = 0$$

$$(6) \quad -a^k f_{ij,k} + \frac{\partial}{\partial t} f_{ij} - f_{ik}\, a^k_{,j} - f_{kj}\, a^k_{,i} = 0$$

Of these, (4) and (5) show that f_{ij} must be the curl of a vector. Moreover, n, the number of dimensions of the space, must be even, if the skew-symmetric determinant $|f_{ij}|$ is not to vanish identically. Writing the fundamental tensor as $f_{ij} = \phi_{i,j} - \phi_{j,i}$ where ϕ_i is a function of x, t, we see from (6) that $(f_{ir}a^r)_{,j} - (f_{jr}a^r)_{,i} = \frac{\partial}{\partial t} f_{ij}$ which indicates that $f_{ir}a^r - \frac{\partial}{\partial t}\phi_i$ is a partial derivative, $-\psi_{,i}$. The integrand of the variational problem is obtained at once, and the paths are the extremals of $\delta\int(\phi_r \dot{x}^r + \psi)dt = 0$.