Parallelism and Path-Spaces.

By

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1. The present paper is devoted to the geometrical study of an arbitrary system of second order differential equations of the form:

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0 \qquad (i = 1, 2, \ldots, n). \tag{I}$$

The integral curves of (1) are assumed to be such that in some continuous n-dimensional region of the space (x^i) they possess the property of convexity: one and only one such curve — which we shall call *path* — passes through any two points of the region. The parameter t can be considered as an additional timelike coordinate, or an arbitrary arc-like parameter, the latter point of view being rarely stressed. Besides the tensor summation convention, I use the following notation for partial differentiation:

$$\frac{\partial A::}{\partial x^k} = A::_{,k}, \qquad \frac{\partial A::}{\partial \dot{x}^k} = A::_{;k}.$$

In a previous memoir[1]) (the knowledge of which is not assumed here) I attempted to investigate the possibility of deducing the system (1) from a variational principle

$$\delta \int f(x, \dot{x}, t)\, dt = 0 \tag{1. 1}$$

this is equivalent to finding a metric for the path-space. It is seen at once that the integrand f must be a solution of

$$-\alpha^i f_{;i;j} + \dot{x}^i f_{,i;j} + \frac{\partial}{\partial t} f_{;j} - f_{,j} = 0. \tag{1. 2}$$

These equations as they stand are tensor-invariant, but without simple or even geometrically interpretable compatibility conditions. If,

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[1]) D. D. Kosambi, *The existence of a metric and the inverse variational problem*, Bull. U. P. Acad. of Science, vol. 2. The main ideas of the investigation were set forth in a lecture to the Aligarh Mathematical Seminar on March 5. 1931. Some of the results of the present paper have also been given in a note in the Rendiconti R. Accad. dei Lincei 16 (1932), S. 410—415.

however, an additional condition is imposed to the effect that f be a constant over the paths i. e. $\frac{df}{dt} = 0$ the system (1. 2) reduces at once to

$$\begin{aligned} -\alpha^i f_{;i} + \dot{x}^i f_{,i} + \frac{\partial f}{\partial t} &= 0, \\ \tfrac{1}{2}\alpha^i_{;j} f_{;i} - f_{,j} &= 0. \end{aligned} \tag{1. 3}$$

The compatibility conditions for this first order system are easily worked out, and if $f_{,k}$ be eliminated wherever they occur, we obtain the following differential invariants as coefficients in successive equations:

$$\begin{aligned} \varepsilon^i &= \alpha^i - \frac{1}{2}\alpha^i_{;k}\dot{x}^k, \\ -2P^i_j &= \alpha^\sigma \alpha^i_{;\sigma;j} - \dot{x}^\sigma \alpha^i_{,\sigma;j} - \frac{1}{2}\alpha^\sigma_{;j}\alpha^i_{;\sigma} - \frac{\partial}{\partial t}\alpha^i_{;j} + 2\alpha^i_{,j}, \\ 3R^i_{jk} &= P^i_{j;k} - P^i_{k;j}. \end{aligned}$$

The ordinary Riemann-Christoffel curvature tensor is $R^i_{jk;l}$. M. E. Cartan[2]) has been kind enough to point out that one differential invariant, namely, $\alpha^i_{;j;k;l}$ does not so appear; but this may be regarded essentially as $\varepsilon^i_{;j;k}$. To see the geometrical bearing of these invariants it is necessary to develop the concept of parallelism for our spaces, and this is the main purpose of the present work.

Two fundamental ideas are sufficient to develop the full use of a parallelism: that the paths be autoparallel lines, and that the operator of the parallelism should have the tensorial character. From these, it is shown that a non-distributive parallelism results, but that a distributive bi-parallelism can be obtained by omitting an additive vector. If on the other hand, we keep consistently to the point of view that the inverse variational problem is essential, then the parallelism must give an invariantive form to the equations of variation of the paths. And that too leads to much the same result. These methods show no way of obtaining an additional index for a given tensor, in a fashion analogous to covariant differentiation in the Ricci-calculus. But it is seen that since only the biderivate can be defined for the general tensor, this is not a fundamental question.

In the first paper referred to, there appeared a curious result in the guise of the theorem:

A necessary and sufficient condition that the integral of any twice differentiable function $\varphi(f)$ be stationary over the extremals of $\delta\int f\,dt = 0$ is that the integrand f be a constant along the extremals i. e. $\frac{df}{dt} = 0$.

[2]) In a personal letter, an extract of which is published after this paper.

This gives the same reduction as in the system (1. 3). To account for it, it is necessary to develop the analogue of the equations of Killing. The metric f can be regarded as an invariant of a certain fundamental group, and any function thereof will naturally be an invariant also. The second order equations given by Davis[3]) in his treatment of the inverse variational problem can be deduced from (1. 2) by requiring the metric to be a relative invariant in place of an absolute invariant, which implies that $\frac{df}{dt} = \lambda f$. The arbitrary constant λ is then eliminated from the resulting compatibility conditions.

2. In the metric case, where the vanishing of the first variation gives the usual system of Euler's equations, or (as we then assume) the equivalent system (I), the important part played by the second variation of the integrand is well-known; this leads to the equations of variation, which we have to consider in order to restrict the end-points of the integral in (1. 1) to lie between two conjugate foci on the extremal. On the other hand, in the metric as well as in the non-metric case, the equations of variation can be obtained by taking

$$\bar{x}^i = x^i + u^i \delta\tau$$

where $\delta\tau$ is an infinitesimal, and $\bar{x}^i$ as well as x^i are assumed to lie on nearby paths or extremals. Substituting in (I) and neglecting higher powers of $\delta\tau$, we have:

$$\text{(II)} \qquad \ddot{u}^i + \dot{u}^k \alpha^i_{;k} + u^k \alpha^i_{,k} = 0.$$

It will be seen that these are equivalent to the equations of second variation in the metric case, provided f is non-trivial, i. e.

$$\Delta \equiv |f_{;k;l}| \neq 0.$$

We shall take (I) *and* (II) *as fundamental systems of equations for affine path-spaces, and proceed to investigate the possibility of deducing both from an unifying basis of parallelism.*

3. The *derivate* $D(u)^i$ of a set of n quantities shall be defined along a given curve as

$$\text{(3. 1)} \qquad D(u)^i = \dot{u}^i + \beta^i(x, \dot{x}, t).$$

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[3]) D. R. Davis in the Bull. Amer. Math. Soc. (1929), pp. 371—380. The equations given there would seem to be necessary, but not sufficient.

[4]) For a detailed bibliography of the subject, and in particular for references to the numerous papers of Berwald, I refer the reader to the article of Koschmieder, Jahresber. d. D. M. V. **40**, pp. 109—132. Other papers related to the present investigation are D. R. Davis, l. c. and Trans. Amer. Math. Soc. **33**, p. 246 and J. Douglas, Ann. of Math. (II) **29**.

We neglect here the possibilities of derivates containing differential coefficients of x and u of higher order than the first. Even so, a more general form might seem to be

$$D(u)^i = f^i(x, \dot{x}, u, \dot{u}, t).$$

But if the derivate is to be fundamental, we must deduce (I) from

$$f^i(x, \dot{x}, \dot{x}, \ddot{x}, t) = 0.$$

This implies that the equations

$$f(x, \dot{x}, u, \dot{u}, t) = 0$$

thought of as equations in five variables, are soluble for $\dot{u}$. Hence rather than attempt an investigation of all possible linear or functional combinations of the f^i, we might as well postulate a derivate as in (3. 1). Since an invariantive form of equations is to be desired, we restrict the derivate to be such that on a non-singular transformation of coordinates, the transformed derivate also vanishes with the original, at least when the set derived forms the components of a contravariant vector. That is to say,

$$\overline{D}(u)^i = F^i_j D(u)^j. \tag{3. 2}$$

The coefficients F^i_j must be non-singular for $D(u) = 0$, and their determinant $F = |F^i_j| \neq 0$. The simplest possible assumption that gives the result desired is that the derivate of a contravariant vector is itself a contravariant vector. The coefficients F^i_j are then $\frac{\partial \bar{x}^i}{\partial x^j}$, functions of the transformation itself, and the non-vanishing of F is equivalent to the non-singularity of the transformation.

The assumption is more restrictive than others that can be made, but it leads directly to the tensor-invariance of all our fundamental equations.

The parallel displacement of a vector or even a set of n functions along a given curve will be said by definition to take place when and only when the derivate along the curve vanishes. The vector curvature of any curve will be defined as the derivate $D(\dot{x})$ along the curve itself. Finally, the paths are to be autoparallel, or curves of zero vector curvature:

$$D(\dot{x})^i \equiv \ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0. \tag{3. 3}$$

This gives at once, a restriction on the form of the derivate,

$$\beta^i(x, \dot{x}, \dot{x}, t) = \alpha^i(x, \dot{x}, t). \tag{3. 4}$$

4. Since the equations (II) are fundamental in the metric K-space, it should be also possible to reduce them by an application of the principle of derivation to a tensor-invariant form. To this end, we shall assume $u = \frac{\bar{x} - x}{\delta \tau}$ to be a contravariant vector, which is displaced parallely along the path that is the *base* of the variation. The equations (II) must reduce to the form

$$(4.1) \qquad D^2 (u)^i \equiv D [D (u)]^i = \varphi^i (x, \dot{x}, u, t).$$

To compute φ^i equate the two forms of (II).

$$\ddot{u}^i + \dot{u}^k \alpha^i_{;k} + u^k \alpha^i_{,k}$$
$$\equiv \ddot{u}^i - \alpha^k \beta^i_{;k} + \dot{x}^k \beta^i_{,k} + \dot{u}^k \frac{\partial \beta^i}{\partial u^k} + \frac{\partial \beta^i}{\partial t} + \beta^i (x, \dot{x}, \dot{u} + \beta, t) - \varphi^i = 0.$$

It follows immediately by observing the fashion in which $\dot{u}$ enters the identity that

$$(4.2) \qquad \begin{gathered} \beta^i (x, \dot{x}, u, t) = \gamma^i_k u^k + \varepsilon^i, \\ \gamma^i_k = \gamma^i_k (x, \dot{x}, t), \quad \varepsilon^i = \varepsilon^i (x, \dot{x}, t), \\ \tfrac{1}{2} \alpha^i_{;k} = \gamma^i_k. \end{gathered}$$

Furthermore, recalling the provious identity (3.4) we have our complete formula for the derivate;

$$(4.3) \qquad \begin{gathered} \varepsilon^i = \alpha^i - \tfrac{1}{2} \dot{x}^k \alpha^i_{;k}, \\ D (u)^i = \dot{u}^i + \tfrac{1}{2} u^k \alpha^i_{;k} + [\alpha^i - \tfrac{1}{2} \dot{x}^k \alpha^i_{;k}]. \end{gathered}$$

Note that the residual coefficients ε^i vanish identically if and only if $\alpha^i (\lambda \dot{x}) = \lambda^2 \alpha^i (\dot{x})$. Whenever the residual coefficients are zero, we have the derivate of the null vector also vanish, and the operation of the derivate becomes distributive;

$$(4.4) \qquad D (u + v) - \{D (u) + D (v)\} = - \varepsilon^i = 0.$$

The vanishing of ε^i is seen to be a necessary as well as a sufficient condition for the last. Postulating a distributive law a priori for the derivate would have greatly restricted the spaces for which the operation had a meaning.

5. The equations (II) or (4.1) can now be written in the invariantive form

$$D^2 (u)^i = u^r S^i_r + D (\varepsilon)^i$$

where

$$(5.1) \qquad S^i_r = \gamma^i_k \gamma^k_r - \alpha^i_{,r} - \gamma^i_{r;k} \alpha^k + \dot{x}^k \gamma^i_{r,k} + \frac{\partial \gamma^i_r}{\partial t} = P^i_r.$$

It can be seen that S_r^i is a mixed tensor. For if $D(u)^i$ is to be a vector with u^i then ε^i must be the components of a vector, which we call the residual vector. And we have assumed that u is a vector, and $D(u)$, $D^2(u)$ are then vectors also. The chief usefulness of our equations is seen to be in their normal form. That is, when the equations can be reduced by means of some change of coordinates that is non-singular, and brings (5.1) to

$$\ddot{u}^i = P_r^i u^r. \tag{5.2}$$

This implies that we can make

$$\ddot{u} = D^2(u) - D(\varepsilon)$$

along the path that forms the base. The transformation must therefore be that particular one which makes

$$\gamma_k^i = \tfrac{1}{2}\alpha^i_{\ ;k} = 0$$

along the base. We can thus state a theorem.

A necessary and sufficient condition for the reduction of the equations of variation to the normal form is the existence of a non-singular transformation of coordinates for which $\gamma_k^i = \alpha^i_{\ ;k} = 0$ *along the given base*[5]).

Thus, we see the need for what amounts to an extended theorem of Fermi for our K-spaces. This is proved in a later section though when the residual vector is a identically null, the proof for symmetric affine connections is immediately extensible.

In the normal form, the equations (II) give the geodesic deviation of Levi-Cività. Successive roots of $|u_j^i| = 0$ give conjugate foci on the base. $|u_j^i|$ is the determinant of n independent solution of (5.2), or of (II). Dynamical stability for systems that are given by (I) would mean that u can be made arbitrarily small for all values of the parameter by choosing reasonably small values of u und $\dot{u}$ initially. But here again, the various definitions of stability will have to be considered separately.

By taking $v(x)$ to be a vector field one can consider the existence of a covariant derivative $v^i_{\ |r}$

$$\begin{aligned} D(v)^i &= v^i_{\ |r}\dot{x} = \dot{x}^r [v^i_{\ ,r} + \gamma^i_{k\,r} v^k + \varepsilon^i_r], \\ \dot{x}^r \gamma^i_{k\,r} &= \tfrac{1}{2}\alpha^i_{\ ;k}, \quad \dot{x}^r \varepsilon^i_r = \alpha^i - \tfrac{1}{2}\dot{x}^k \alpha^i_{\ ;k}. \end{aligned} \tag{5.3}$$

[5]) Note that for other parallelisms, where $\gamma_k^i \neq \tfrac{1}{2}\alpha^i_{\ ;k}$, we must have transformations that make both γ_k^i and $\alpha^i_{\ ;k}$ vanish simultaneously on the base, for reduction to the normal form; this is the general necessary and suff. condition.

To be of any use at all the covariant derivative must be independent of the direction of derivation, which gives

$$(5.4) \qquad \alpha^i = \dot{x}^k \dot{x}^j \Gamma^i_{jk}(x,t), \quad \Gamma^i_{jk} = \Gamma^i_{kj}$$

for the most general K-spaces in which a proper covariant derivative exists: the symmetric affine connections.

If, however, we use the general parallelism γ^i_k and ε^i where $\varepsilon^i = \alpha^i - \dot{x}^k \gamma^i_k \quad 2\gamma^i_k \neq \alpha^i_{;k}$ we can still get valid results; these are seen to be;

$$(5.5) \qquad \begin{aligned} \gamma^i_k &= \dot{x}^r \gamma^i_{kr}(x,t) \\ \varepsilon^i_r(x,t) &= \alpha^i_{;k} - [\gamma^i_{kr} + \gamma^i_{rk}]\,\dot{x}^k \end{aligned}$$

i.:

$$\alpha^i = \gamma^i_{rk}\dot{x}^r \dot{x}^k + \varepsilon^i_r \dot{x}^r.$$

These are the most general parallelisms admitting a covariant derivative independent of direction. And it is seen that the torsion tensor is given by

$$(5.6) \qquad \Omega^i_{jk} = \gamma^i_{jk} - \gamma^i_{kj} = \gamma^i_{j;k} - \gamma^i_{k;j}.$$

6. As ε^i is a vector on the assumption that $D(u)$ is always a vector with u, we can have a restricted derivate, or the "biderivate"

$$(6.1) \qquad \mathfrak{D}(u)^i = \dot{u}^i + \tfrac{1}{2}\alpha^i_{;k}\, u^k.$$

The operation so defined becomes distributive, and is also a vector with u. We get correspondingly bipaths and a biparallelism and the reduction of the equations of variation is simplified, though for the canonical form, the necessary and sufficient condition reads as before.

The same results are seen to be true even for a more general tensor analysis, of the sort suggested in the first paper. For instance, let the vector law of transformation be

$$(6.2) \qquad \bar{u}^i = F^i_j(x,\dot{x},t)\, u^j, \quad F = |F^i_j| \neq 0.$$

where the coefficients F^i_j are functions of the transformation, as well as of any particular curve along which the vector may be defined. We again assume that the derivate $D(u)$ is a vector with u and that the paths are given by $D(\dot{x}) \equiv \ddot{x}^i + \alpha^i(x,\dot{x},t) = 0$ as before. Then the identities following must hold for all vectors u.

$$(6.3) \qquad \begin{gathered} D(u)^i = \dot{u}^i + \beta^i(x,\dot{x},u,t), \\ \bar{D}(\bar{u})^i = F^i_j D(u)^j, \\ \dot{u}^j F^i_j + u^j \frac{d}{dt}F^i_j + \beta^i(\bar{x},\dot{\bar{x}},u^j F^i_j,t) \equiv F^i_j[\dot{u}^j + \beta^j(x,\dot{x},u,t)]. \end{gathered}$$

The following results are then read off by inspection:

a) The β^i are linear in u;

$$\beta^i(x, \dot{x}, u, t) = u^k \gamma^i_k(x, \dot{x}, t) + \varepsilon^i(x, \dot{x}, t).$$

b) The residual coefficients ε^i still form a vector;

$$\varepsilon^i = \alpha^i - \gamma^i_k \dot{x}^k,$$
$$\bar{\varepsilon}^i = F^i_j \varepsilon^j.$$

c) The law of transformation for γ^i_k is

$$\bar{\gamma}^i_k F^k_j + \frac{d}{dt} F^i_j = F^i_k \gamma^k_j.$$

We have again a vector biderivate. The complete determination of the γ^i_k requires further conditions, which we can impose as before, with the same results.

The most general transformation laws, which involve the vector itself, are not feasible, for

$$\bar{u}^i = u^j F^i_j(x, \dot{x}, u, t)$$

implies that

$$\bar{D}(\bar{u})^i = D(u)^j F^i_j(x, \dot{x}, D(u), t).$$

To solve the resulting equations, we ought to assume that

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0$$

when $D(\dot{x}) = 0$ but without the identical relationship.

The analysis becomes complicated, and as yet, I see no elegant presentation, or even any particularly interesting results that can be included here.

7. To revert to the discussion of reduction to a normal form of the equations (II), we note first of all that the standard proof by Eisenhart[6]) of the theorem of Fermi can be extended as it stands to the case $\varepsilon^i = 0$. Furthermore, the formula (6.3) c shows that the $\bar{\gamma}^i_k$ will all vanish provided only

$$\gamma^i_k \bar{F}^k_j + \frac{d}{dt} \bar{F}^i_j = 0. \tag{7.1}$$

If this is scrutinized accurately, one can see at once that a sufficient condition for the reducibility in question is the existence of n independent families of vectors $\lambda^i_{(j)}$ that are all *biparallel* along the base, and equal there to the coefficients of the inverse transformation to F^i_j. We must have

$$\lambda^i_{(j)} = \bar{F}^i_j \qquad \bar{F}^k_j F^i_k = \delta^i_j = \bar{\lambda}^i_{(j)} \tag{7.2}$$
$$\mathfrak{D}(\lambda^i_{(j)}) = 0.$$

[6]) See Eisenhart, *Non-Riemannian Geometry*, Am. Math. Soc. Coll. (1927), pp. 64—67.

For the ordinary laws of point transformation, this can always be done, as is seen from the work of Eisenhart cited, formula (25.10), where such a transformation is built up for any curve as base; the process can be reproduced merely by using biparallelism fo the K-space in question. Stability thus comes to discussing the roots of the characteristic equation

(7.3) $$\left| \lambda \delta^i_j - S^i_j \right| = 0.$$

If these are all real and negative, we have a transformation for our normal coordinates, and finite oscillations in these.

$S^i_j = P^i_j$ is a mixed tensor, and the roots of (7.3) will therefore be invariants under point transformation.

With the general $F^i_j(x, \dot{x}, t)$ for transformation coefficients, reduction is not always possible, as there may not be a transformation corresponding to a given set of coefficients even along a curve; $\dot{x}$ is not in general a vector. The condition (7.1) and its interpretation are unchanged. Matrix laws of combination apply to the coefficients F, when two or more transformations are performed in succession. Lastly, a covariant biderivate can be defined;

(7.4) $$\mathfrak{D}\, u_i = \dot{u}_i - \gamma^k_i u_k.$$

The coefficients of covariance being φ^i_j

$$\bar{u}_i = \varphi^j_i u_j.$$

For self-consistence and simplicity $\bar{F}^i_j = \varphi^i_j$ can be assumed, using the upper index for summation, though this is merely a sufficient condition. Biderivation for tensors of any rank can be defined by analogy with the usual formulae for covariant differentiation, giving always a tensor of the same type as the original.

8. The foregoing deductions can be motivated by considering groups of deformations of the space. If the infinitesimal transformation of the group be

(8.1) $$\bar{x}^i = x^i + \xi^i \delta\tau$$

invariants of the group of the form $f(x, \dot{x})$ will be given by

$$\delta f \equiv \delta\tau [\xi^i f_{,i} + \dot{\xi}^i f_{;i}] = 0$$

we demand that the transformation be parallel in the path-space:

(8.2) $$\dot{\xi}^i + \gamma^i_k \xi^k + \varepsilon^i = 0.$$

This gives us at once

(8.3) $$\xi^i [f_{,i} - \gamma^k_i f_{;k}] - \varepsilon^i f_{;i} = 0.$$

Sufficient conditions for invariance are

(8.4) (a) $$\varepsilon^i \frac{\partial f}{\partial \dot{x}^i} = 0,$$

(b) $$\frac{\partial f}{\partial x^i} - \gamma^k_i \frac{\partial f}{\partial \dot{x}^k} = 0.$$

In addition to this, if f contains the parameter t explicitly, we can expand the group by adding to (8. 1)

$$\bar{t} = t + \delta\tau.$$

By this, we shall say that t is an *affine parameter* for the path-space. There is then added an extra term $\frac{\partial f}{\partial t}$ to (8. 3) and (8. 4) become

$$f_{,i} - \gamma_i^k f_{;k} = 0, \quad \varepsilon^i f_{;i} - \frac{\partial f}{\partial t} = 0, \tag{8. 5}$$

If the parallel transformations are $n+1$ in number, it is seen that (8. 5) are necessary as well as sufficient conditions for invariance. If the stream-lines of the transformation defined by the congruence $\dot{x}^i = \xi^i$ are paths of the space, as would be expected from (8. 2) we see from the relation $\varepsilon^i = \alpha^i - \gamma^i \dot{x}^k$ and from the condition for invariance that

$$\frac{df}{dt} \equiv -\alpha^i \frac{\partial f}{\partial \dot{x}^i} + \dot{x}^i \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial t} = 0, \tag{8. 6}$$

i. e. the function $f(x, \dot{x}, t)$ is constant along the path-streamline. The $n+1$ transformations imply that every path can be made a streamline in some sufficiently restricted n dimensional manifold of the space: we should expect the same result from (8. 5). This indeed is seen at once to be true by eliminating γ_k^i. And this property of the invariant is independent of the particular γ_k^i — or parallelism — chosen. Our reduction of the equations of variation gave us $\gamma_k^i = \frac{1}{2}\alpha^i{}_{;k}$, the transformation being one which carried paths into paths[7]).

9. If then, we are to deduce our geometry from some fundamental group to which the space is subjected, and the space has one or more metrics attached to it, we should expect the metric $f(x, \dot{x}, t)$ to be an invariant of the group, and the paths to be the geodesics of the metric, i. e. extremals of the variational principle

$$\delta \int f(x, \dot{x}, t)\, dt = 0.$$

This implies that f be a solution of

$$\delta_i f \equiv \alpha^j \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j} - \dot{x}^j \frac{\partial^2 f}{\partial \dot{x}^i \partial x^j} - \frac{\partial^2 f}{\partial \dot{x}^i \partial t} + \frac{\partial f}{\partial x^i} = 0 \tag{9. 1}$$

such that

$$\Delta \equiv \left| \frac{\partial^2 f}{\partial \dot{x}^i \partial \dot{x}^j} \right| \neq 0.$$

[7]) And the group, if solutions of the equation of variation define one as would be expected, will be valid in that neighbourhood of a point within which no conjugate focus exists on any extremal through the point.

Now the actual condition of invariance $\frac{df}{dt} = 0$ along the paths reduces this system to one of the first order;

$$\frac{1}{2}\frac{\partial \alpha^i}{\partial \dot{x}^j}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^j} = 0. \tag{9.2}$$

The parallelism is then determined at once[8]) as $\gamma^i_k = \frac{1}{2}\alpha^i_{;k}$. The same reduction can be obtained by imposing on f a property of the invariants: that any $\varphi(f)$ be also a possible metric with f.

As is well known from Hamilton's principle, or as can be proved directly, any solution of (9.1) has the property

$$\frac{d}{dt}\left[\dot{x}^i\frac{\partial f}{\partial \dot{x}^i} - f\right] + \frac{\partial f}{\partial t} = 0. \tag{9.3}$$

If f does not contain explicitly the parameter t, then $\dot{x}^i f_{;i} - f$ is a constant along the paths.

Whence any $f(x, \dot{x})$, homogeneous of any degree except one in $\dot{x}$, and a solution of (9.1) is also a solution of (9.2), and constant along a path-extremal. There will in general be a finite number of independent solutions if equations (9.2) are compatible with $\frac{df}{dt} = 0$; or else none. And just as all invariants of the group can be expressed as a function of a finite number of them, so also any invariant metric will be expressible in terms of these fundamental solutions, the analogy being complete. Of course, a metric proper would need certain other conditions to determine it completely.

Differential invariants of the space, including the two curvature tensors, appear as coefficients in successive compatibility conditions of (8.5) or (9.2) depending on the choice of parallelism.

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[8]) Or else extra compatibility conditions are introduced $(\alpha^i_{;k} - \gamma^i_k) f_{;i} = 0$.

(Eingegangen am 2. Dezember 1932.)