Zusammenfassung

Die Stabilität der laminaren Strömung zwischen zwei koaxialen Kreiszylindern, die durch Rotation des inneren Zylinders und Wirkung eines konstanten Druckgradienten in Umfangsrichtung erzeugt wird, wird untersucht mit Hilfe eines Näherungsverfahrens, welches nur die bekannten Stabilitätskriterien für Couette-Strömung zwischen zwei gegenrotierenden Zylindern und Poiseuille-Strömung in einem gekrümmten Kanal benutzt. Es ergibt sich, dass solch eine Methode kritische Werte für das Auftreten der Instabilität liefert, die in guter Übereinstimmung mit denen von DI PRIMA [2] sind und dass die Methode eine zufriedenstellende Erklärung für die ziemlich unerwartete Abhängigkeit des Taylor-Görtler-Parameters und der kritischen Wellenzahl von der Grösse des azimutalen Druckgradienten gibt.

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Similar Solutions of the Boundary Layer Equations for Power Law Fluids

By JAGAT N. KAPUR and RAMESH C. SRIVASTAVA, Kanpur, India[1])

1. Introduction

Non-Newtonian fluids are of great and increasing importance in chemical engineering and technology and their flow behaviour has been discussed by METZNER [1, 2][2]), WILKINSON [3], BHATNAGAR [4], KAPUR [5] and others. Out of these the simplest and most useful are the power law fluids for which the rheological equation of state between stress components τ_{ij} and strain rate components e_{ij} is given by

$$\tau_{ij} = \mu \left| \sum_{n=1}^{3} \sum_{l=1}^{3} e_{lm} e_{ml} \right|^{\frac{n-1}{2}} e_{ij} , \tag{1}$$

where μ and n are the consistency and flow behaviour indices of the fluid. The boundary layer flows for such fluids have been discussed recently in a number of papers. Thus BOGUE [6] used the boundary layer equations to extend SHILLERS [7] method for discussing the inlet length for a circular pipe for a Newtonian fluid, to the case of power law fluids. KAPUR and GUPTA [8] have used a similar method for discussing the two dimensional flow in the inlet length of a straight channel. ACRIVOS, PATERSON, and SHAH [9] have integrated the boundary layer equations for a flat plate. KAPUR [10, 11] has integrated boundary layer equations for two dimensional and axially symmetric jets of incompressible non-Newtonian fluids.

In the present paper, we have developed the theory for similar solutions of the boundary layer equations on the same lines as is usually done [12] for Newtonian fluids. We obtain in this manner a generalisation of the Falkner Skan equation. Important particular cases like the boundary layer flow along a wedge, along a flat plate, in a convergent channel and two dimensional stagnation point flow have been briefly discussed.

2. The Basic Equation

The boundary layer equations for the two dimensional flow of a power law fluid are:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \left[\frac{\partial}{\partial y} \left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) \right], \tag{2}$$

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[1]) Indian Institute of Technology.

[2]) Numbers in brackets refer to References, page 388.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

where u and v are the components of velocity along and perpendicular to the wall, U is the free stream velocity, ϱ is the density of the fluid and $\nu = 2^{(n-1)/2} \cdot \mu/\varrho$.

The continuity equation is integrated by introducing the stream function $\psi(x, y)$ which is such that

$$u = +\frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4}$$

The boundary layer equations then reduce to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U \frac{dU}{dx} + \nu \frac{\partial}{\partial y}\left[\left|\frac{\partial^2 \psi}{\partial y^2}\right|^{n-1} \frac{\partial^2 \psi}{\partial y^2}\right]. \tag{5}$$

We introduce new variables and functions

$$\xi = \frac{x}{L}, \quad \eta = \frac{y\, R^{1/(n+1)}}{L\, g(x)}, \tag{6}$$

$$f(\xi,\eta) = \frac{\psi(x, y)\; R^{1/(n+1)}}{L\, U(x)\, g(x)}, \tag{7}$$

where R is the modified Reynold number defined by

$$R = \frac{L^n}{\nu\, U_\infty^{n-2}} \tag{8}$$

and L and U_∞ are reference length and velocity respectively, and $g(x)$ is a suitable scaling function to be chosen later. We get

$$u = \frac{\partial \psi}{\partial y} = U(x) \frac{\partial f}{\partial \eta}, \tag{9}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{R^{1/(n+1)}}\left[L\, f \frac{d}{dx}(U\, g) + U\, g\left(\frac{\partial f}{\partial \xi} - \frac{g'(x)}{g(x)} L\, \eta \frac{\partial f}{\partial \eta}\right)\right], \tag{10}$$

$$\frac{\partial u}{\partial x} = U' \frac{\partial f}{\partial \eta} + \frac{U}{L}\left[\frac{\partial^2 f}{\partial \xi\, \partial \eta} - L\, \eta \frac{\partial^2 f}{\partial \eta^2}\frac{g'(x)}{g(x)}\right], \tag{11}$$

$$\frac{\partial u}{\partial y} = U(x) \frac{\partial^2 f}{\partial \eta^2}\frac{\eta}{y}. \tag{12}$$

Substituting in (2) and remembering that for similar solutions f should be independent of ξ, we get after considerable simplification, the basic equation

$$|f''|^{n-1} f''' + \alpha f f'' + \beta (1 - f'^2) = 0, \tag{13}$$

where

$$\alpha = \frac{L\, g^n(x)\, (U_\infty)^{n-2}}{n\, U^{n-1}} \frac{d}{dx}[U\, g], \tag{14}$$

and

$$\beta = \frac{1}{n} \frac{L\, g^{n+1}(x)\, (U_\infty)^{n-2}\, U'.}{U^{n-1}} \tag{15}$$

We choose $U(x)$ and $g(x)$ so that α and β are constants. For $n = 1$, Equation (13) reduces to the well-known Falkner Skan equation:

$$f''' + \alpha f f'' + \beta (1 - f'^2) = 0 \tag{16}$$

with

$$\alpha = \frac{L\, g(x)}{U_\infty} \frac{d}{dx}[U\, g], \quad \beta = \frac{L\, g^2(x)}{U_\infty} U'. \tag{17}$$

In general the velocity component u will increase from its zero value at the wall to the value U at the edge of the boundary layer and thus in this case $\partial u/\partial y$ would be non-negative. From (6) and (12) it would then appear that if $g(x)$ can be chosen to be a non-negative function, we can take f'' to be non-negative. Thus we can make the assumption that f'' is a non-negative function and integrate:

$$f''^{n-1} f''' + \alpha f f'' + \beta (1 - f'^2) = 0 \tag{18}$$

subject to the boundary conditions

$$\eta = 0,\ f = 0,\ f' = 0;\ \eta = \infty,\ f' = 1\ . \tag{19}$$

The assumption made can then be tested against the solution so obtained.

3. Solution for $U(x)$ and $g(x)$

From (14) and (15)

$$\frac{d}{dx} [g^{n+1} U^{2-n}] = \frac{n}{L\, U_\infty^{n-2}} [(n+1)\,\alpha - (2n-1)\,\beta]\,. \tag{20}$$

Integrating for the case $(n+1)\,\alpha - (2n-1)\,\beta \neq 0$, we get

$$g^{n+1}(x)\ U^{2-n}(x) = \frac{n}{L\, U_\infty^{n-2}} [(n+1)\,\alpha - (2n-1)\,\beta]\, x\,. \tag{21}$$

Also from (14) and (15)

$$\frac{U'}{U} (\alpha - \beta) = \beta \frac{g'}{g}\ , \tag{22}$$

which gives on integration

$$\left(\frac{U}{U_\infty}\right)^{\alpha-\beta} = k\, g^{\beta}\ , \tag{23}$$

where k is a dimensionless constant.

Solving for $U(x)$ and $g(x)$ we get

$$\left(\frac{U}{U_\infty}\right)^{(\alpha-\beta)/\beta + (2-n)/(n+1)} = k^{1/\beta} \left[\{(n+1)\,\alpha - (2n-1)\,\beta\} \frac{x\,n}{L}\right]^{1/(n+1)}. \tag{24}$$

and

$$[g(x)]^{1 + [\beta(2-n)/(\alpha-\beta)\,(n+1)]} = k^{-(2-n)/(n+1)\,(\alpha-\beta)} \left[\{(n+1)\,\alpha - (2n-1)\,\beta\} \frac{x\,n}{L}\right]^{1/(n+1)}. \tag{25}$$

From (14) and (15) it is seen that the result is independent of any common factor of α and β as it can be included in g. Therefore as long as $\alpha \neq 0$, we can put $\alpha = 1$ without loss of generality. Also introducing a new parameter m defined by

$$m = \frac{\beta}{(n+1) - (2n-1)\,\beta} \quad \text{or}\ \beta = \frac{m\,(n+1)}{1 + m\,(2n-1)}\ . \tag{26}$$

We get

$$\frac{U}{U_\infty} = k^{1+(2n-1)\,m} \left[\frac{n\,(n+1)}{1+m\,(2n-1)} \frac{x}{L}\right]^m, \tag{27}$$

$$g = \left[\frac{n\,(n+1)}{1+m\,(2n-1)} \frac{x}{L} \left(\frac{U_\infty}{U}\right)^{2-n}\right]^{1/(n+1)}\ . \tag{28}$$

Also from (6) and (28)

$$\eta = y \left[\frac{1+m\,(2n-1)}{n\,(n+1)} \frac{U^{2-n}}{x\,\nu}\right]^{1/n+1}\ . \tag{29}$$

4. Particular Cases

Case I: Flow Past a Wedge

If the angle of the wedge is $\pi\,\theta$, the potential flow is given by

$$U(x) = c\,x^{\theta/2-\theta} = c\,x^m \tag{30}$$

From (26) and (30)

$$m = \frac{\theta}{2-\theta} = \frac{\beta}{(n+1)-(2n-1)\,\beta} \tag{31}$$

or

$$\beta = \frac{(n+1)\theta}{2+2\,(n-1)\,\theta}\,, \quad \theta = \frac{2\,\beta}{(n+1)-2\,(n-1)\,\beta}\,. \tag{32}$$

Also we have from (29), (28), (6), (9), (10), and (16)

$$\eta = y\left[\frac{1+m\,(2n-1)}{n\,(n+1)}\;\frac{c}{\nu}\right]^{1/(n+1)} x^{m\,(2-n)-1/n+1}\,, \tag{33}$$

$$\psi\,(x, y) = \left[\frac{n\,(n+1)}{1+m\,(2n-1)}\;\nu\,c^{2n-1}\right]^{1/(n+1)} x^{(1+m\,(2n-1)/(1+n)}\,, \tag{34}$$

$$u = c\,x^m\,f(\eta)\,, \tag{35}$$

$$v = -\frac{1}{n+1}\left[\frac{n\,(n+1)}{1+m\,(2n-1)}\;\nu\,c^{2n-1}\right]^{1/n+1} x^{m\,(2n-1)-n/n+1}$$
$$\times\left[\{m\,(2n-1)+1\}\,f + \{m\,(2-n)-1\}\,\eta\,\frac{\partial f}{\partial\eta}\right], \tag{36}$$

$$f''^{\,n-1}\,f''' + f\,f'' + \frac{(n+1)\,\theta}{2+2\,(n-1)\,\theta}\,(1-f'^2) = 0\,. \tag{37}$$

Integration of (37) subject to (19) would give $f(\eta)$ and $\partial f/\partial\eta$ and then (35) and (36) would give u and v. We can discuss the flow both in accelerating flow ($m > 0$) and decelerating flow ($m < 0$).

Case II: Two Dimensional Stagnation Point Flow

In the above analysis, we put $m = \beta = \theta = 1$, then we get

$$\eta = y\left(\frac{2}{n+1}\;\frac{c^{2-n}}{n\,\nu}\right)^{1/(n+1)} x^{(1-n)/(1+n)}, \quad \psi\,(x, y) = \left(\frac{n+1}{2}\;\nu\,c^{2n-1}\right)^{1/(n+1)} x^{2\,n/(1+n)} \tag{38}$$

$$u = c\,x\,f\,(\eta), \quad v = -\left(\frac{n+1}{2}\;\nu\,c^{2\,n-1}\right)^{1/(n+1)} x^{(n-1)/(n+1)}\left[(1-n)\,\eta\,\frac{\partial f}{\partial\eta} + 2\,n\,f\right]\frac{1}{n+1} \tag{39}$$

$$f''^{\,n-1}\,f''' + f\,f'' + \frac{n+1}{2\,n}\,(1-f'^2) = 0\,. \tag{40}$$

We thus get the stagnation point boundary layer, though unlike the Newtonian case, this may not satisfy the complete equations of motion.

Case III: Flow Past a Flat Plate

In this case we put $m = \beta = \theta = 0$ then

$$U = c\,x^0 = c = U_\infty \text{ (say).}$$

This gives us the case of a flat plate with zero incidence. We get

$$\eta = y \left[\frac{U_\infty^{2-n}}{\nu\, n\, (n+1)}\right]^{1/(n+1)} x^{-1/(n+1)}, \tag{41}$$

$$\psi(x, y) = [n\,(n+1)\,\nu\, c^{2n-1}]^{1/(n+1)}\, x^{1/(1+n)}\, f \tag{42}$$

$$u = U_\infty f\,(\eta) \tag{43a}$$

$$v = -\frac{1}{n+1}\, [n\,(n+1)\,\nu\, c^{2n-1}]^{1/(n+1)}\, x^{-n/(n+1)} \left[f - \eta\, \frac{\partial f}{\partial \eta}\right] \tag{43b}$$

$$f''^{\,n-2} f''' + f = 0. \tag{44}$$

Case IV: Flow in a Convergent Channel

This case corresponds to $\alpha = 0$, $\beta = 1$. From (14) we get $U g =$ const and from (15),

$$U(x) \propto x^{-1/2n-1}, \quad g(x) \propto x^{1/2n-1} \tag{45}$$

The basic differential equation becomes

$$f''^{\,n-1} f''' + 1 - f'^2 = 0\,. \tag{46}$$

Multiplying by f'', integrating and using the condition that as $\eta \to \infty$, $f' = 1$, $f'' = 0$; we get

$$f''^{\,n+1} = \frac{n+1}{3}\,(1 - f')\,(2 + f')\,. \tag{47}$$

Integrating again

$$\eta = \left(\frac{3}{n+1}\right)^{1/(n+1)} \int_0^{f'} \frac{d f'}{[(1-f')^2\,(2+f')]^{1/(n+1)}}\,. \tag{48}$$

Since at $\eta = 0$, $f' = 0$. Further if $n \leqslant 1$ as $f' \to 1$, η would tend to infinity. If $\eta > 1$ when $f' \to 1$, η would be finite and after that f' would remain 1.

We may also note that since

$$\frac{\partial u}{\partial y} = \frac{\partial^2 f}{\partial \eta^2}\, \frac{R^{1/(n+1)}}{L\, g(x)}\,, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 f}{\partial \eta^3}\, \frac{R^{2/(n+1)}}{L^2\, g^2(x)}\, \frac{1}{U(x)}\,. \tag{49}$$

for $n \leqslant 1$, we shall require both f'' and f''' tend to zero in such a way that $f''^{\,n-1} f'''$ also tends to zero as $n \to \infty$. If $n > 1$, we need not insist on f''' tending to zero.

From (48)

$$\eta = \left(\frac{3^{n-1}}{n+1}\right)^{1/(n+1)} \int_{2/3}^{(f'+2)/3} z^{-1/(n+1)}\, (1 - z)^{-2/(n+1)}\, dz \tag{50}$$

or

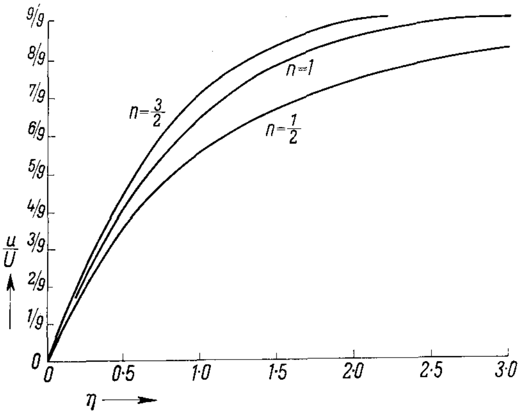
$$\eta = \left(\frac{3^{n\,-\,v\,e\,1}}{n+1}\right)^{1/(n+1)} \left[B_{(f'+2)/3}\left(\frac{n}{n+1}, \frac{n-1}{n+1}\right) - B_{2/3}\left(\frac{n}{n+1}, \frac{n-1}{n+1}\right)\right], \tag{51}$$

where the incomplete beta function is defined by

$$B\, x\,(p, q) = \int_0^x x^{p-1}\,(1 - x)^{q-1}\, dx. \tag{52}$$

(50) or (51) enable us to plot u/U as a function of η. If $n > 1$, the tables of incomplete beta functions [13] can be used. If $0 < n \leqslant 1$, numerical integration has to be used.

The figure below gives u/U as a function of η for a pseudo plastic fluid ($n = 1/2$), a Newtonian fluid ($n = 1$) and a dilatant fluid ($n = 3/2$).



Figur 1

Case V: $(n + 1)\,\alpha - (2\,n - 1)\,\beta = 0$

In this case from (20)

$$g^{n+1}\,U^{2-n} = \text{const}\,. \tag{53}$$

Using (15), we get

$$U(x) \propto e^{px}\,, \tag{54}$$

where p is a positive or negative constant. This flow is the same as for Newtonian fluids

Case VI: Exact and Asymptotic Solution

The exact and asymptotic solutions of (13), (18), (37), (40), and (44) will be discussed in a subsequent paper.

Acknowledgement

We are grateful to Mr. R. C. Gupta for carrying out the numerical integrations of (50).

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Zusammenfassung

Grenzschichten in nicht-newtonschen Medien werden für den Fall berechnet, dass die Geschwindigkeit der Potentialströmung, ausgehend von einem Staupunkt, sich nach einem Potenzgesetz entlang der Oberfläche ändert.

Als spezielle Fälle werden behandelt: Strömungen 1) an einem Keil, 2) entlang einer ebenen Platte, 3) in einem konvergenten Kanal, 4) in der Nähe einer ebenen Staulinie.

(Received: March 3, 1962.)

Varia – Miscellaneous – Divers

Theodor von Kármán

Am 7. Mai ist Professor Dr. TH. VON KÁRMÁN im 83. Lebensjahr in Aachen verschieden. Der Tod dieses «Grand Old Man» wird den wissenschaftlichen Mechanikern noch einmal in Erinnerung rufen, wieviel sie VON KÁRMÁN verdanken.

Seine Abhandlungen und Vorträge sind in ansprechender Form gesammelt[1]), eingeleitet durch ein Vorwort seines Freundes H. L. DRYDEN, das gegen Ende den treffenden Satz enthält: «No technique has yet been invented which would bring an adequate appreciation of his scientific stature to one who does not know him personally.»

Wie ist nur der eigentümliche Zauber seiner Persönlickeit zu verstehen? Ausgestattet mit ungewöhnlich tiefen Kenntnissen in Mathematik und Physik und einem ausgezeichneten Gedächtnis für Fakten und Personen war er in jeder Diskussion imstande, den springenden Punkt rasch zu erfassen und mit pädagogischem Geschick den Zuhörern darzulegen – sehr oft ohne Formeln und nicht selten in charakteristisch gutmütig-witziger Form. Irgendwie stand er hoch über Dingen, die andere allzu ernst nahmen, in klarer Erkenntnis der Schwachheit unserer Bemühungen gegenüber der Unerschöpflichkeit der Natur. Freilich war er philosophischen Betrachtungen im üblichen Stil eher abhold, die Worteweisheit (wie GAUSS sich ausdrückte) konnte ihm kein Ersatz sein für ehrliche harte Arbeit. Ähnlich wie sein Freund ALBERT EINSTEIN war er zutiefst überzeugt von einer grossartigen Harmonie hinter all den heute noch unverstandenen Dingen. Er erzählte dann gerne die Geschichte von einem offenbar etwas unbeholfenen mathematischen Kollegen, der den Spruch: «Wir Deutschen fürchten Gott, sonst nichts in der Welt» für sich abwandelte: «Ich fürchte Gott nicht, sonst alles in der Welt». Und wenn auch VON KÁRMÁN hinsichtlich des Nachsatzes sich recht gut zu wehren wusste, glaube ich doch, dass er mit dem Anfang völlig einigging.

J. ACKERET

3rd European Regional Conference on Electron Microscopy, Prague 1964

This Conference on Electron Microscopy, organized by the National Committee for Electron Microscopy of the Czechoslovak Academy of Science under the auspices of the International Federation of Societies for Electron Microscopy, will be held in Prague, August 26th to September 3rd, 1964.

Further information about the Conference will be given by Professor JAN WOLF, Organizing Committee of the 3rd European Regional Conference on Electron Microscopy, Albertov 4, Prague 2, Czechoslovakia.

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[1]) *Collected Works of Theodore von Kármán*. 4 Bände (Butterworths Scientific Publications; London 1956).

Einen guten Einblick in einige Arbeiten VON KÁRMÁNS gibt auch sein Buch «Aerodynamics, Selected Topics in the Light of their Historical Development» (Cornell University Press; Ithaca, 1954).