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COMBINATORIAL ANALYSIS AND SCHOOL MATHEMATICS

I. INTRODUCTION

"I hope you are not going in for 'modern mathematics'. Please don't." This was the spontaneous remark of Professor Morris Kline of the Courant Institute of Mathematical Sciences, when he was told that I was convenor of the Mathematics Study Groups of the Government of India. My reply was simple: "We are definitely 'modernizing' our curriculum, but we are not going all out for Baby Bourbaki. We are trying to find a proper balance between abstractions and applications."

Professor Kline was obviously relieved, but I was left thinking. Why is it that persons like Professor Kline, who have a clear vision of the nature and history of mathematics, are against 'new mathematics'? Why is it that Professor Hammersley of the U.K. goes on preaching against 'modern mathematics and similar other trash'? Why is it that prospective employers get puzzled by the abstract language used by the student trained in modern mathematics? (*Mathematics Teaching*, 1968). Obviously no mathematician can feel happy about the traditional mathematics curriculum which is so much out-of-step with the spirit of the times. In spite of this, why should so many sincere mathematicians feel disturbed about the present trends in modernization of the curriculum?

II. POLARIZATION IN THINKING

One can easily discern signs of polarization in thinking, among professional mathematicians, about the new curricula in schools and colleges.

There is one group which believes that we must first build up fundamental structures of mathematics axiomatically, rigorously, abstractly and logically and then we can superpose on these the structures of applications. This group believes that no non-trivial applications can be built up before a good grounding in mathematics is given and it is a disservice to mathematics to give trivial applications. For this group, mathematics and applications of mathematics form two distinct disciplines, and inevitably the structures of pure thought must precede the structures of Nature and Society. This group would like to see 'applications' as an 'appendix' in the book on mathematics, and from its point of view, no serious harm would be done even if this

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appendix is left out by the student. There are some members of this group who would like to see applications postponed even beyond the Ph.D stage. They believe sincerely that mathematics is now a self-contained and a self-sustaining discipline which can keep its vitality intact only by pure inbreeding. They consider it natural that physicists, biologists and social scientists should construct the superstructures of applications on the basis of the structure of mathematics built up in the mathematics curriculum. They even argue that these mathematically-oriented scientists can even do the job better as they are closer to reality.

There is a second group of professional mathematicians which believes, on the other hand, that mathematics and its applications should always keep in step throughout the curriculum. It believes that students should see the need for some particular segment of mathematics before it is created for them (or better still before the students create it for themselves!) and they should be able to see how the mathematics created has served the need felt in the first instance. This group would thus like to see applications (internal and external) both precede and succeed the creation of mathematics. In fact, some need of understanding of Nature and Society should lead to the creation of some mathematics, and the creation of this new mathematics should pave the way for solving more complex problems of Society and thus create the need for more mathematics and so on. This group would like to make mathematics appear as a spontaneous and natural response of the human mind (and human genius) to the environment, physical, biological and social, in which man lives and has his being. This group would like the students to see for themselves that axiomatization, generalization and abstraction in mathematics are necessary for the purpose of understanding of problems of Nature and Society. This group would like to isolate some fundamental structures of Nature and Society and build up the fundamental structures of mathematics around these concepts so that a more-or-less perfect intertwining takes place. The views of this group are supported by the history of mathematics and the psychology of learning of mathematics.

There is a third group of professional mathematicians which is in sympathy with both points of view, which would like to avoid both extremes and which wants to 'graft' applications as best as possible.

In framing the curriculum, the task of the first group of mathematicians is relatively easier. This group has to deal with a realm of pure thought, undisturbed and undefiled by the 'noise' of the external world, though arising from it by having its roots in the external world. However in course of time, 'noise' has been filtered out by the process of abstraction. This group can accordingly build up a nice, elegant, aesthetically satisfying and almost perfect curriculum. This group has a model in 'Bourbaki' at the highest level and it can produce a 'Baby Bourbaki' at the school level. The objectives are clear-cut, the methods are clear-cut and it is only a matter of time before its goals are achieved.

The task of the second group is not so easy. This group has no 'Bourbaki', on the model of which it can develop a 'Baby Bourbaki'. The structures in physical, biological, social and management sciences are relatively more complex and the recognition of the isomorphisms of these structures with the purely mathematical structures is not easy and also requires a deep insight into both types of structures. There is a wealth of scattered material to be drawn upon and proper choice and integration are not easy tasks. The number of persons who can possibly achieve this integration is also relatively limited and these persons are scattered over departments of mathematics, statistics, physics, biology, social sciences, computer science and industrial establishments.

The result of all this has been that while the first group has produced some coherent curricula, the second group (or possibly the third group) has produced isolated books and articles.

The members of the second group face a really challenging task. They have to do some collective thinking and produce concrete alternatives. Simply being unhappy with 'too much abstraction' will not do. In fact, unless this is done, these members run the risk of being dubbed as reactionaries for opposing 'modernization.'

III. THE ROLE OF COMBINATORIAL MATHEMATICS

The applications of mathematics for school curriculum can come from the following areas:

(a) Physical Sciences and Engineering.
(b) Biological Science.
(c) Social Sciences.
(d) Management Sciences.
(e) Others.

In the modern context, the applications have to be necessarily computer-oriented.

Alternatively we may say that applications have to belong to either (i) the mathematics of the continuous or (ii) the mathematics of the discrete.

The first category of applications require the knowledge of calculus which can be built only upon the foundations of real numbers and so inevitably comes later in the curriculum. However the methods of calculus are powerful enough and so applications of calculus can be done even by 'idiots'. On the other hand, in the mathematics of the discrete not many such powerful methods are available and ingenuity is always required.

Combinatorial mathematics is an essential component of the mathematics of the discrete and as such it has an important role to play in school mathematics. This role has been little exploited so far.

Some of the reasons which make combinatorial analysis important in school mathematics are the following:

(a) Since it does not depend on calculus, its problems can be taken up at an early stage in the school curriculum. In fact it has problems suitable for all grades.

(b) It can be used to train students in the concepts of enumeration (counting with counting through counting without counting), making conjectures, generalizations, optimization, existence, systematic thinking etc.

(c) Applications to physics, chemistry, biology, network analysis, design of experiments, communication theory, symmetry, probability, dynamic programming, number theory, topology, recreational mathematics etc. can be indicated.

(d) The need for creation of more mathematics can be created in the minds of the students. A large number of challenging problems can be indicated to them.

(e) Distinction between plausible and rigorous proofs can be brought out.

(f) Enough motivation for working with computers can be provided.

(g) Students can appreciate the powers and limitations of mathematics as well as the powers and limitations of computers through combinatorial mathematics.

(h) It can help in the development of concepts of mapping, relations, functions, equivalence relations, equivalence classes, isomorphisms etc. rather clearly.

(i) Some of the great victories of the human mind over challenging problems can be indicated.

(j) Most of the combinatorial problems and their applications have been developed recently and so students can get a feeling for the growth of mathematics.

(k) This can help in developing the combinatorial attitude of mind which examines all possibilities, enumerates them and finds out the best possibility and thus leads to clearheaded thinking.

IV. SOME ILLUSTRATIVE PROBLEMS

The following problems illustrate the types of combinatorial problems which can be discerned at the school level. The problems are of varying degrees of difficulty and are suitable for different grades. In some cases, the same problem can be taken up in different grades, in a spiral approach. Some of these have been actually tried by the author with children of age groups 9 to 13.

(1) *Partitions of a given number*

A given positive integer can be regarded as the sum of a number of integers in a number of ways e.g.

$$4=3+1=2+2=1+1+1+1;$$
$$5=4+1=3+2=3+1+1=2+2+1=2+1+1+1=$$
$$=1+1+1+1+1.$$

Thus 5 can be partitioned into one part in one way, into two parts in two ways, into three parts in three ways, into four parts in one way, and into five parts in one way. The following table can now be easily prepared (see Table I).

TABLE I

Number of parts

n	1	2	3	4	5	6	7	Number of partitions $p(n)$
1	1							1
2	2	11						2
3	3	21	111					3
4	4	31	211	1111				5
		22						
5	5	41	311	2111	11111			7
		32	221					
6	6	51	411	3111	21111	111111		11
		42	321	2211				
		33	222					
7	7	61	511	4111	31111	211111	1111111	15
		52	421	3211	22111			
		43	331	2221				
			322					

There is no exact formula for $p(n)$ but by using ingeneous methods, tables for $p(n)$ for even large values of n have been prepared e.g.

$$p(35)=14{,}883, \qquad p(56)=526823,$$
$$p(68)=3087735, \qquad p(78)=12132164,$$
$$p(99)=169229875.$$

In general, the number of unrestricted partitions of n is asymptotic to

$$\frac{1}{4n\sqrt{3}}\exp\left[\pi\left(\frac{2n}{3}\right)^{1/2}\right].$$

From these tables, the following results can be easily verified:

(i) the partitions with unequal parts are equinumerous with those with all parts odd;

(ii) the number of partitions of n with no part greater than k equals the number of partitions of $n+k$ with exactly k parts;

(iii) the number of partitions of n with at most k parts $=$ the number of partitions of n with n parts greater than k $=$ number of partitions of $n+k$ with exactly k parts.

Of course, proofs for all these results and others are available.

(2) *Compositions of a given number*

In this case, the number is partitioned with a number of parts and the order in which different parts are taken is important. Thus there are 8 compositions for the number 4 viz.

$$4=3+1=1+3=2+2=2+1+1=1+2+1=1+1+2= \\ =1+1+1+1.$$

These can be geometrically represented as follows (see Figure 1).

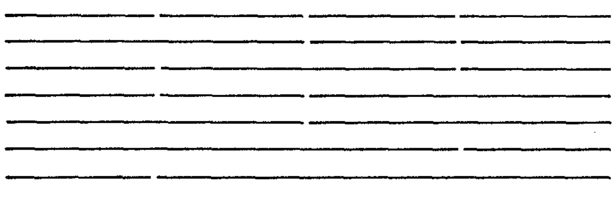

Fig. 1.

The following table is now easily prepared (see Table II).

The table provides nice examples of patterns which can be discovered by careful observations and which can be used to continue the table. Children

TABLE II

Number of parts

n	1	2	3	4	5	6	7	Total
1	1	–	–	–	–	–	–	1
2	1	1	–	–	–	–	–	2
3	1	2	1	–	–	–	–	4
4	1	3	3	1	–	–	–	8
5	1	4	6	4	1	–	–	16
6	1	5	10	10	5	1	–	32
7	1	6	15	20	15	6	1	64

can be led to discover the binomial coefficients and the Pascal triangle patterns from here.

(3) *Maximum number of points of intersection of n lines*

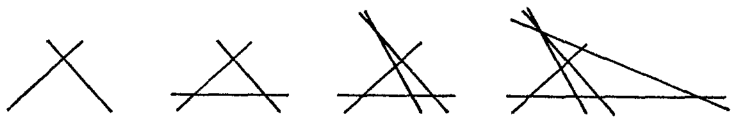

Fig. 2.

The following table is easily prepared:

n	2	3	4	5	6	7	8	9	10
Max. no. of pts. of intersection	1	3	6	10	15	21	28	36	45

The pattern is obvious. The general formula $n(n-1)/2$ can be conjectured and proved. The pattern for the number of triangles formed can also be found.

(4) *Linear graphs. Friendship patterns*

For 4 persons, the possible friendship patterns are illustrated below (see Figure 3).

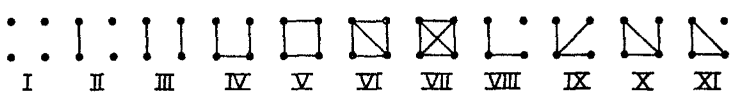

Fig. 3.

In (IV), (IX) and (XI), there are three pairs of friends, but these are distinct patterns because:

in (IV), 2 persons have one friend each and 2 persons have 2 friends each;
in (IX), 3 persons have one friend each and one person has 3 friends;
in (XI), 3 persons have two friends each and one person has no friend.

On the other hand the following patterns (see Figure 4) have the same form as (XI) since in each case 3 persons have two friends each and one person has no friend.

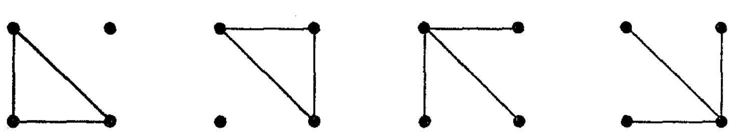

Fig. 4.

Patterns having the same form are called *isomorphic*. For each of the eleven patterns given above, isomorphic forms can be found. We have included only one pattern for each class of isomorphic forms and we say that for four persons, there are only eleven patterns, *up to isomorphism*. These eleven also represent all the possibilities when 4 teams are playing games e.g. (VI) represents the state when two teams have played two games each and two teams have played three teams each. (VI) is called a *linear graph* with 4 points and 5 lines. The following table (see Table III) of linear graphs with n points and k lines can now be easily prepared.

TABLE III

k/n	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
2		1	2	2	2	2	2
3		1	3	4	5	5	5
4			2	6	9	10	11
5			1	6	15	21	24
6			1	6	21	41	56
7				4	24	65	115
8				2	24	97	221

(5) *Directed graphs or digraphs*

Here each edge of the graph has a direction e.g. with three vertices we get the patterns as illustrated in Figure 5.

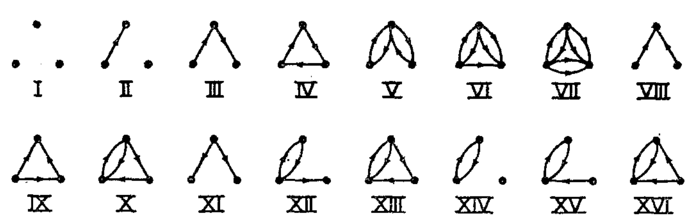

Fig. 5.

We may interpret a directed edge as 'one person liking another' or as 'one team winning over another'.

Thus (III) means that a person likes an other and is liked by the third. Similarly (XIV) means that in two games played, two teams have defeated each other while the third team has not played any game.

The cases $n=4$, 5 can be similarly investigated. Digraphs can be strong, unilateral, weak or disconnected and different enumeration problems arise.

(6) *Trees*

A tree is a connected subgraph of a connected graph which contains all the vertices of the graph but does not contain any circuits. With 5 points we get the patterns as illustrated in Figure 6.

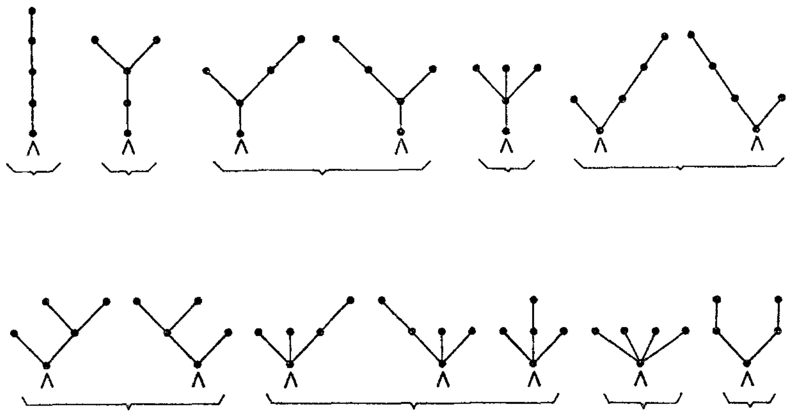

Fig. 6.

We get 14 trees. The following table can now be easily verified.

n	2	3	4	5	6	7
Number of trees	1	2	5	14	42	132

The general formula is $\frac{1}{n}\binom{2n-2}{n-1}$.

If we regard the trees bracketed together as the same i.e. we make no distinction between right and left, we get the table

n	1	2	3	4	5	6	7
Number of rooted trees	1	1	2	4	9	20	48

If we consider trees with single degree for the root, we get

n	1	2	3	4	5	6	7
Number of trees with single degree for root	1	1	1	2	4	6	11

We have also interesting combinatorial problems for trees with a given diameter, trees with a given partition, directed trees, oriented trees, signed trees, trees of given strength etc.

The theory of trees is useful in

(i) probability theory,

(ii) sorting of letters in post offices,

(iii) forming family trees,

(iv) forming genetic trees,

(v) connecting cities by roads with minimum length (the choice is restricted to trees, since any circuits can be eliminated).

(7) *Parenthesising expressions*

The problem concerns finding the number of ways in which brackets for binary operations can be introduced in an expression, e.g. we have

for $n=3$, we have two ways viz. $(a_1+a_2)+a_3$ and $a_1+(a_2+a_3)$
for $n=4$, we have five ways viz. $a_1+((a_2+a_3)+a_4)$, $a_1+(a_2+(a_3+a_4))$,
$(a_1+a_2)+(a_3+a_4)$, $((a_1+a_2)+a_3)+a_4$, $(a_1+(a_2+a_3))+a_4$,
for $n=5$, we have 5 ways with a_1 and 5 combinations of the other four,
5 ways with 5 combinations of (a_1, a_2, a_3, a_4) and a_5,
2 ways with (a_1+a_2) and two combinations of (a_3, a_4, a_5),
2 ways with two combinations of (a_1, a_2, a_3) and a_4+a_5,
giving a total of 14.

The general formula for n is $\frac{1}{n}\binom{2n-2}{n-1}$.

(8) *Latin squares*

Suppose we have 4 drugs A, B, C, D to be tested on 4 days, Monday, Tuesday, Wednesday and Thursday on 4 subjects I, II, III, and IV, then the possible arrangements are as given in Table IV.

TABLE IV

	Mon.	Tues.	Wed.	Thurs.		Mon.	Tues.	Wed.	Thurs.
I	A	B	C	D	I	A	B	C	D
II	B	C	D	A	II	B	D	A	C
III	C	D	A	B	III	C	A	D	B
IV	D	A	B	C	IV	D	C	B	A
	Mon.	Tues.	Wed.	Thurs.		Mon.	Tues.	Wed.	Thurs.
I	A	B	C	D	I	A	B	C	D
II	B	A	D	C	II	B	A	D	C
III	C	D	A	B	III	C	D	B	A
IV	D	C	B	A	IV	D	C	A	B

In each of these arrangements, each drug is tested on every day on every subject only once so that if there are any differences in drug effects on different days and on different subjects, the differences are shared equally by all the drugs. The arrangements are called Latin squares in standard form with the first row and the first column being A, B, C, D. All the other

4×4 Latin squares can be obtained by permutations of rows and columns of these.

Again suppose we have four varieties of wheat A, B, C, D to be tested for their productivity. We can grow these in either of the above patterns. The advantage is, as before, that if rows (or columns) differ in fertility, this difference is shared equally by all the varieties.

For four varieties, we have a choice of 4 distinct 4×4 Latin squares; for five varieties, we have a choice of 56 distinct 5×5 Latin squares; for six varieties, we have a choice of 9408 distinct 6×6 Latin squares; for seven varieties, we have a choice of 16,942,080 distinct 7×7 distinct Latin squares and for eight or more varieties, we do not know!

(9) *Orthogonal Latin squares*

Suppose we have three varieties of wheat and three varieties α, β, γ of corn, we can use the Latin squares

$$\begin{matrix} A & B & C \\ B & C & A \\ C & A & B \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{matrix}$$

If we superpose these, we get

$$\begin{matrix} A\,\alpha & B\,\beta & C\,\gamma \\ B\,\gamma & C\,\alpha & A\,\beta \\ C\,\beta & A\,\gamma & B\,\alpha \end{matrix}$$

in which each of the nine pairs of varieties occur once and once only. Such pairs of Latin squares are called orthogonal Latin squares or Graeco-Latin squares.

How many mutually orthogonal pairs of Latin squares of order $n \times n$ are there? We may need a pair e.g. if we have to test six fever remedies, six headache remedies and we want to test these on six days of the week on six subjects such that each pair of remedies occurs once on every day and once on each subject. Euler faced a similar problem when he tried to arrange 36 officers, 6 from each of 6 ranks and 6 from each of 6 regiments, in a square in such a way that each row and column will have one officer of each rank and one officer for each regiment. Obviously this is a problem of constructing a pair of 6×6 orthogonal Latin squares. Euler could not find a pair and conjectured that no such pair existed. Tarry proved that his conjecture was right.

Euler further conjectured that no orthogonal pair of order $4t+2$ existed. This was proved false by Parker, Bose and Shirkharde. The following can be easily verified to be orthogonal 10×10 Latin squares.

0	4	1	7	2	9	8	3	6	5	0	7	8	6	9	3	5	4	1	2
8	1	5	2	7	3	9	4	0	6	6	1	7	8	0	9	4	5	2	3
9	8	2	6	3	7	4	5	1	0	5	0	2	7	8	1	9	6	3	4
5	9	8	3	0	4	7	6	1	0	9	6	1	3	7	8	2	0	4	5
7	6	9	8	4	1	5	0	3	2	3	9	0	2	4	7	8	1	5	6
6	7	0	9	8	5	2	1	4	3	8	4	9	1	3	5	7	2	6	0
3	0	7	1	9	8	6	2	5	4	7	8	5	4	2	4	6	3	6	1
1	2	3	4	5	6	0	7	8	9	4	5	6	0	1	2	3	7	8	9
2	3	4	5	6	0	1	8	9	7	1	2	3	4	5	6	0	9	7	8
4	5	6	0	1	2	3	9	7	8	2	3	4	5	6	0	1	8	9	7

The proof of the falsity of Euler's Conjecture was a remarkable achievement indeed.

(10) *Cell growth and polyminoes*

Let a one-celled animal with square shape grow in the plane by adding a square cell of the same size to any one of its sides. The problem is to find the number, up to isomorphism, of animals with r cells. The animals are simply connected i.e. there are no 'holes'.

Up to 5 cells, the shapes of the animals are given below (see Figure 7).

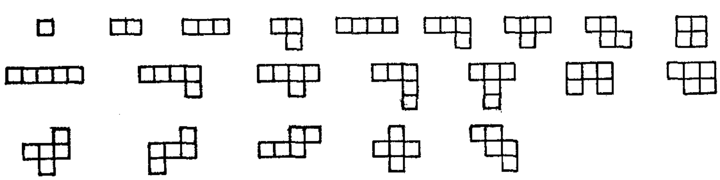

Fig. 7.

By mathematical arguments, the following table can be obtained:

n	1	2	3	4	5	6	7	8	9	10
A_n	1	1	2	5	12	35	107	363	1248	4271

If we use a computer and allow for holes, we get

n	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	1	2	3	12	35	108	369	1285	4655	17073	63600

(11) *Transformations on finite sets*

There are n one-one mappings of a finite set of n points onto itself. In how many of these does no points remain fixed, in how many does one point remain fixed, in how many do two points remain fixed and so on? The answers are given in the following table (see Table V).

TABLE V

Number of fixed points points $= r$

n	0	1	2	3	4	5	6	7	Total
3	2	3	0	1					6
4	9	8	6	0	1				24
5	44	45	20	10	0	1			120
6	265	264	135	40	15	0	1		720
7	1855	1854	924	315	70	20	0	1	5040

The number of transformations in which $n-r$ points remain fixed is

$$n(n-1)-(n-r+1)\left[\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{(-1)^r}{r!}\right].$$

The number of transformations which keep 0 or 1 points fixed differ by unity between themselves and account for about two-thirds of the total number of transformations.

The problem can be formulated in many different ways e.g.

(i) n gentlemen check their hats at a coatroom and are given back their coats at random. In how many ways can the hats be returned so that only r gentlemen get their own hats back?

(ii) A secretary types n letters and puts them in n addressed envelopes at random. In how many ways can exactly r letters go in the right envelopes?

(12) *Symmetries in polygons*

A transformation in a plane which keeps distance between points unchanged is called an isometry. Possible isometries in the plane are translations, rotations about points, reflections in line and glide reflections. An isometry that brings a figure into self-coincidence is called a symmetry of the figure.

An equilateral triangle has 6 symmetries viz three reflections in the medians and three rotations through angles $2\pi/3$, $4\pi/3$ and 2π about the centroid.

A square has eight symmetries viz. two reflections in diagonals, two reflections in lines joining middle points of opposite sides and four rotations through angles $\pi/2$, π, $3\pi/2$ and 2π about the center.

In the same way, it can be shown that every regular polygon of n sides has $2n$ symmetries.

The symmetries of three dimensional figures like cubes, tetrahedron, regular primes, regular pyramids etc. can be discussed in the same way.

(13) *Other symmetries*

The following combinatorial results require group theory for their investigation, but are remarkable achievements of human genius. At least the first two can be easily discussed at the school level:

(a) There are 32 crystallographic point groups of symmetry in a plane.

(b) There are 7 infinite one-dimensional plane groups of symmetry (frieze groups).

(c) There are 17 infinite plane groups of symmetries with two independent directions of translation (wall-paper groups).

(d) There are 230 infinite three dimensional groups of symmetries with three independent directions of translations.

(e) There are 1651 black and white groups with a fourth 'direction' of anti-symmetry.

(14) *Assignment problem*

There are n men and n jobs and if the ith man does the ith job, the production is h_{ij}. The problem is to assign one job to each man so that the total production is optimized. The direct method is to write all the $n!$ assignments systematically, calculate the total productivity for each assignment and thus find the total optimum assignment. For $n=3, 4, 5$, this is feasible, but with $n>5$, this becomes a time-consuming job as is shown in the following which gives the time required by a high-speed computer doing 10^6 assignments per second.

n	10	12	16	18	20	30
Time	0.05 min	0.1 hour	0.6 years	200 years	10^5 years	10^{19} years

The first stage in teaching this problem is to show the students how to write the assignments systematically, the second stage is to let them practice with the assignment algorithm and the final stage is to give them the proof of the algorithm.

A similar problem arises in preparing a computer programme for arranging in ascending order n given distinct numbers.

In this connection, the following theorems are of great interest. Given a sequence of n^2+1 integers, it is possible to find a subsequence of $n+1$ entries which is either increasing or decreasing.

If a set of n^2+1 objects is partitioned into n or fewer blocks, at best one block should contain $n+1$ or more objects.

If a set of n objects is partitioned into k blocks and $n>k$, then at least one block will contain two or more objects (pigeon-hole principle).

The proof of the first theorem and applications of the other two are of great interest. An interesting problem is to find the number of times one will have to verify the first theorem for $n=2, 3, 4, 5, \ldots$.

(15) *Transportation problems*

Some wagons are available at some stations and are required at other stations. We have to transport them in such a way as to minimize the total cost of transportation, the cost of transportation of a wagon from any origin to any destination being given.

The first stage is to find some feasible solutions.

The second stage is to find the total number of feasible solutions.

The third stage is to find costs of each of these feasible solutions.

The fourth stage is to find the minimum cost by trial and error.

The fifth stage is to give the algorithm for finding the minimum cost.

The final stage is to give the proof of the algorithm.

(16) *Sequencing problems*

If we have a number of books to be typed and printed and times of typing and printing are given, the problem is to find the sequence in which books should be taken up so that the idle time on the machine is minimized. For n books, there are $n!$ possibilities and for small values of n, the optimum sequence can be found by trial and error, but for large values of n, a definite algorithm is available which can be done at the school level.

We have also extensions of the problems when there are n machines and m jobs to be done.

(17) *Travelling salesman problem*

A travelling salesman has to visit n cities once and once only and to return to his starting point. The problem is to find his route so that his travelling cost is minimized. Again some simple problems can be done by enumeration. For large values of n, some plausible algorithms are available and these have to be worked on the computer.

(18) *Coding theory*

We can use symbols 0 and 1 and our messages are in words of length n symbols each. How many different words can we form so that the minimum distance between every two words (i.e. the number of places in which the symbols differ) is d? The total number of words in 2^n and the subset of these words in which the minimum distance is d is called an n-d code. An n-d code is called a full code if every word in it has a distance less than d from at least one word of the code. An n-d code is called maximal code if it is a full code and no other full code has more words than the given code. The number of words in it is denoted by $A(n, d)$. Students may find $A(n, d)$ for small values of n and d. There is no formula for $A(n, d)$, but bounds for it can be found.

The set of 2^n words is interesting because it can be equipped with group structure, ring structure, normed vector space structure, partial order structure etc. and thus can be used effectively for teaching algebraic structures.

Interesting combinatorial problems arise in the construction of Hadamard matrices whose elements are 1 and -1 and which are such that any two rows (or two columns) differ in n places and agree in n places.

(19) *Isling problems*

There is a rectangular $m \times n$ grid of unit squares. Each square has a blue or red color. The problem is to find the number of different color patterns if the number of boundary edges between red squares and blue squares is prescribed. For small values of m and n and the number of boundaries, the problem can be easily solved, but for the general case, the solution is not available.

(20) *Networks and dynamic programming*

There are a number of alternative routes from a student's house to his school. The map being given, it is desired to find the shortest route. This simple-looking problem can be used to introduce the concepts of recurrence relations, functional equations, dynamic programming, flow charts and computer programming. The techniques developed can also lead to the scientific solution of some of the classical puzzles like the problem of three jars, river crossings etc.

V. IMPLICATIONS OF THE COMBINATORIAL PROBLEMS

Some of these problems look trivial in their statements and they present great difficulties in their complete solutions, but most of these can be discussed at the school level.

The problems have deep implications for science and technology and we give some indication of it below:

(a) *Physics.* The theory of partitions has fundamental applications in thermodynamics and statistical mechanics. The Ising problem is important for the explanation of the macroscopic behavior of matter on the basis of known facts at molecular or atomic levels. The enumeration of symmetries is fundamental to crystallography and to the group-theoretic classification of fundamental particles. The enumeration of random walks is basic to the theory of diffusion, understanding of Brownian motion and solid state physics. Combinatorial problems are basic to the story of ferromagnetism in substances. Polya's theorem has been used to solve a difficult problem in Lie algebra which has a deep impact on contemporary physics.

(b) *Chemistry.* The enumeration of possible organic molecules, finding the isomers of a given molecule and the topology of molecular structure are all basic problems here.

(c) *Biological sciences.* Enumeration of possible types of organisms, spread of epidemics, testing of medicines and crops through designs of experiments like Latin squares, orthogonal Latin squares, balanced incomplete block designs etc., combinatorial problems in genetics are all important problems here.

(d) *Probability theory.* Probability theory is essentially combinatorial and as such all applications of probability theory also depend on combinatorial analysis. An important unsolved problem here is to find the probability that when N points are chosen at random in a cube of side L no two of them will be closer than a prescribed distance d. This problem has implications for statistical mechanics.

(e) *Management sciences.* These deal with the areas of organized complexity or with problems of finite but large numbers of degrees of freedom while classical mathematics dealt with either finite and small numbers of degrees of freedom or with almost infinite degrees of freedom. Most of the optimization problems are combinatorial in nature.

(f) *Logic and automata theory.* The theory of first state sequential machines give rise to many interesting combinatorial problems.

It will require a great deal of investigation, research and experimentation to find to which problems of combinatorial mathematics can go to school, but it is obvious that along with algebraic structures and geometric transformations, combinatorial problems provide a rich storehouse for modernizing and revitalizing our school curriculum. It is the author's hope that this source will be fully exploited in the future.

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