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Complexity reduction in a telephone switching system

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Abstract. This paper reviews some of the recent developments in complexity theory as applied to telephone-switching. Some of these techniques are suitable for practical implementation in India.

Keywords. Complexity theory; telephone switching; congestion theory.

1. Introduction

1.1 *Complexity theory and telephone switching*

Switching networks in telephone systems essentially consist of a large number of basically simple components. These networks provide paths for originating calls between the subscribers. Complexity theory helps us to minimise the number of switches in such a network. The following are the three types of telephone switching systems which find extensive applications in modern communication: (i) Strowger system, (ii) cross-bar system and (iii) electronic system.

Although we would in this paper confine ourselves to the application of complexity theory as applicable to cross-bar switching, extension to other types of switching is not, in principle, difficult.

1.2 *Background*

Complexity theory as applied to telephone switching has its roots in a number of fields, basically in computer science and related probability theory, set theory and graph theory. During the period 1900–1950, the most complex digital system was the telephone network. For the system engineers in the field, Erlang (Syski 1960) derived a formula under the following assumptions:

(i) The traffic incident onto the network is a pure chance traffic.
(ii) The number of contacts in a particular switching stage is equal to the number of contacts in the preceding stage.
(iii) The total traffic offered to the exchange is calculated from the average of the number of busy hour traffic.
(iv) When all switches are busy, the traffic incident on the exchange is lost.

The formula is

$$B = \frac{A^N/N!}{1 + A + A^2/2! + \ldots + A^N/N!}, \tag{1}$$

Where B is the proportion of the calls lost, A the traffic offered and N, the number of trunks. When N is large,

$$B = A^N/N! \, (\exp A). \tag{2}$$

When the traffic offered to a telephone network is known, the number of trunks required for a given proportion of lost calls can be obtained using this formula. The theories developed by Lee (1955) are discussed in detail in § 3. Another major step was the development of probablistic model suggested by Benes (1967).

In 1975 Pippenger of the University of Massachusetts has given a new shape to the complexity theory in telephone switching (Pippenger 1975).

1.3 *Purpose and method*

In this paper we provide a tutorial introduction to complexity theory as applicable to telephone switching. This paper first introduces the basic pre-requisites of complexity theory developed by Clos & Contor (Wolman 1965) and Bassalygo & Pinsker (1973). It then analyses simple probablistic models developed by Lee and Benes. The main body of the paper is the analysis and application of complexity theory to the cross-bar switching system.

2. Review

2.1 *Cross-bar switch*

An $n \times m$ cross-bar switch is a device with n inlets and m outlets. It consists of n horizontal and m vertical bars and the contacts are situated at the junction points of these bars as shown in figure 1.

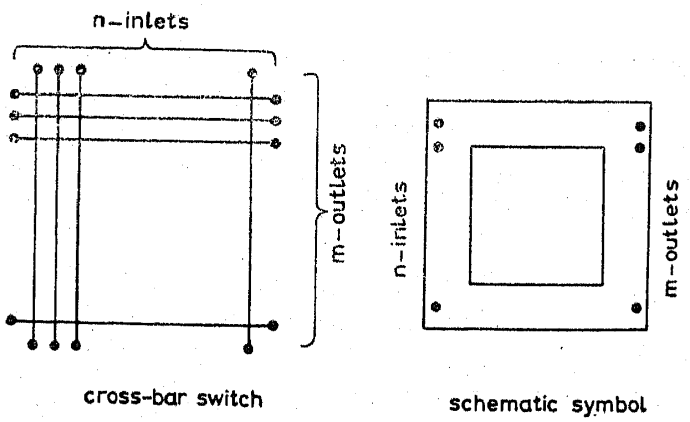

Figure 1. Details of the cross-bar telephone switch

When a call is made (in a telephone exchange) through a route, a called subscriber should be connected to a calling subscriber. A route is a set of contacts, one in each stage of switching; when closed simultaneously establish a continuous path from an input to an output. Two routes are compatible if they have no contacts in common. Compatible roots avoid fear of cross-talk. If there is an admissible route from the input to the output, then we say that they are linked, otherwise blocked. State is a set of routes, each two of which are compatible. The set of links interconnecting consecutive stages will be called rank. The inputs, outputs and the links will be referred to collectively as nodes in the case of a network. The set of inputs, the set of outputs, or a rank will be called a column. The definitions are illustrated in figure 2.

2.2 *Role of complexity theory*

To estimate the efficiency of cross-bar network design, consider a network capable of handling N calls, that is one with N trunks and N subscribers. In this case $N \times N$, or N^2 different calls can be made through the network and so there are N^2 switches in a cross-bar. To double the number of calls, the increase in the number of switches is four times. This kind of disproportionate increase in switches (called the diseconomy of scale of switching) can be overcome by studying the complexity theory of telephone switching.

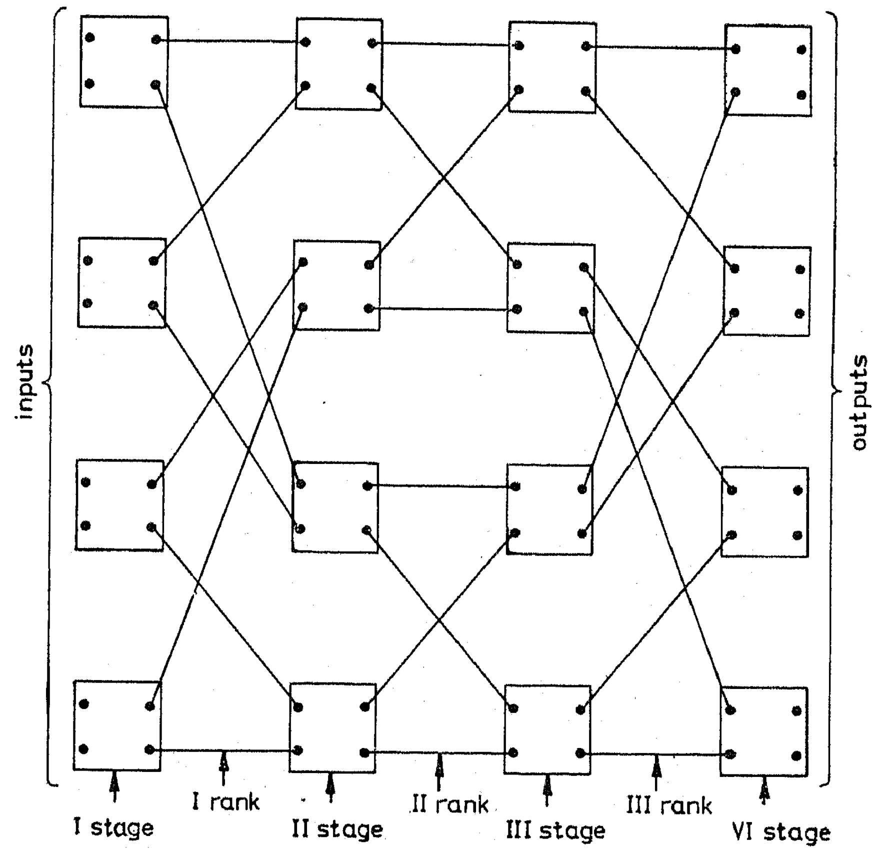

Figure 2. Illustration of definitions for telephone networks.

2.3 *Early studies*

In 1950 Claude E Shannon of Bell Laboratories proved that diseconomy of scale is an intrinsic feature of switching networks (see *e.g.* Pippenger 1978). He showed that the minimum number of switches per call is $\log_2 N$ when N calls are being originated. Hence the dependency of switches on the number of calls cannot be avoided.

Clos (see Pippenger 1978) constructed networks out of smaller sub-networks and proved that for N calls, the total number of switches required is $6\ N^{1.5}$ switches. In 1970 David G Contor (see Pippenger 1978) of the University of California found that $O\ [N(\log N)^2]$ switches are sufficient to construct a network to handle N calls (O denotes the order of the number of switches, leaving the importance of the multiplying constant). Subsequently Bassalygo & Pinsker (1973) of the Institute for Problems of Information Transmission in Moscow, proved that a network capable of handling N calls can indeed be built with $O\ (N \log N)$ switches. Figure 3 illustrates the relation between the number of calls and number of switches required for the different arrangements suggested.

2.4 *Telephone network connections*

Before dealing with the complexity theory of telephone switching some important definitions about the network connections are presented.

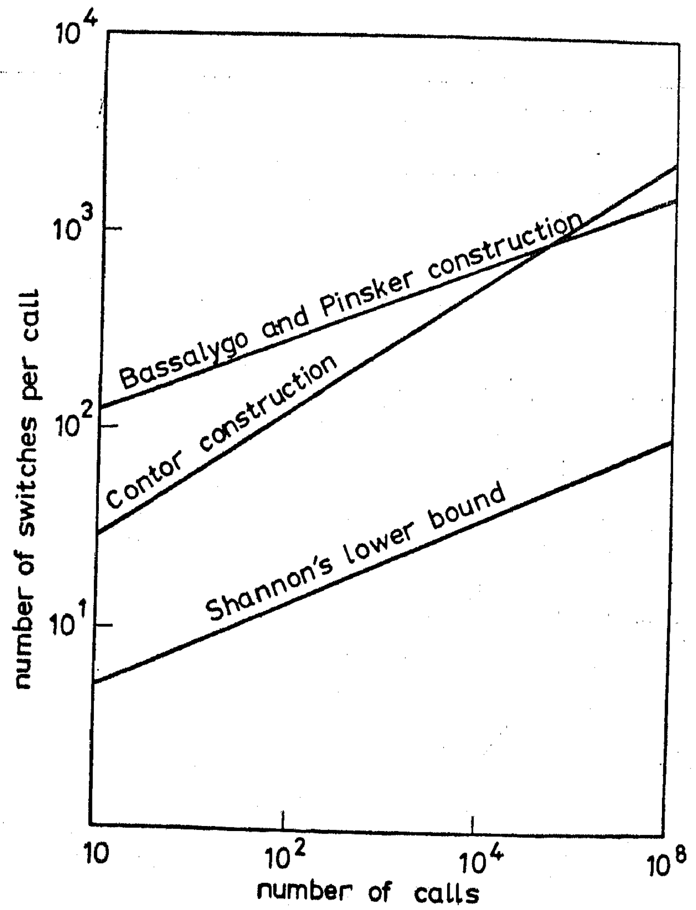

Figure 3. Switch requirements for different connections.

A switch with equal number of inlets and outlets is called a square switch. Any network which has square switches is called uniform network. If every switch is square, the total number of inlets on all of the switches in a given stage is equal to the total number of outlets on all these switches. To define series parallel networks, two ways of combining smaller networks to form a larger one are to be considered. Suppose there are two arbitrary networks U_1 and U, and suppose that U_1 has n inputs and n_1 outputs while U has m inputs and m_1 outputs. We construct a new network from these components called the series connection of U_1 and U denoted $[U_1, U]$ according to figure 4. We make m copies of U_1, one for each input of U, and these are called primary networks (or simply primaries). We make n_1 copies of U, one for each output of U_1 and these are called secondary networks (or simply secondaries). We interconnect these as shown, so that there is one link between a given primary and a given secondary.

Now we are given $n \times n_1$ switches, arbitrary networks U with m inputs and m_1 outputs and $n_1 \times n_2$ switches. From these components we construct a new network called a parallel connection of U and denoted $[n \times n_1, U, n_1 \times n_2]$. This scheme is shown in figure 5. We take $mn \times n_1$ switches and call them primaries. We take n_1 copies of U and call them as secondaries. We take $m_1 n_1 \times n_2$ switches and call them tertiaries. We interconnect these as shown, so that there is one link between a given primary and a given secondary, and one link between a given secondary and a given tertiary A series parallel network is one that can be constructed by starting with switches

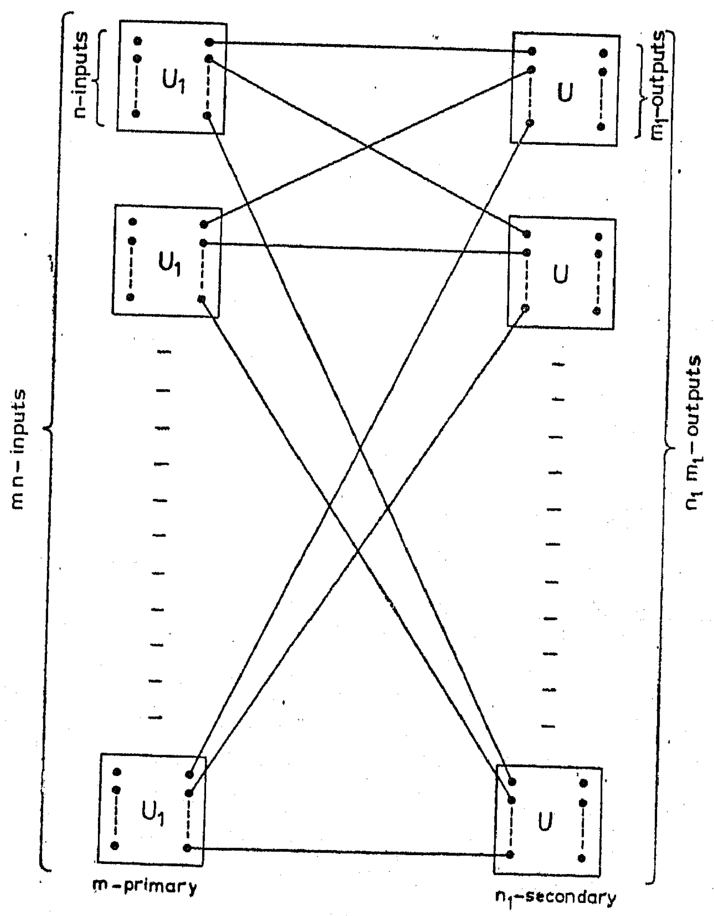

Figure 4. Series connection.

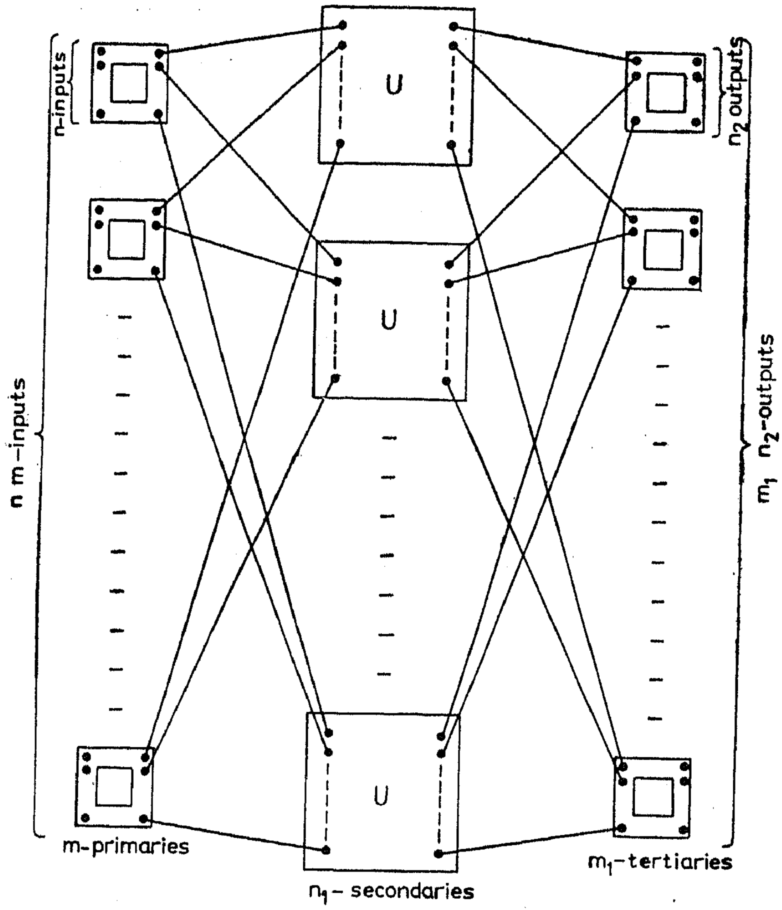

Figure 5. Parallel connection.

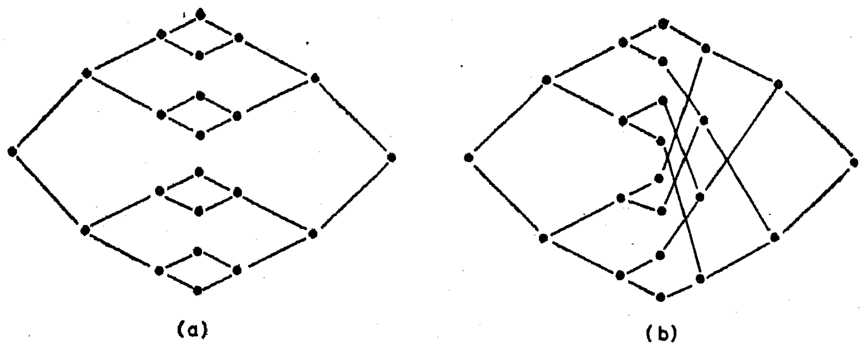

Figure 6. **a.** The path structure of series-parallel network. **b.** The path structure of non-series parallel network.

and applying the rules for series and parallel connections recursively any number of times in any order. For example, the network shown in figure 2 is a series parallel network. The term 'series parallel' refers to the structure obtained by considering all of the paths connecting a given input to a given output as shown in figure 6a. In this the points represent nodes and the lines represent connection through contacts. An example of a non-series parallel path structure is the so-called spider-web (see figure 6b). A uniform series parallel network is one that is both uniform and series parallel. Figure 2 shows such a network.

3. *Lee's probabilistic model*

Lee has constructed a very simple model of telephone network. It has been used to generate random states for simulation, to derive expressions for blocking probability and to study the asymptotic behaviour of network cost. Lee's model is based on the following assumptions.

3.1 *Assumption* 1

For every node v there is a probability $p(v)$ that v is busy and a complementary probability $q(v) = 1 - p(v)$, that v is idle.

3.2 *Assumption* 2

The condition of different nodes are independent.

The drawback of Lee's probabilistic models is that since all the nodes on a route are made busy or idle simultaneously when the route is added or dropped from the state, the conditions of nodes in different columns cannot, in general, be different.

4. Pippenger's probability model

Pippenger's (1975, 1978) probabilistic model is based on the following assumptions.

4.1 *Assumption P* 1

There are complementary probabilities p and $q=1-p$ such that every input is busy with probability p and idle with probability q.

4.2 *Assumption P* 2

The conditions of different inputs are independent.

4.3 *Assumption P* 3

If a given $n \times n$ switch has K busy inlets, all the $n(n-1) \ldots (n-K+1)$ ways in which they might be connected to K busy outlets are equally likely.

4.4 *Assumption P* 4

The conditions of (connection established by) different switches in a given stage are independent. The condition of a switch in one stage influences the condition of a switch in the later stage by affecting the busy inlets of the later switch, but not by affecting the way in which they are connected to the busy outlets.

These assumptions can be used to generate random states. Assumptions P 1 and P 2 can be used to determine which inputs are busy. Then assumptions P 3 and P 4 can be applied to each stage in turn, from the first to last. This results in the determination of closed contacts in the first stage and the busy links in the first rank. This process can be continued until the last stage. For various values of n and K, this particular process requires randomising device that will make independent choices from $n(n-1)\ldots(n-k+1)$ equally likely possibilities.

With particular values of random variables, if the process described is executed, it results in a state; similarly other states can be generated. All the assumptions and the description of generation of random states are based on the idea of working from the inputs to the outputs. The considerations discussed is applicable to uniform networks.

A network has the random routing property, if a busy input is equally likely, to be connected to any output and a busy output is equally likely to be connected to any input. We cannot generalise that all the uniform network has the random routing property because it may not even be possible for any input to be connected to any output. Using the following important propositions suggested by Pippenger any random routing problem can be solved.

4.5 *Proposition* 1

Every uniform series parallel network has the random routing property.

4.6 *Proposition* 2

In any series or parallel connection the states of different primaries or of different secondaries, or of different tertiaries are independent.

4.7 *Proposition* 3

In any series or parallel connection, the states of a primary and a secondary or of a secondary and tertiary, are independent, given the condition of the link connecting them.

5. Blocking probability

In this section the relations concerning the blocking probabilities of different kinds of network connection are discussed.

Pippenger has derived a relation according to Lee's probabilistic model as

$$P([U_1, U]) = 1 - q\,[1 - P(U_1)]\,[1 - P(U_1)], \tag{3}$$

where $P([U_1, U])$ is the blocking probability of series

connection of networks U_1 and U

$$q : \mathscr{P}(w \text{ idle}/v \text{ idle}, v' \text{ idle}) = \mathscr{P}(w \text{ idle})$$

$\mathscr{P}$ denotes the probability of the variable.

According to his own probabilistic model Pippenger has derived a relation for blocking probability as

$$P([U_1, U]) = 1 - Q\,[1 - P(U_1)]\,[1 - P(U)], \tag{4}$$

where $$Q = \frac{\mathcal{P}(v \text{ idle}/w \text{ idle})\, \mathcal{P}(v' \text{ idle}/w \text{ idle})\, \mathcal{P}(w \text{ idle})}{\mathcal{P}(v \text{ idle}, v' \text{ idle})}.$$

Here v denotes an input node, v' an output node and w the link connecting them. In practice it has been shown by Pippenger that $Q \geqslant q$.

An important special case of the relations (3) and (4) is that U_1 is a single $n \times n$ switch. A single square switch is non-blocking

$$P(n \times n) = 0. \tag{5}$$

Then (3) becomes

$$P([n \times n, U]) = 1 - q\,[1 - P(U)]. \tag{6}$$

and (4) becomes

$$P([n \times n, U]) = 1 - Q\,[1 - P(U)]. \tag{7}$$

In general

$$P([n \times n, U]) = F[P(U)]. \tag{8}$$

Similar expressions have been derived by Pippenger for series parallel network as

$$P([n \times n, U, n \times n]) = g[P(U)], \tag{9}$$

where $$g\,[P(U)] = \{(n-1)\,[1 - q\,(1 - P(U))]^2\,[1 - q^2\,(1 - P(U))]^{n-2} + P(U)\,[1 - q^2\,(1 - P(U))]^{n-1}\}/n. \tag{10}$$

Relations (3), (4), (8) and (9) are useful to calculate the blocking probability of series and parallel connected networks.

6. Recursion techniques in the determination of blocking probability

In this section we explain how blocking probability of telephone switching system with a number of series and parallel connections of networks can be determined. The recursion technique to be explained here is an effective tool for calculating the blocking probability of a switching system.

Let us take a telephone switching system consisting entirely of $n \times n$ switches. When it contains s number of series connection and t number of parallel connections, the switching network as a whole can be represented by a symbol $C_n^{s,t}$. Each series connection adds one stage and each parallel connection adds two stages with the existing system. With the help of definitions discussed in § 2 we can see that in a $C_n^{s,t}$ connection system, we will have

$s + 2t + 1$ stages,

n^{s+t+1} nodes/column,

n^{s+t+2} contacts per stage,

$(s + 2t + 1)\, n^{s+t+2}$ contacts in all.

The following are particular cases in $C_n^{s,t}$

$C_n^{0,0}$: Single $(n \times n)$ switch (with no series or parallel connection),

$C_n^{s+1,0}$: $(n \times n,\ C_n^{s,0})$,

$C_n^{s,t+1}$: $(n \times n,\ C_n^{s,t},\ n \times n)$.

Now using (5), (8) and (9) we can arrive at the relation

$$P(C_n^{0,0}) = 0, \tag{11}$$

$$P(C_n^{i,0}) = F_i\,[P(C_n^{i-1,0})], \tag{12}$$

$$P(C_n^{s,j}) = G_j\,[P(C_n^{s,j-1})], \tag{13}$$

$$P\,[C_n^{s,t}] = G_t\,[\ldots\, G_1\,(F_s\,(\ldots\, F_1\,(0) \ldots)) \ldots]. \tag{14}$$

Equations (11), (12), (13) and (14) are convenient forms for simulation in the digital computer.

7. Asymptotic behaviour of cost

Pippenger (1975) pointed out that the measure of cost is the number of contacts, and the average number of calls is the measure of traffic carried out. He has shown that the network $C_n^{s,t}$, where

$$s = \log_2 4/3.\ \log_3 N + O(1),$$

$$t = [\log_2 (3/2)]\ \log_3 N + O(1),\ n = 3,$$

and $p=\frac{1}{2}$, has $6N \log_2 N - O(N)$ contacts and carries N erlang* with a blocking probability at most E (for some $E < 1$, independent of N). This particular relation helps in determining the number of contacts in the switching system if the amount of traffic to be carried out is known.

*A traffic intensity of one erlang is obtained for any specified period when the average number of calls simultaneously in progress during that period is unity.

When a series parallel network $C_n^{s,t}$ has to be connected with $m \times m$ switches in parallel, the theories developed by Pippenger (1975) can be applied. In this particular case the number of contacts required to carry out N erlangs of traffic was found out to be $6\,N \log_2 N + O\,[N \log 1/\epsilon]$ where $m = O\,[\log 1/\epsilon]$, s, t, n and p are as defined previously and ϵ is the blocking probability.

8. Conclusions

An introduction to the complexity theory of telephone switching is presented utilising the theories developed by Erlangs, Clos, Contor, Bassalygo and Pinsker. Then the probabilistic models of the switching network explained by Lee and Pippenger are described. Finally the asymptotic behaviour of the cost in telephone switching network is explained with the help of Pippenger's theories. Many of the theories explained in this paper will be useful for a telephone engineer to design an efficient switching system.

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