PHOTO-ELASTIC EFFECT IN CRYSTALS

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Received October 16, 1942

1. *Introduction*

THE optical coefficients of a crystal give us the relation between the three components of an incident light vector and those of the induced optic moment vector. They should accordingly be 9 in number but reduce to 6 on account of the relation $C_{rs} = C_{sr}$. While their number remains at 6 in crystals of the triclinic system, a further reduction takes place when higher symmetry is present as the latter involves mutual relationships between the various coefficients. Similarly, the moduli of elasticity give the relation between the six components of the stress tensor on the one hand and those of the strain tensor on the other and these could be 36 in number but reduce to 21 even in triclinic crystals on account of the relation $C_{rs} = C_{sr}$. The stress-optical coefficients deal with the photo-elastic behaviour and give the relation between the optical coefficients of the crystal and the components of an applied stress tensor. Their maximum number is 36 and this number remains undiminished in the triclinic system of crystals because the relation $C_{rs} = C_{sr}$ does not generally hold good for stress-optical coefficients. Just as in the case of optical coefficients, the largest number of moduli of elasticity and of stress-optical coefficients that are required in each case depends on the symmetry properties of the crystal in question. The usual methods of ascertaining the number of coefficients required for each crystal system and of writing them out are known and described in standard treatises.[1]

A Group theoretical treatment of the subject in so far as it relates to the elastic properties has recently been given by Jahn.[2] In this paper, it is proposed to give an easy method based on the theory of groups for obtaining the number of optical, elastic and stress-optical coefficients necessary for

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[1] Reference may be made to Love's *Mathematical Theory of Elasticity* for literature relating to the elastic moduli. The subject of photo-elasticity in crystals is dealt with by Coker and Filon in their *Treatise on Photo-Elasticity* and by Szivessy in *Handbuch Der Physik*, 1929, **21**, 832. These authors have, however, only quoted the earlier and pioneering work on the subject by Pockels contained in *Lehrbuch der Kristalloptik*, 1906.

[2] *Z. Kristall.*, 1937, **98**, 191.

describing these properties for each of the 32 classes of crystals. There are some discrepancies between the results thus obtained and those given earlier by Pockels in respect of photo-elasticity. With a view to clarify these, detailed tables containing the stress-optical coefficients are worked out and given for all the classes of crystals.

2. *Formulation of the Method*

Let a_{xx}, a_{yy}, a_{zz}, a_{yz}, a_{zx} and a_{xy} represent the set of 6 independent optical coefficients of a crystal. Under an operation R_ϕ consisting of a rotation through ϕ or a rotation reflection through ϕ, these coefficients which constitute a tensor transform as products of cartesian co-ordinates. If R_ϕ is a covering operation for the crystal, the equations connecting the optic moments with the components of the incident light vector should remain invariant under such an operation. This requirement imposes certain restrictions on the coefficients and those or such combinations of those which remain invariant for all the covering operations alone will survive. The problem before us is to find the number of such surviving terms for each class of crystal symmetry.

Relations (1) and (2) show respectively how the cartesian co-ordinates and the tensor components transform. The plus and the minus sign where the alternative occurs refer respectively to cases of pure rotation and rotation reflection in the order in which they are given.

$$x \to x \cos\phi + y \sin\phi;\quad y \to -x \sin\phi + y \cos\phi;\quad z \to \pm z \tag{1}$$

$$\begin{aligned}
a_{xx} &\to a_{xx} \cos^2\phi + a_{yy} \sin^2\phi + a_{xy} \sin\phi \cos\phi \\
a_{yy} &\to a_{xx} \sin^2\phi + a_{yy} \cos^2\phi - a_{xy} \sin\phi \cos\phi \\
a_{zz} &\to a_{zz} \\
a_{yz} &\to \pm a_{yz} \cos\phi \mp a_{zx} \sin\phi \\
a_{zx} &\to \pm a_{yz} \sin\phi \pm a_{zx} \cos\phi \\
a_{xy} &\to -a_{xx} \sin\phi \cos\phi + a_{yy} \sin\phi \cos\phi + a_{xy} (\cos^2\phi - \sin^2\phi)
\end{aligned} \tag{2}$$

(2) may be regarded as a linear substitution. It is easily seen that the character of the transformation matrix in (2) works out as $2 \cos\phi\ (\pm 1 + 2 \cos\phi)$. These linear substitutions constitute a reducible representation of the group G. Six mutually orthogonal and independent linear combinations of the above variables may now be found in such a way that they fall into six or less number of sets, the members in each set transforming among themselves by every operation of the group G. These will constitute the basis for a new and completely reducible representation of the group G. The character appropriate to any element R_ϕ in this representation will be

the same as that obtained before since the two are equivalent. It is now quite easy to find n_i, the number of times a particular irreducible representation repeats itself in the representation consisting of the new variables with the help of the following formula[3]:—

$$n_i = \frac{1}{N} \sum_j h_j \chi_i(R) \chi_j'(R) \quad \ldots\ldots \qquad (3)$$

Since we want to know the number of combinations that remain invariant for all operations R, we need only find the value of n_i appropriate to the total symmetric irreducible representation. This is characterised by the fact that $\chi_i(R) = 1$ for all R. In this case $\chi_j'(R)$ has already been shown to be equal to $2 \cos \phi\, (\pm 1 + 2 \cos \phi)$ and h_j is the number of elements in the *j*th class of the symmetry group.

As has already been said, the elastic moduli which connect the components of the stress tensor with those of the strain tensor are 21 in number. Under a change of axes, these coefficients transform as products of tensor components. The transformation matrix can easily be written down with the help of (2) and it can be shown that the character appropriate to R_ϕ is $1 - 4 \cos^2 \phi \pm 8 \cos^3 \phi + 16 \cos^4 \phi$, the plus or the minus sign being used according as the operation is a pure rotation or a rotation reflection. Similar remarks as in the case of the optical coefficients apply here also and formula (3) will enable us to know the number of combinations that remain invariant for all operations R if we put $\chi_i(R) = 1$ for all R and carry the summation over all the elements of the symmetry group. This result represents the number of moduli required to describe the elastic properties of the particular crystal.

The stress-optical coefficients are 36 in number and connect the optical coefficients with the components of the stress tensor. Under change of axes, they transform as the elastic moduli but differ from them in that C_{sr} has to be distinguished from C_{rs}. If this is taken into account, the character of the transformation matrix works out as $4 \cos^2 \phi \pm 16 \cos^3 \phi + 16 \cos^4 \phi$. Use of this character in formula (3) will enable us to know the number of coefficients required to describe the photo-elastic properties of particular crystals.

The general formula for obtaining the numbers of coefficients in each case along with the appropriate character is thus given as follows:—

$$n_i = \frac{1}{N} \sum_j h_j \chi_j'(R)$$

[3] For further elucidation of the formula and the notation employed in this paper, reference may be made to Bhagavantam, *Scattering of Light and the Raman Effect*, 1940.

$$\begin{aligned} \text{where } \chi_j'(\mathrm{R}) &= 2\cos\phi\,(\pm 1 + 2\cos\phi) \ldots\ldots\ldots\ldots \text{for optical} \\ &= 1 - 4\cos^2\phi \pm 8\cos^3\phi + 16\cos^4\phi \ \text{ for elastic} \\ &= 4\cos^2\phi \pm 16\cos^3\phi + 16\cos^4\phi \ldots\ldots \text{for stress-optical} \end{aligned} \quad (4)$$

3. *Application to Crystals*

Results of applying formula (4) to all the 32 classes of crystals are given in Table I.

TABLE I

Crystal system	Symbol	Symmetry operations	Optical	Elastic	Stress-optical
Triclinic ..	C_1	E	6	21	36
	C_i	E i	6	21	36
Monoclinic ..	C_s	E σ_h	4	13	20
	C_2	E C_2	4	13	20
	C_{2h}	E C_2 i σ_h	4	13	20
Orthorhombic ..	C_{2v}	E C_2 σ_v σ_v'	3	9	12
	D_2	E C_2 C_2' C_2''	3	9	12
	D_{2h}	E C_2 C_2' C_2'' i σ_h σ_v' σ_v''	3	9	12
Tetragonal ..	C_4	E $2C_4$ C_2	2	7	10
	S_4	E $2S_4$ C_2	2	7	10
	C_{4h}	E $2C_4$ C_2 i $2S_4$ σ_h	2	7	10
	C_{4v}	E $2C_4$ C_2 $2\sigma_v$ $2\sigma_v'$	2	6	7
	D_{2d}	E C_2 C_2' C_2'' σ_v $2S_4$ σ_v'	2	6	7
	D_4	E $2C_4$ C_2 $2C_2$ $2C_2'$	2	6	7
	D_{4h}	E $2C_4$ C_2 $2C_2$ $2C_2'$ i $2S_4$ σ_h $2\sigma_v$ $2\sigma_v'$	2	6	7
Hexagonal ..	C_3	E $2C_3$	2	7	12
	S_6	E $2C_3$ i $2S_6$	2	7	12
	C_{3v}	E $2C_3$ $3\sigma_v$	2	6	8
	D_3	E $2C_3$ $3C_2$	2	6	8
	D_{3d}	E $2C_3$ $3C_2$ i $2S_6$ $3\sigma_v$	2	6	8
	C_{3h}	E $2C_3$ σ_h $2S_3$	2	5	8
	C_6	E $2C_6$ $2C_3$ C_2	2	5	8
	C_{6h}	E $2C_6$ $2C_3$ C_2 i $2S_6$ $2S_3$ σ_h	2	5	8
	D_{3h}	E $2C_3$ $3C_2$ σ_h $2S_3$ $3\sigma_v$	2	5	6
	C_{6v}	E $2C_6$ $2C_3$ C_2 $3\sigma_v$ $3\sigma_v'$	2	5	6
	D_6	E $2C_6$ $2C_3$ C_2 $3C_2$ $3C_2'$	2	5	6
	D_{6h}	E $2C_6$ $2C_3$ C_2 $3C_2$ $3C_2'$ i $2S_3$ $2S_6$ σ_h $3\sigma_v$ $3\sigma_v'$	2	5	6
Cubic ..	T	E $3C_2$ $8C_3$	1	3	4
	T_h	E $3C_2$ $8C_3$ i 3σ $8S_6$	1	3	4
	T_d	E $8C_3$ $3C_2$ 6σ $6S_4$	1	3	3
	O	E $8C_3$ $3C_2$ $6C_2$ $6C_4$	1	3	3
	O_h	E $8C_3$ $3C_2$ $6C_2$ $6C_4$ i $8S_6$ 3σ 6σ $6S_3$	1	3	3

The number of optical coefficients in each case is quite familiar and can easily be verified. Numbers in respect of elastic properties agree with the known results in all cases. Numbers in respect of photo-elasticity agree with those given by Pockels in all cases except C_4, S_4 and C_{4h} of the tetragonal system, C_3, S_6, C_{3h}, C_6 and C_{6h} of the trigonal system and T and T_h of the cubic system. With a view to obtain confirmation of the results reported in the present paper and to clarify the discrepancies, the stress-optical coefficients have also been worked out for all crystal classes by the direct method explained by Coker and Filon in their book. The results are given below and they show that the numbers of surviving coefficients in all cases are in agreement with those given in Table I. These may be compared with the results of Pockels quoted by Szivessy in the *Handbuch Der Physik* already referred to as the notation employed is the same.

First group.—(Triclinic Hemihedral and triclinic Holohedral) (36 coefficients)

q_{11}	q_{12}	q_{13}	q_{14}	q_{15}	q_{16}
q_{21}	q_{22}	q_{23}	q_{24}	q_{25}	q_{26}
q_{31}	q_{32}	q_{33}	q_{34}	q_{35}	q_{36}
q_{41}	q_{42}	q_{43}	q_{44}	q_{45}	q_{46}
q_{51}	q_{52}	q_{53}	q_{54}	q_{55}	q_{56}
q_{61}	q_{62}	q_{63}	q_{64}	q_{65}	q_{66}

Second group.—(Monoclinic Hemihedral, monoclinic Hemimorphic and monoclinic Holohedral) (20 coefficients)

q_{11}	q_{12}	q_{13}	0	0	q_{16}
q_{21}	q_{22}	q_{23}	0	0	q_{26}
q_{31}	q_{32}	q_{33}	0	0	q_{36}
0	0	0	q_{44}	q_{45}	0
0	0	0	q_{54}	q_{55}	0
q_{61}	q_{62}	q_{63}	0	0	q_{66}

Third group.—(Rhombic Hemimorphic, rhombic Hemihedral and rhombic Holohedral) (12 coefficients)

q_{11}	q_{12}	q_{13}	0	0	0
q_{21}	q_{22}	q_{23}	0	0	0
q_{31}	q_{32}	q_{33}	0	0	0
0	0	0	q_{44}	0	0
0	0	0	0	q_{55}	0
0	0	0	0	0	q_{66}

Fourth group.—(Trigonal Tetartohedral and trigonal Paramorphic) (12 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & q_{14} & -q_{25} & 2q_{62} \\ q_{12} & q_{11} & q_{13} & -q_{14} & q_{25} & -2q_{62} \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \\ q_{41} & -q_{41} & 0 & q_{44} & q_{45} & 2q_{52} \\ -q_{52} & q_{52} & 0 & -q_{45} & q_{44} & 2q_{41} \\ -q_{62} & q_{62} & 0 & q_{25} & q_{14} & q_{11}-q_{12} \end{matrix}$$

Fifth group.—(Trigonal Hemimorphic, trigonal Enantiomorphic and trigonal Holohedral) (8 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & q_{14} & 0 & 0 \\ q_{12} & q_{11} & q_{13} & -q_{14} & 0 & 0 \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \\ q_{41} & -q_{41} & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{44} & 2q_{41} \\ 0 & 0 & 0 & 0 & q_{14} & q_{11}-q_{12} \end{matrix}$$

Sixth group.—(Tetragonal Tetartohedral I, tetragonal Tetartohedral II and tetragonal Paramorphic) (10 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & 0 & 0 & q_{16} \\ q_{12} & q_{11} & q_{13} & 0 & 0 & -q_{16} \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & q_{45} & 0 \\ 0 & 0 & 0 & -q_{45} & q_{44} & 0 \\ q_{61} & -q_{61} & 0 & 0 & 0 & q_{66} \end{matrix}$$

Seventh group.—(Tetragonal Hemimorphic, tetragonal Hemihedral II, tetragonal Enantiomorphic and tetragonal Holohedral) (7 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & 0 & 0 & 0 \\ q_{12} & q_{11} & q_{13} & 0 & 0 & 0 \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{66} \end{matrix}$$

Eighth group.—(Hexagonal trigonal Paramorphic, hexagonal Terartohedral and hexagonal Paramorphic) (8 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & 0 & 0 & 2q_{62} \\ q_{12} & q_{11} & q_{13} & 0 & 0 & -2q_{62} \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & q_{45} & 0 \\ 0 & 0 & 0 & -q_{45} & q_{44} & 0 \\ -q_{62} & q_{62} & 0 & 0 & 0 & q_{11}-q_{12} \end{matrix}$$

Ninth group.—(All other classes of the Hexagonal system) (6 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & 0 & 0 & 0 \\ q_{12} & q_{11} & q_{13} & 0 & 0 & 0 \\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{11}-q_{12} \end{matrix}$$

Tenth group.—(Cubic Tetartohedral and cubic Paramorphic) (4 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{13} & 0 & 0 & 0 \\ q_{13} & q_{11} & q_{12} & 0 & 0 & 0 \\ q_{12} & q_{13} & q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{44} \end{matrix}$$

Eleventh group.—(All other classes of the cubic system) (3 coefficients)

$$\begin{matrix} q_{11} & q_{12} & q_{12} & 0 & 0 & 0 \\ q_{12} & q_{11} & q_{12} & 0 & 0 & 0 \\ q_{12} & q_{12} & q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{44} \end{matrix}$$

It is not clear how the extra coefficients in the classes cited above have been regarded by Pockels as vanishing. An important result of the present investigation relates to the T and T_h classes of the cubic system. It will be noticed that they require 4 stress-optical coefficients for the description of their photo-elastic behaviour while the rest of the classes under the cubic system require only 3. Such a distinction does not occur in respect of elastic moduli.

4. *Summary*

An easy method, based on the theory of groups, for obtaining the number of optical, elastic and stress-optical coefficients necessary for describing these properties for each of the 32 classes of crystals is given. The stress-optical coefficients are worked out in detail for all the 32 classes. It is noticed that the Tetartohedral and the Paramorphic hemihedral classes of the cubic system require four stress-optical coefficients for the description of their properties while the rest of the classes under this system require only three.