A METHOD OF CALCULATING THE WALL CORRECTION FOR ELLIPTIC TUNNELS

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Received December 1, 1944

The first solution of the problem of finding the wall-interference of a wind tunnel of elliptic cross-section was given by Sanuki and Tani.[1] The method used in this investigation does not give a physical picture of the phenomenon and does not appear to be very suitable for engineers, to handle. I. Lotz[2] gave a general method for the correction of downwash in wind tunnels. Using Lotz's method of converting the elliptic boundary of the tunnel into two parallel planes by the use of elliptic co-ordinates, Gavin and Hensel[3] have worked out the upwash due to the walls of the M.I.T. Tunnel. This method has got the advantage of representing the effect of the wall as equivalent to a system of images due to the tip vortices of the airfoil placed inside. However, in getting the value of the velocity induced by the image vortices, the series requires the values of these image positions in elliptic co-ordinates. The method used in this paper avoids the exponential functions, by using a different function for transforming the ellipse, and the image system is obtained by using simple algebraic functions only.

Let the elliptical boundary be represented by the equation.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

in the Z-plane; then by using a transformation function;

$$t=\frac{Z \pm \sqrt{Z^2-4c}}{2}$$

the ellipse is converted into two concentric circles of radii r_1 and r_2 such that $r_1=\frac{a+b}{2}$; $r_2=\frac{a-b}{2}$ and $c=\frac{a^2-b^2}{4}$ where a and b are the semi-major and semi-minor axes of the ellipse. The entire space inside the ellipse is now enclosed in the annular space between the two circles r_1 and r_2 in the t-plane. The major axis of the ellipse, which is chosen as the real axis in the Z-plane has got its corresponding points on the real axis of the t-plane where $Z^2 > 4c$. For $Z^2 = 4c$ the two values in the t-plane coincide and the corresponding value is given by the equation

$$t=\sqrt{c} \text{ for } Z=2\sqrt{c},$$

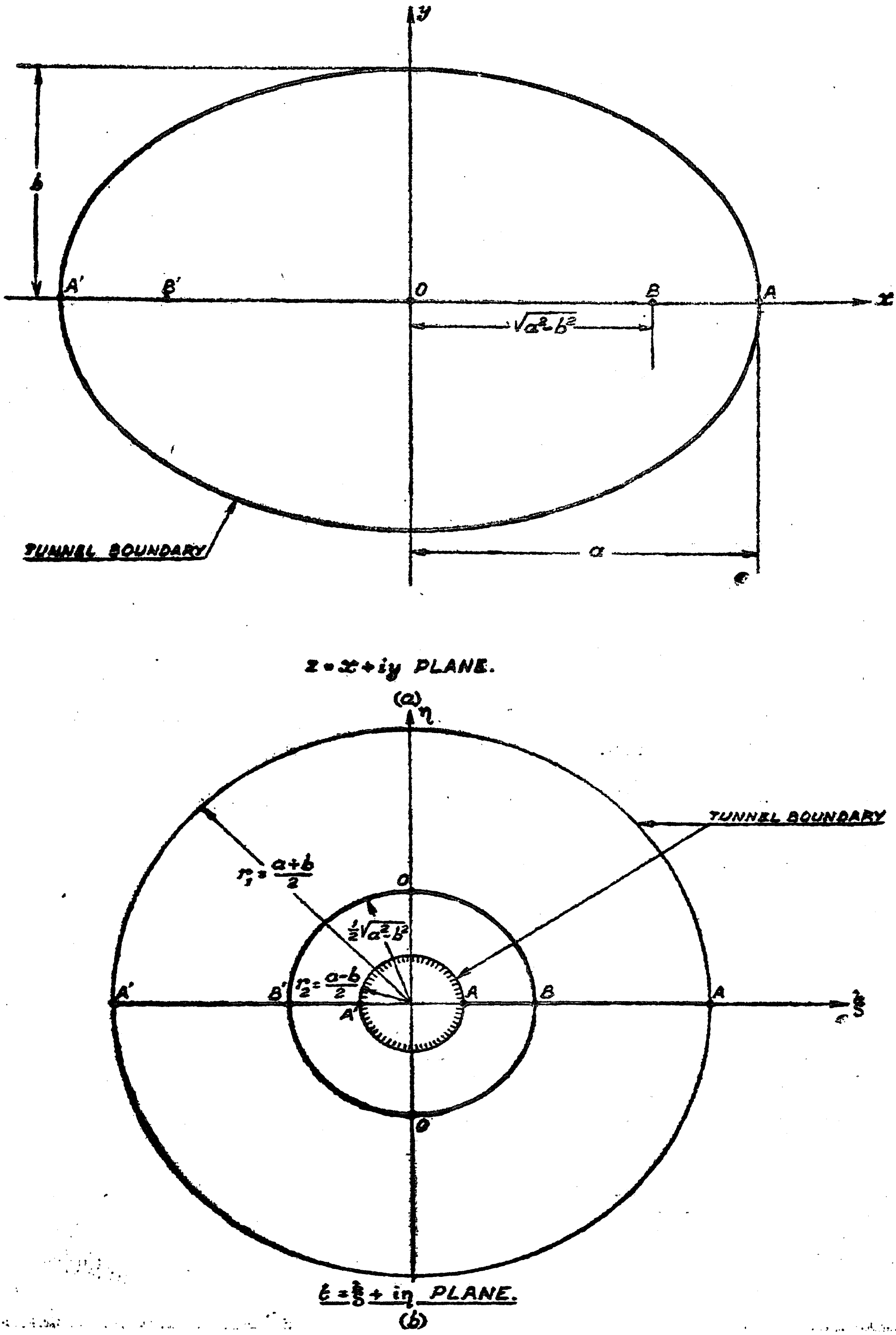
y
b
A'
B'
O
B
A
x
$\sqrt{a^2-b^2}$
a
TUNNEL BOUNDARY
z = x + iy PLANE.
(a)
η
TUNNEL BOUNDARY
$r_1 = \frac{a+b}{2}$
O
$\frac{1}{2}\sqrt{a^2-b^2}$
$r_2 = \frac{a-b}{2}$
A'
B'
A'
A
B
A
ξ
O
ζ = ξ + iη PLANE.
(b)

Fig. 1

this particular value of Z gives the distance of the focus from the centre of the ellipse. When $Z^2 < 4c$ the corresponding values in the t-plane lie on a circle of radius $\sqrt{c}$. The t values for this case are of the form, $t = \xi + i\eta$ and the two values of t make an angle $\pm\ \theta$ with the real axis, where $\tan\theta = \frac{\eta}{\xi}$.

The picture of the elliptical boundary with the major axis as transformed to the t-plane is shown in Fig. 1 (*b*). If we consider the airfoil, situated in the tunnel with its span along the major axis and symmetrical about the centre, to be replaced by a lifting line with two tip vortices at the end of the line situated at $\pm\, x$, we can get the corresponding position of these vortices in the t-plane. Three cases arise.

$$\text{(i)}\ \ x < 2\sqrt{c}; \quad \text{(ii)}\ \ x = 2\sqrt{c}; \quad \text{(iii)}\ \ x < 2\sqrt{c}.$$

In the first case, x has two values ξ_1 and ξ_2 in the t-plane and a vortex at x is now represented by two vortices in the same sense at ξ_1 and ξ_2.

Using the laws of reflection in the circular boundary, we can get an infinite series of images for each of these vortices at ξ_1 and ξ_2. If we consider the direction of rotation of the tip vortex to be positive and any image having an opposite sense of rotation as negative then the image positions for the vortices in the t-plane are given by following values.

$$-\frac{1}{\xi_1}\left(\frac{r_1^{2n+2}}{r_2^{2n}}\right);\ -\frac{1}{\xi_1}\left(\frac{r_2^{2n+2}}{r_1^{2n}}\right);\ +\xi_1\left(\frac{r_1}{r_2}\right)^{2n+2};\ +\xi_1\left(\frac{r_2}{r_1}\right)^{2n+2} \tag{1}$$

$$-\frac{1}{\xi_2}\left(\frac{r_1^{2n+2}}{r_2^{2n}}\right);\ -\frac{1}{\xi_2}\left(\frac{r_2^{2n+2}}{r_1^{2n}}\right);\ +\xi_2\left(\frac{r_1}{r_2}\right)^{2n+2};\ +\xi_2\left(\frac{r_2}{r_1}\right)^{2n+2} \tag{2}$$

$$n = 0, 1, 2, 3, \ldots\ldots\ldots$$

These image positions can be transformed back to the Z-plane by the inverse function $Z = t + \frac{c}{t}$ and we get the image system for the vortex situated at x in the Z-plane. Fortunately the two systems corresponding to the two values need not be calculated as it can be shown that

$$Z_{n1} = \frac{1}{\xi_1}\frac{r_1^{2n+2}}{r_2^{2n}} + \frac{c\xi_1 r_2^{2n}}{r_1^{2n+2}} = Z_{n2} = \frac{1}{\xi_2}\frac{r_2^{2n+2}}{r_1^{2n}} + \frac{c\xi_2 r_1^{2n}}{r_2^{2n+2}}.$$

This avoids the calculation with one of the values ξ_1 and ξ_2 and any one of them can be neglected. The images all lie on the real axis and in the space outside the ellipse.

In the second case x is represented by a single point ξ on the real axis of the t-plane but additional simplification occurs as for this particular value

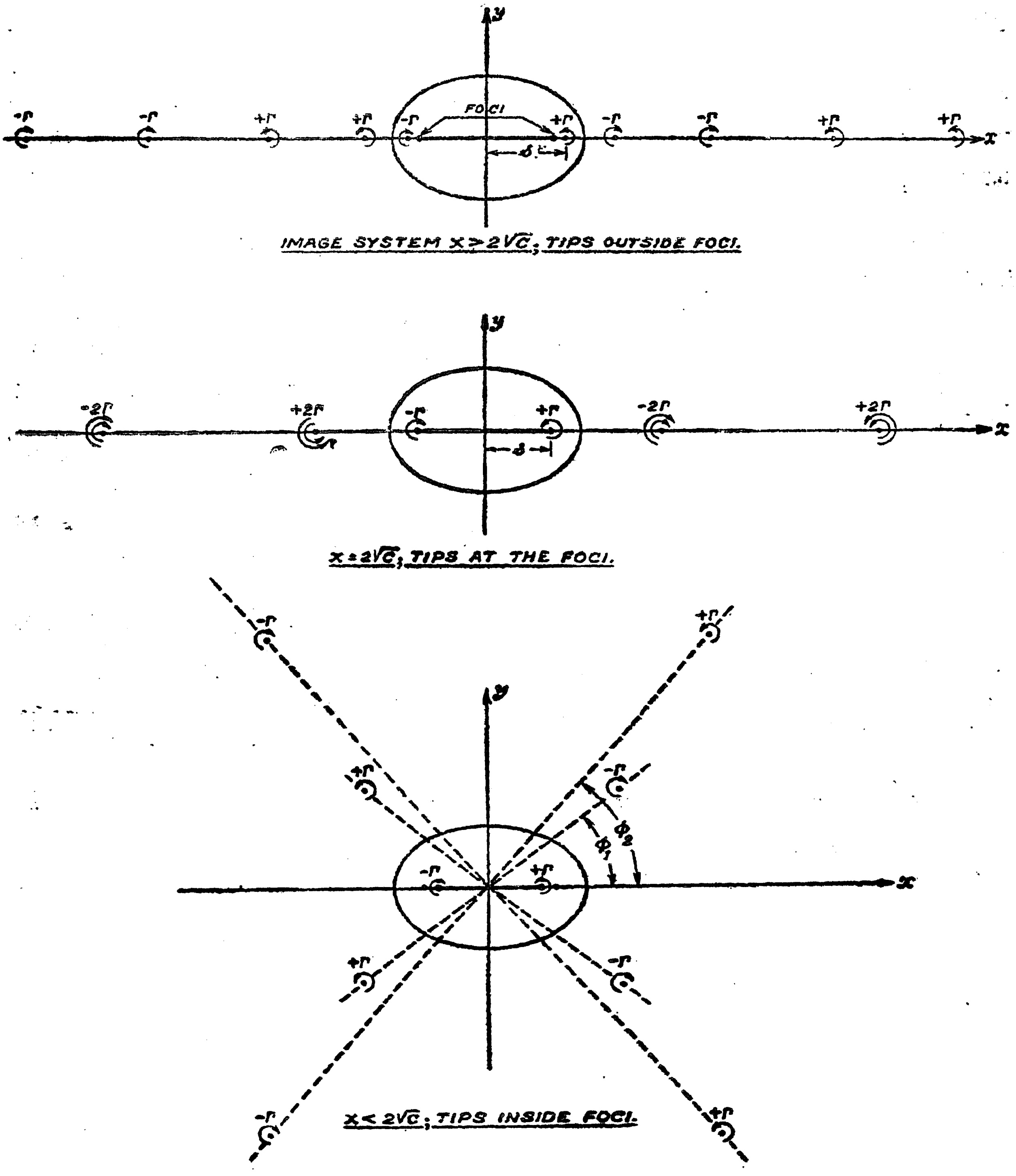
y
x
FOCI
-Γ
+Γ
δ
IMAGE SYSTEM X>2√C; TIPS OUTSIDE FOCI.
-2Γ
+2Γ
X=2√C; TIPS AT THE FOCI.
Φ1
Φ2
X<2√C; TIPS INSIDE FOCI.

Fig. 2

of $\xi = \sqrt{c}$

$$\frac{r_1^{2n+2}}{\sqrt{c}\, r_2^{2n}} + \frac{c^{3/2} r_2^{2n}}{r_1^{2n+2}} = \frac{r_2^{2n+2}}{\sqrt{c}\, r_1^{2n}} + \frac{c^{3/2} r_1^{2n}}{r_2^{2n+2}}$$

and we get half the number of images of each kind with double the strength of the original vortex.

When $x < 2\sqrt{c}$, the vortex in the Z-plane has two positions in the t-plane given by $t_1 = \dfrac{x + i\sqrt{4c - x^2}}{2}$ and $t_2 = \dfrac{x - i\sqrt{4c - x^2}}{2}$ and one is the reflection of the other in the real axis. The angle subtended by these positions at the centre is $\pm\,\theta$ and is given by the equation, $\tan\theta = \pm\dfrac{\sqrt{4c - x^2}}{x}$. All the images due to the vortices at t_1 and t_2 lie on radial lines making an angle $\pm\,\theta$ with the real axis. It is interesting to note that the absolute values of t_1 and t_2 are the same for all values of $x < 2\sqrt{c}$; only the value of θ changes in each case so that the radial distances of the images are the same, the change being shown by the values of θ.

If we denote the image due to a vortex at either t_1 or t_2 by $t_n = 1/r_n\, e^{\pm i\theta}$ then in the Z-plane we get

$$Z_n = x_n + iy_n = r_n e^{\pm i\theta} + \frac{c}{r_n} e^{\mp i\theta}. \tag{3}$$

Expanding $e^{i\theta}$ we get,

$$Z_n = x_n + iy_n = \left(r_n + \frac{c}{r_n}\right)\cos\theta \pm \left(r_n - \frac{c}{r_n}\right)\sin\theta$$

and Z_n makes an angle ϕ in the Z-plane given by the relation

$$\tan\phi = \frac{y_n}{x_n} = \pm\frac{r_n^2 - c}{r_n^2 + c}\tan\theta. \tag{4}$$

It will be seen from equation (4) that ϕ is different for different images in the Z-plane though θ is the same. As r_n increases in magnitude, ϕ approaches the value of θ.

The positions of the images in the three cases are represented in Fig. 2.

The induced upward velocity ω at any point can now be written down in the three cases as

$$\omega = \frac{\Gamma}{2n}\left\{\frac{1}{2}\left(-\frac{1}{x_1 - x} - \frac{1}{x_1 + x} - \frac{1}{x_2 - x} - \frac{1}{x_2 + x} + \frac{1}{x_3 - x} + \frac{1}{x_3 + x} + + - -\right)\right\} \tag{5}$$

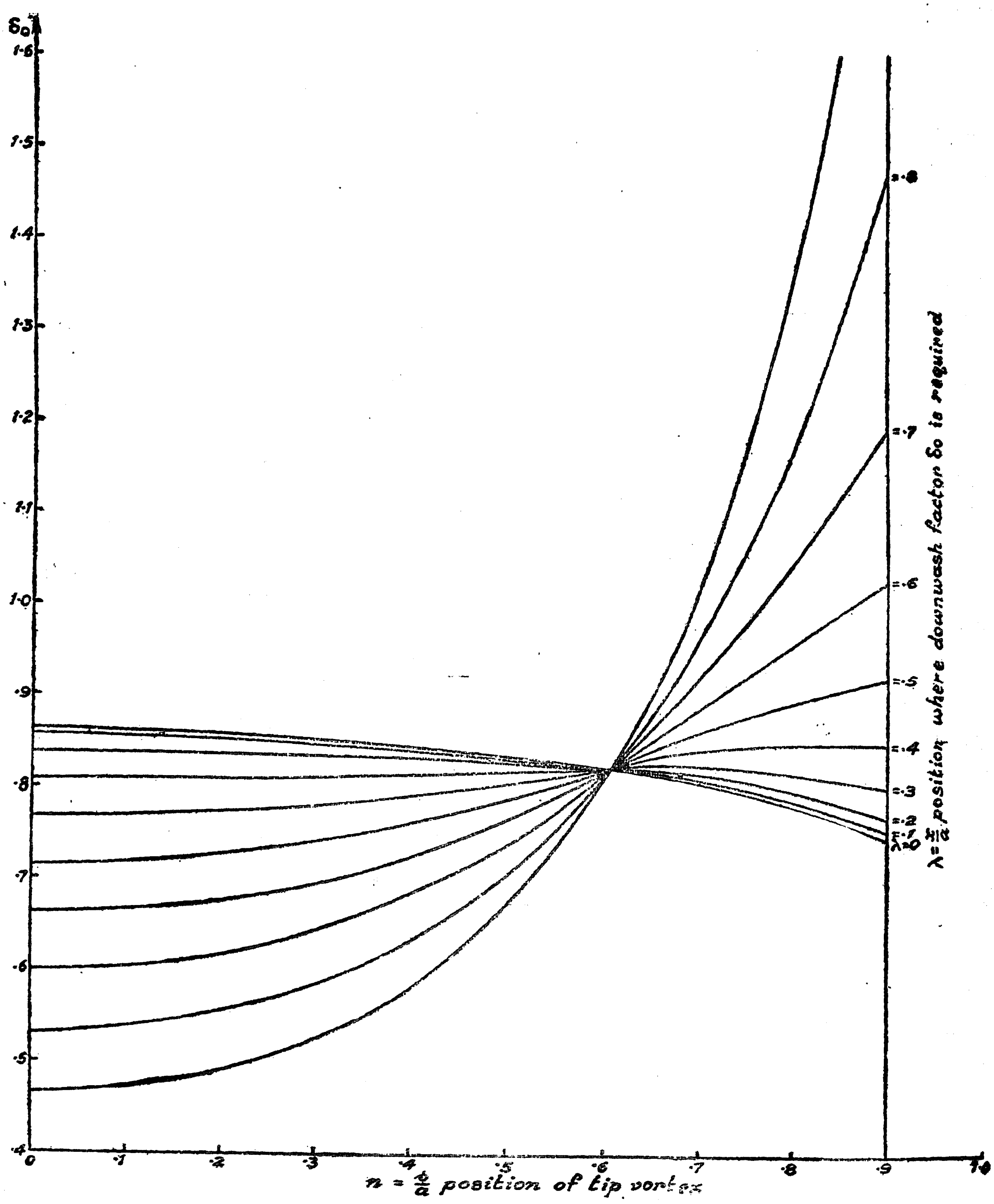
δ_0
1·6
1·5
1·4
1·3
1·2
1·1
1·0
·9
·8
·7
·6
·5
·4
0
·1
·2
·3
·4
·5
·6
·7
·8
·9
1·0
$n = \frac{c}{a}$ position of tip vortex
=·8
=·7
=·6
=·5
=·4
=·3
=·2
=·1
λ=0
$\lambda = \frac{x}{a}$ position where downwash factor δ_0 is required

Fig. 3

where $x_1, x_2, \ldots. x_n$ are the image points for the case $x < 2\sqrt{c}$, *i.e.*, the wing tips outside the foci

$$\omega = \frac{\Gamma}{2n}\left\{-\frac{1}{x_1 - x} - \frac{1}{x_1 + x} + \frac{1}{x_2 - x} + \frac{1}{x_2 + x} - - ++ ..\right\} \quad (6)$$

for the case $x = 2\sqrt{c}$ or the tips at the foci.

$$\omega = \frac{\Gamma}{2n}\left\{-\frac{x_1 \cos\phi_1 - x}{x_1^2 + x^2 - 2x_1 x \cos\phi_1} - \frac{x_1 \cos\phi_1 + x}{x_1^2 + x^2 + 2x_1 x \cos\phi_1}\right.$$

$$\left. + \frac{x_2 \cos\phi_2 - x}{x_2^2 + x^2 - 2x_2 x \cos\phi_2} + \frac{x_2 \cos\phi_2 + x}{x_2^2 + x^2 + 2x_2 x \cos\phi_2} - - ++ ..\right\} \quad (7)$$

for $x < 2\sqrt{c}$, *i.e.*, the wing tips inside the foci. Downward induced velocity is taken as positive and so the negative sign indicates that the image system induces an upwash in a closed tunnel. If we express the positions of the images in terms of the semi-major axis of the tunnel and use the tunnel constants to introduce the cross-sectional area of the tunnel, we get

$$\omega = \frac{\Gamma}{2\pi a}\left\{-\frac{1}{x_1' - x'} - \frac{1}{x_1' - x'} + \frac{1}{x_2' - x'} + \frac{1}{x_2' + x'} - - ++ \ldots\right\}$$

In the case $x = 2\sqrt{c}$

where $$x_1' = \frac{x_1}{a};\; x' = \frac{x}{a}$$

If the expression within the brackets is denoted by δ (δ is $-$ve for closed tunnel)

$$\omega = \frac{\Gamma}{2\pi a}\cdot\delta = \frac{\Gamma \cdot n\alpha a}{2\,\pi\alpha a^2}\cdot\frac{\delta}{n} = \frac{\Gamma s}{2S_T}\,\delta_0$$

where $\delta_0 = \frac{\alpha\delta}{n}$; $na = s$ the semispan of the lifting line, and $S_T = \pi\alpha a^2$ the tunnel area. Using the expression for the induced upwash angle $\frac{\omega}{V}$ in radians, where V is the velocity of the relative wind we get

$\left(\frac{\omega}{V}\right)_{nx} = \frac{\Gamma na}{2V S_T}\,\delta_{0nx}$ where n represents the position of the tip vortex and x represents the place where the upwash is calculated. The series for δ converges rapidly and it is necessary to calculate only three or four terms to get an accuracy of one per cent.

To get the upwash due to an airfoil of semispan $s = na$ and having a given lift distribution, the circulation along the semispan under the influence of the tunnel wall is broken into a number of components represented by lifting lines of suitable spans and appropriate strength of circulation uniformly distributed. If Γ_0 represents the circulation at the centre and Γ_m at the tip of the mth component lifting line in any given case, then,

$$\left(\frac{\omega}{V}\right)_{nx} = \frac{\Gamma_0\, na}{2V\, S_T} \cdot \Sigma\, K_\gamma \cdot K_s \cdot \delta_{0mx}$$

where $\frac{\Gamma_m}{\Gamma_0} = K_\gamma$; and $\frac{s_m}{s} = K_s$.

For elliptic distribution of circulation it can be shown that

$$\frac{\Gamma_0}{V} = \frac{C_L\, S}{\pi na}, \tag{8}$$

where S is the wing area.

Substituting this value we get,

$$\left(\frac{\omega}{V}\right)_{nx} = \frac{C_L}{2\pi} \frac{S}{S_T} \sum_m K_\Gamma \cdot K_s \cdot \delta_{0mx} \tag{9}$$

This variation of the upwash along the span of the airfoil can now be plotted and an average value obtained for any case to get a correction of the wall influence

$$\left(\frac{\omega}{V}\right)_{av} = \frac{C_L}{2\pi} \frac{S}{S_T} \left(\sum_m K_\gamma \cdot K_s \cdot \delta_{0mx}\right)_{av} \tag{10}$$

The average value of the wall correction factor Δ is then defined by the equation

$$\left(\frac{\omega}{V}\right) = \Delta \frac{S}{S_T} \cdot C_L.$$

Comparing this with equation (10), we have

$$\Delta = \frac{1}{2\pi} \left(\sum_m K_\gamma \cdot K_s \cdot \delta_{0mx}\right)_{av}$$

If a distribution other than elliptical is used, a comparison may be made by assuming that both airfoils have the same aspect ratio and carry the same total load. Under this condition equation (8) can be modified with the result that

$$\Delta = \frac{1}{2\pi} \cdot \frac{\Gamma_9}{\Gamma_{0ell}} \left(\sum_m K_\gamma \cdot K_s \cdot \delta_{0mx}\right)_{av}.$$

Using this method, the average correction factor is calculated for rectangular airfoils for the tunnel of the Department of Aeronautical Engineering, Indian Institute of Science, Bangalore. Fig. 3 gives the values of δ_0 for different positions of the tip vortices at different places on the major axis of the tunnel, both expressed in the non-dimensional form. The average values of Δ for rectangular airfoils are plotted in Fig. 4 which also gives the tunnel constants.

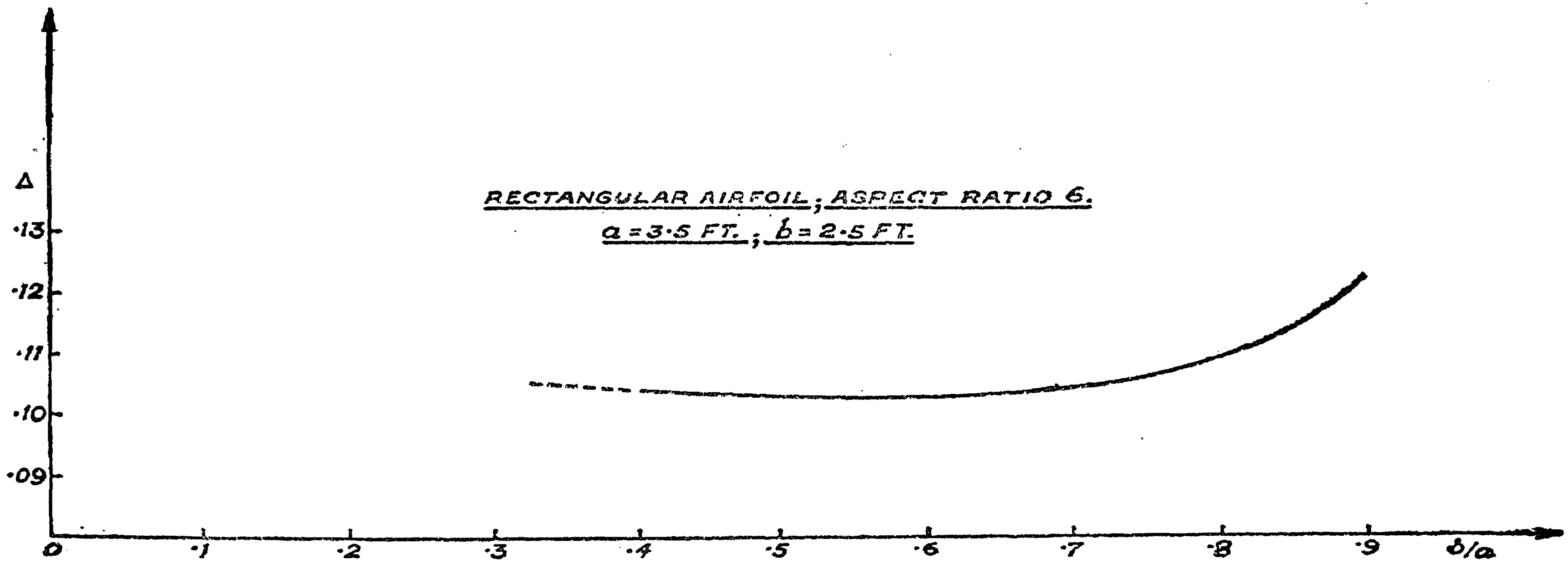

Fig. 4

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3. J. G. Gavin Jr. and R. W. Hensel "Elliptic Tunnel Wall Corrections on Drag and Stall," *Jour. Ae. Sc.*, 1942, **9**, No. 14, 533–37.